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On some properties of the bibasic Humbert hypergeometric functions Ξ_1 and Ξ_2

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Abstract. The main object of this paper is to deduce the bibasic Humbert functions Ξ_1 and Ξ_2 Some interesting results and elementary summations technique that was successfully employed, $q-$recursion, $q-$derivatives relations, the $q-$differential recursion relations, the $q-$integral representations for Ξ_1 and Ξ_2 are given. The summation formula derives a list of $p-$analogues of transformation formulas for bibasic Humbert functions that have been studied, also some hypergeometric functions properties of some new interesting special cases have been given.

§1 Introduction

The bibasic summation formula and multivariable hypergeometric functions have been investigated and have a long history see, for example, [3, 11, 13, 36], also for two variables or more multivariable basic hypergeometric functions see, for example, [1, 16, 21, 29, 33, 40]. Recently, considerable attention from several researchers and mathematicians has been developed and studied the bibasic hypergeometric functions in [2, 4, 14, 20, 23, 29] were presented, the basic hypergeometric functions, which namely $q-$analogue generalizations of the ordinary hypergeometric series have a long history [23]. The derivatives, $q-$derivatives of the generalization of the hypergeometric series and basic hypergeometric series with respect to the parameters see, also for example, [14,24–26]. Bibasic $q-$Appell functions with contiguous bibasic $q-$Appell functions, $q-$statistical summability method [17] and certain $q-$partial derivative relations have been derived in [36]. Also, the authors [36, 37] introduce some summation formulas for the bibasic $q-$Appell function and the relation between the bibasic Appell series and continued fractions of Ramanujan. Some multivariable generalizations of the bibasic summation formula have been studied in [3, 11, 22]. Also, some transformation formulas expressing certain sums of bibasic series by using contour integrals and the calculus of residues defined by Agarwal and Verma, see [11], for $q-$ calculus and $q-$analogue, see some examples as [5,6,8,9,18]. Some extensions of the bibasic indefinite summation formula to derive bibasic extensions of Euler's transformation formula and of fields and Wimp expansion formula have been obtained by Gasper [13].

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Because of their utility and applications in a variety of research fields, the bibasic hypergeometric functions associated with special matrix functions, matrix series have received recent attention see, e.g., [4, 31, 32], and also see, e.g., [10, 19, 27, 28, 30, 34, 35, 39] for some q-function of different type of matrix functions.

The q-analogue of τ, also known as the q-number or q-bracket number $[\tau]_q$ is defined by [12]

$$[\tau]_q = 1 + q + q^2 + \ldots + q^{\tau-1} = \frac{1-q^\tau}{1-q}; \tau \in \mathbb{C}. \tag{1}$$

For r the factorial $[r]_q!$ is defined by [12]

$$[r]_q! = \begin{cases} [1]_q + [2]_q + \ldots + [r]_q, & r \in \mathbb{N}; \\ 1, & r = 0. \end{cases} \tag{2}$$

The q-shifted factorial is defined by [12]

$$(q^\chi; q)_\tau = \begin{cases} \prod_{r=0}^{\tau-1}(1-q^{\chi+r}), & \tau \geq 1; \\ 1, & \tau = 0. \end{cases}$$
$$= \begin{cases} (1-q^\chi)(1-q^{\chi+1})\ldots(1-q^{\chi+\tau-1}), & \tau \in \mathbb{N},\ q^\chi \in \mathbb{C} \ \setminus \{1, 1/q, 1/q^2, \ldots, 1/q^{\tau-1}\}; \\ 1, & \chi \in \mathbb{C},\ 0 < |q| < 1,\ q \in \mathbb{C} - \{1\}. \end{cases} \tag{3}$$

The p-analogue of gamma functions $\Gamma_p(c)$ is defined by the q-integral representations as follows

$$\Gamma_p(c) = \int_0^{\frac{1}{1-p}} E_p(-pu)u^{c-1}d_pu, \tag{4}$$

where

$$E_p(x) = \sum_{\tau=0}^{\infty} \frac{p^{\binom{\tau}{2}}}{[\tau]_p!} x^\tau.$$

The q-binomial theorem is

$$\phi_1(p^a; -; p, x) = \frac{(xp^a; p)_\infty}{(x; p)_\infty}, |x| < 1. \tag{5}$$

In [12], the basic or q-hypergeometric function has been defined as follows

$$\phi_1(p^a, p^b; p^c; p, x) = \sum_{\tau=0}^{\infty} \frac{(p^a; p)_\tau (p^b; p)_\tau}{(p^c; p)_\tau (p; p)_\tau} x^\tau, p^c \neq 1, q^{-1}, q^{-2}, \ldots, |x| < 1. \tag{6}$$

Now we may define a bibasic Humbert functions Ξ_1 and Ξ_2 on the two independent bases p and q, for $0 < |p| < 1$, $0 < |q| < 1$, $p, q \in \mathbb{C}$, as follows:

$$\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y)$$
$$= \sum_{k,\ell=0}^{\infty} \frac{(p^a; p)_k (q^b; q)_\ell (p^c; p)_k}{(p^d; p)_{\ell+k} (p; p)_k (q; q)_\ell} x^\ell y^k, p^d \neq 1, q^{-1}, q^{-2}, \ldots, |x| < 1, |y| < 1 \tag{7}$$

and

$$\Xi_2(p^a, q^b; p^c; p, q, x, y)$$
$$= \sum_{k,\ell=0}^{\infty} \frac{(p^a; p)_k (q^b; q)_\ell}{(p^c; p)_{\ell+k} (p; p)_k (q; q)_\ell} x^\ell y^k, p^c \neq 1, q^{-1}, q^{-2}, \ldots, |x| < 1, |y| < 1. \tag{8}$$

Also, q-difference operator defined by [15]

$$D_{y,q} f(y) = \big[f(y) - f(yq)\big]\big[(1-q)y\big]^{-1}, \tag{9}$$

where f is a function real or complex -valued and $D_{y,q}f(0) = f'(0)$ provided $f(0)$ exists.

§2 Some new auxiliary results

Here we utilize the interesting results and elementary summations technique that was successfully employed in the bibasic Humbert confluent hypergeometric functions Ξ_1 and Ξ_2 on two independent bases p and q of two variables.

Theorem 2.1. *The following q-recursion relations for Ξ_1 and Ξ_2 are true*

$$\begin{aligned}\Xi_1(p^{a+1},q^b,p^c;p^d;p,q,x,y) =&\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\ &+\frac{p^a(1-p^c)y}{1-p^d}\Xi_1(p^{a+1},q^b,p^{c+1};p^{d+1};q,p,x,y), p^d\neq 1,\end{aligned} \tag{10}$$

$$\begin{aligned}\Xi_1(p^a,q^b,p^{c+1};p^d;q,p,x,y) =& \Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\ &+\frac{p^c(1-p^a)y}{1-p^d}\Xi_1(p^{a+1},q^b,p^{c+1};p^{d+1};q,p,x,y), q^d\neq 1,\end{aligned}$$

$$\begin{aligned}&\Xi_1(p^a,p^b,p^c;p^{d-1};q,p,x,y)\\ &=\frac{1}{1-p^{d-1}}\Xi_1(p^a,q^b,p^c;p^d;q,p,x,y)-\frac{p^{d-1}}{1-p^{d-1}}\Xi_1(p^a,q^b,p^c;p^d;q,p,px,y)\\ &+\frac{p^{d-1}(1-p^a)(1-p^c)y}{(1-p^{d-1})(1-p^d)}\Xi_1(p^{a+1},q^b,p^{c+1};p^{d+1};q,p,px,y), p^d,p^{d-1}\neq 1,\\ &=\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)+\frac{p^{d-1}(1-p^a)(1-p^c)y}{(1-p^{d-1})(1-p^d)}\Xi_1(p^{a+1},q^b,p^{c+1};p^{d+1};q,p,x,y)\\ &+\frac{p^{d-1}}{1-p^{d-1}}\Xi_1(p^a,q^b,p^c;p^d;q,p,x,py)-\frac{p^{d-1}}{1-p^{d-1}}\Xi_1(p^a,q^b,p^c;p^d;q,p,px,py), p^d,p^{d-1}\neq 1\end{aligned} \tag{11}$$

and

$$\begin{aligned}&\Xi_2(p^{a+1},q^b;p^c;p,q,x,y)\\ &\quad=\Xi_2(p^a,q^b;p^c;p,q,x,y)+\frac{p^a y}{1-p^c}\Xi_2(p^{a+1},q^b;p^{c+1};q,p,x,y), p^c\neq 1,\end{aligned}$$

$$\begin{aligned}&\Xi_2(p^a,p^b;p^{c-1};q,p,x,y)\\ &\quad=\frac{1}{1-p^{c-1}}\Xi_2(p^a,q^b;p^c;q,p,x,y)-\frac{p^{c-1}}{1-p^{c-1}}\Xi_2(p^a,q^b;p^c;q,p,px,y)\\ &\quad+\frac{p^{c-1}(1-p^a)y}{(1-p^{c-1})(1-p^c)}\Xi_2(p^{a+1},q^b;p^{c+1};q,p,px,y), p^c,p^{c-1}\neq 1,\\ &\quad=\Xi_2(p^a,q^b;p^c;p,q,x,y)+\frac{p^{c-1}(1-p^a)y}{(1-p^{c-1})(1-p^c)}\Xi_2(p^{a+1},q^b;p^{c+1};q,p,x,y)\\ &\quad+\frac{p^{c-1}}{1-p^{c-1}}\Xi_2(p^a,q^b;p^c;q,p,x,py)-\frac{p^{c-1}}{1-p^{c-1}}\Xi_2(p^a,q^b;p^c;q,p,px,py), p^c,p^{c-1}\neq 1.\end{aligned} \tag{12}$$

Proof. To prove (10), we proceed as follows

$$(p^{a+1};p)_k=\frac{1-p^{a+k}}{1-p^a}(p^a;p)_k=\left[1+p^a\frac{1-p^k}{1-p^a}\right](p^a;p)_k,$$

and

$$\begin{aligned}(p^a;p)_{k+1}&=(1-p^a)(p^{a+1};p)_k,\\ (p^c;p)_{k+1}&=(1-p^c)(p^{c+1};p)_k,\\ (p^d;p)_{\ell+k+1}&=(1-p^d)(p^{d+1};p)_{\ell+k},\end{aligned}$$

we have

$$\begin{aligned}
&\Xi_1(p^{a+1},q^b,p^c;p^d;p,q,x,y)-\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\
&=p^a\sum_{\ell,k=0}^{\infty}\left[\frac{1-p^k}{1-p^a}\right]\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k\\
&=\frac{p^a}{1-p^a}\sum_{\ell=0,k=1}^{\infty}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_{k-1}(q;q)_\ell}x^\ell y^k\\
&=\frac{p^a}{1-p^a}\sum_{\ell,k=0}^{\infty}\frac{(p^a;p)_{k+1}(q^b;q)_\ell(p^c;p)_{k+1}}{(p^d;p)_{\ell+k+1}(p;p)_k(q;q)_\ell}x^\ell y^{k+1}\\
&=\frac{p^a(1-p^c)y}{1-p^d}\Xi_1(p^{a+1},p^b;,p^{c+1};p^{d+1};q,p,x,y).
\end{aligned}$$

Similarly, by same manner the relations (11)-(12) can be obtained. □

Theorem 2.2. *The following relations for Ξ_1 and Ξ_2 are true*

$$\begin{aligned}
&\Xi_1(p^a,q^{b+1},p^c;p^d;q,p,x,y)=\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\
&+\frac{q^bx}{1-p^d}\Xi_1(p^a,q^{b+1},p^c;p^{d+1};q,p,x,y),p^d\neq 1,
\end{aligned}\tag{13}$$

$$\begin{aligned}
&(1-q^b)\Xi_1(p^a,q^{b+1},p^c;p^d;q,p,x,y)+q^b\Xi_1(p^a,q^b,p^c;p^d;q,p,qx,y)\\
&\quad=\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y),\\
&\Xi_1(p^a,q^{b+1},p^c;p^d;q,p,x,y)\\
&\quad=\Xi_1(p^a,q^b,p^c;p^d;q,p,qx,y)+\frac{x}{1-p^d}\Xi_1(p^a,q^{b+1},p^c;p^{d+1};q,p,x,y),p^d\neq 1
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
&(1-q^b)\Xi_2(p^a,q^{b+1};p^c;q,p,x,y)+q^b\Xi_2(p^a,q^b;p^c;q,p,qx,y)=\Xi_2(p^a,q^b;p^c;p,q,x,y),\\
&\Xi_2(p^a,q^{b+1};p^c;q,p,x,y)\\
&\quad=\Xi_2(p^a,q^b;p^c;p,q,x,y)+\frac{q^bx}{1-p^c}\Xi_2(p^a,q^{b+1};p^{c+1};q,p,x,y),p^c\neq 1,\\
&\Xi_2(p^a,q^{b+1};p^c;q,p,x,y)\\
&\quad=\Xi_2(p^a,q^b;p^c;q,p,qx,y)+\frac{x}{1-p^c}\Xi_2(p^a,q^{b+1};p^{c+1};q,p,x,y),p^c\neq 1.
\end{aligned}\tag{15}$$

Proof. Using the relation

$$(q^b;q)_{\ell+1}=(1-q^b)(q^{b+1};q)_\ell=(1-q^{b+\ell})(q^b;q)_\ell,$$

we have

$$\begin{aligned}
&\Xi_1(p^a,q^{b+1},p^c;p^d;q,p,x,y)-\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\
&=q^b\sum_{\ell,k=0}^{\infty}\left[\frac{1-q^\ell}{1-q^b}\right]\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k\\
&=\frac{q^b}{1-q^b}\sum_{\ell=1,k=0}^{\infty}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_{\ell-1}}x^\ell y^k\\
&=\frac{q^bx}{1-q^b}\sum_{\ell,k=0}^{\infty}\frac{(p^a;p)_k(q^b;q)_{\ell+1}(p^c;p)_k}{(p^d;p)_{\ell+k+1}(p;p)_k(q;q)_\ell}x^\ell y^k.
\end{aligned}$$

Now, by making use of the relation (7), we obtain

$$= \frac{q^b x}{1-p^d}\Xi_1(p^a, q^{b+1}, p^c; p^{d+1}; q, p, x, y), p^d \neq 1.$$

Similarly, we can easily prove the subsequent results (14)-(15). □

Theorem 2.3. *The q-derivatives relations for Ξ_1 and Ξ_2 holds true*

$$D^m_{x,q}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) = \frac{(q^b;q)_m}{(1-q)^m(p^d;p)_m}\Xi_1(p^a, q^{b+m}, p^c; p^{d+m}; q, p, x, y), \tag{16}$$

$$D^n_{y,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) = \frac{(p^a;p)_n(p^c;p)_n}{(1-p)^n(p^d;p)_n}\Xi_1(p^{a+n}, q^b, p^{c+n}; p^{d+n}; q, p, x, y),$$

$$\begin{aligned} &D^m_{x,p}D^n_{y,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) \\ &= \frac{(p^a;p)_n(q^b;q)_m(p^c;p)_n}{(1-q)^m(1-p)^n(p^d;p)_{m+n}}\Xi_1(p^{a+n}, q^{b+m}, p^{c+n}; p^{d+m+n}; q, p, x, y) \end{aligned} \tag{17}$$

and

$$\begin{aligned} D^m_{x,q}\Xi_2(p^a, q^b; p^c; p, q, x, y) &= \frac{(q^b;q)_m}{(1-q)^m(q^c;q)_m}\Xi_2(q^a, q^{b+m}; p^{c+m}; q, p, x, y), \\ D^n_{y,p}\Xi_2(p^a, q^b; p^c; p, q, x, y) &= \frac{(p^a;p)_n}{(1-p)^n(p^c;p)_n}\Xi_2(q^{a+n}, p^b; p^{c+n}; q, p, x, y), \\ D^m_{x,q}D^n_{y,p}\Xi_2(p^a, q^b; p^c; p, q, x, y)& \\ = \frac{(p^a;q)_n(q^b;q)_m}{(1-q)^m(1-p)^n(p^{;c};p)_{m+n}}&\Xi_2(q^{a+n}; p^{b+m}; q^{c+m+n}; q, p, x, y). \end{aligned} \tag{18}$$

Proof. From q-derivative in (9), we yield

$$\begin{aligned} &D_{x,q}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) \\ &= \sum_{\ell,k=0}^{\infty}\frac{1-q^\ell}{1-q}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^{\ell-1}y^k \\ &= \sum_{\ell,k=0}^{\infty}\frac{(p^a;p)_k(q^b;q)_{\ell+1}(p^c;p)_k(1-q)^{-1}}{(p^d;p)_{\ell+k+1}(p;p)_k(q;q)_\ell}x^\ell y^k \\ &= \frac{(1-q^b)}{(1-q)(1-p^d)}\sum_{\ell,k=0}^{\infty}\frac{(p^a;p)_k(q^{b+1};q)_\ell(p^c;p)_k}{(p^{d+1};p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k \\ &= \frac{(1-q^b)}{(1-q)(1-p^d)}\Xi_1(p^a, q^{b+1}, p^c; p^{d+1}; q, p, x, y) \end{aligned} \tag{19}$$

and

$$\begin{aligned} &D_{y,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) \\ &= \sum_{\ell=0,k=1}^{\infty}\frac{1}{1-p}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_{k-1}(q;q)_\ell}x^\ell y^{k-1} \\ &= \frac{(1-p^a)(1-p^c)}{(1-p)(1-p^d)}\sum_{\ell,k=0}^{\infty}\frac{(p^{a+1};q)_k(q^b;q)_\ell(p^{c+1};p)_k}{(p^{d+1};p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k \\ &= \frac{(1-p^a)(1-p^c)}{(1-p)(1-p^d)}\Xi_1(p^{a+1}, q^b, p^{c+1}; p^{d+1}; q, p, x, y). \end{aligned} \tag{20}$$

Now by iterating q-derivative on Ξ_1 for m-times and n-times, we get (16) and (17). By applying the relation of the q-derivative in (8), we obtained (18). □

Theorem 2.4. *The recursion relations of q-differential for* Ξ_1 *and* Ξ_2 *holds true:*

$$\begin{aligned} xD_{x,q}\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y) =& \frac{1-q^b}{(1-q)q^b}\Big[\Xi_1(p^a,q^{b+1},p^c;p^d;q,p,x,y) \\ & -\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\Big], \\ xD_{x,q}\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y) =& \frac{1-q^b}{1-q}\Big[\Xi_1(p^a,q^{b+1},p^c;p^d;q,p,x,y) \\ & -\Xi_1(p^a,q^b,p^c;p^d;p,q,qx,y)\Big], \end{aligned} \tag{21}$$

$$\begin{aligned} yD_{y,p}\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y) =& \frac{1-p^a}{(1-p)p^a}\Big[\Xi_1(p^{a+1},q^b,p^c;p^d;q,p,x,y) \\ & -\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\Big], \\ yD_{y,p}\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y) =& \frac{1-p^c}{(1-p)p^c}\Big[\Xi_1(p^a,q^b,p^{c+1};p^d;q,p,x,y) \\ & -\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\Big] \end{aligned} \tag{22}$$

and

$$\begin{aligned} xD_{x,q}\Xi_2(p^a,q^b;p^c;p,q,x,y) =& \frac{1-q^b}{(1-q)q^b}\Big[\Xi_2(p^a,q^{b+1};p^c;q,p,x,y) \\ & -\Xi_2(p^a,q^b;p^c;p,q,x,y)\Big], \\ xD_{x,q}\Xi_2(p^a,q^b;p^c;p,q,x,y) =& \frac{1-q^p}{1-q}\Big[\Xi_2(p^a,q^{b+1};p^c;q,p,x,y) \\ & -\Xi_2(p^a,q^b;p^c;p,q,qx,y)\Big], \\ yD_{y,p}\Xi_2(p^a,q^b;p^c;p,q,x,y) =& \frac{1-p^a}{(1-p)p^a}\Big[\Xi_2(p^{a+1},q^b;p^c;q,p,x,y) \\ & -\Xi_2(p^a,q^b;p^c;p,q,x,y)\Big]. \end{aligned} \tag{23}$$

Proof. In view of equation (12) and (13), we get (21). In the same way we can also be obtained (22)-(23). □

Corollary 2.5. *The following relations for* Ξ_1 *and* Ξ_2 *holds true:*

$$\begin{aligned} &(1-p^{d-1})\Xi_1(p^a,q^b,p^c;q^{d-1};q,p,x,y) = (1-p^{d-1})\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y) \\ &+(1-p)p^{d-1}yD_{y,p}\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y) + p^{d-1}\Xi_1(p^a,q^b,p^c;q^d;q,p,x,py) \\ &-p^{d-1}\Xi_1(p^a,q^b,p^c;q^d;q,p,px,py), \end{aligned} \tag{24}$$

$$(1-p^{d-1})\Xi_1(p^a,q^b,p^c;q^{d-1};q,p,x,y)$$

$$
\begin{aligned}
&= \Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) - p^{d-1}\Xi_1(p^a, q^b, p^c; p^d; p, q, px, y) \\
&\quad + (1-p)p^{d-1} y D_{y,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, px, y), \\
&(1-p^a)\Xi_1(p^{a+1}, q^b, p^c; p^d; q, p, x, y) = (1-p) y D_{y,p}\Xi_1(p^a, q^b, p^c; p^d; q, p, x, y) \\
&\quad + (1-p^a)\Xi_1(p^a, q^b, p^c; p^d; q, p, x, py), \\
&(1-p^c)\Xi_1(p^a, q^b, p^{c+1}; p^d; q, p, x, y) = (1-p) y D_{y,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) \\
&\quad + (1-p^c)\Xi_1(p^a, q^b, p^c; p^d; q, p, x, py), \\
&(1-q^b)\Xi_1(p^a, q^{b+1}, p^c; p^d; q, p, x, y) = (1-q) x D_{x,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) \\
&\quad + (1-q^b)\Xi_1(p^a, q^b, p^c; p^d; q, p, qx, y)
\end{aligned} \tag{25}
$$

and

$$
\begin{aligned}
&(1-p^{c-1})\Xi_2(p^a, q^b; p^{c-1}; q, p, x, y) \\
&\quad = (1-p^{c-1})\Xi_2(p^a, q^b; p^c; p, q, x, y) + (1-p)p^{c-1} y D_{y,p}\Xi_2(p^a, q^b; p^c; p, q, x, y) \\
&\quad + p^{c-1}\Xi_2(p^a, q^b; p^c; q, p, x, py) - p^{c-1}\Xi_2(p^a, q^b; p^c, q^d; q, p, px, py), \\
&(1-p^{c-1})\Xi_2(p^a, q^b; q^{c-1}; q, p, x, y) = \Xi_2(p^a, q^b; p^c; p, q, x, y) \\
&\quad - p^{c-1}\Xi_2(p^a, q^b; p^c; p, q, px, y) + (1-p)p^{c-1} y D_{y,p}\Xi_2(p^a, q^b; p^c; p, q, px, y), \\
&(1-p^a)\Xi_2(p^{a+1}, q^b; p^c; q, p, x, y) \\
&\quad = (1-p) y D_{y,p}\Xi_2(p^a, q^b; p^c; q, p, x, y) + (1-p^a)\Xi_2(p^a, q^b; p^c, p^c; q, p, x, py), \\
&(1-q^b)\Xi_2(p^a, q^{b+1}; p^c; q, p, x, y) \\
&\quad = (1-q) x D_{x,p}\Xi_2(p^a, q^b; p^c; p, q, x, y) + (1-q^b)\Xi_2(p^a, q^b; p^c; q, p, qx, y).
\end{aligned} \tag{26}
$$

Proof. In view of (11) and (20)yields (24). Similarly, by same manner the relations (25)-(26) can be obtained. □

Theorem 2.6. *The following relations for Ξ_1 and Ξ_2 holds true:*

$$
\begin{aligned}
D_{d,q}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) =& \frac{1}{1-p^d}\bigg[y D_{y,q}\Xi_1(p^{d+1}) + \frac{1}{1-p}\Xi_1(p^{d+1}, py) \\
& - \frac{1}{1-p}\Xi_1(p^{d+1}, px, py)\bigg],
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
D_{a,q}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) =& -\frac{1}{1-p^a} y D_{y,p}\Xi_1, \\
D_{b,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) =& -\frac{1}{1-q^b} x D_{x,q}\Xi_1, \\
D_{c,p}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) =& -\frac{1}{1-p^c} y D_{y,p}\Xi_1, \\
D_{d,q}\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) =& \frac{1}{1-p^d}\bigg[y D_{y,q}\Xi_1(p^{d+1}, px) + \frac{1}{1-p}\Xi_1(p^{d+1}) \\
& - \frac{1}{1-p}\Xi_1(p^{d+1}, px)\bigg]
\end{aligned} \tag{28}
$$

and

$$
D_{a,q}\Xi_2(p^a, q^b; p^c; p, q, x, y) = -\frac{1}{1-p^a} y D_{y,p}\Xi_2,
$$

$$\begin{aligned} D_{b,p}\Xi_2(p^a,q^b;p^c;p,q,x,y) =& -\frac{1}{1-q^b}xD_{x,q}\Xi_2,\\ D_{c,q}\Xi_2(p^a,q^b;p^c;p,q,x,y) =& \frac{1}{1-p^c}\Big[yD_{y,q}\Xi_2(p^{c+1})+\frac{1}{1-p}\Xi_2(p^{c+1},py)\\ &-\frac{1}{1-p}\Xi_2(p^{c+1},px,py)\Big],\\ D_{c,q}\Xi_2(p^a,q^b;p^c;p,q,x,y) =& \frac{1}{1-p^c}\Big[yD_{y,q}\Xi_2(p^{c+1},px)+\frac{1}{1-p}\Xi_2(p^{c+1})\\ &-\frac{1}{1-p}\Xi_2(p^{c+1},px)\Big]. \end{aligned} \tag{29}$$

Proof. Now we have

$$\begin{aligned} &D_{d,p}\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\ &=\sum_{\ell,k=0}^{\infty}\left[\frac{1}{(p^d;p)_k}-\frac{1}{(p^{d+1};p)_{\ell+k}}\right]\frac{1}{(1-p)p^d}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p;p)_k(q;q)_\ell}x^\ell y^k\\ &=\sum_{\ell,k=0}^{\infty}\left[1-\frac{1-p^d}{1-p^{d+\ell+k}}\right]\frac{1}{(1-p)p^d}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k\\ &=\sum_{\ell,k=0}^{\infty}\left[\frac{1-p^{d+\ell+k}-1+p^d}{1-p^{d+\ell+k}}\right]\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(1-p)p^d(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k,\\ &=\frac{1}{1-p^d}\sum_{\ell,k=0}^{\infty}\frac{1-p^{\ell+k}}{1-p}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^c;p)_\ell(q^{d+1};q)_k(p;p)_k(q;q)_\ell}x^\ell y^k\\ &=\frac{1}{1-p^d}\sum_{\ell,k=0}^{\infty}\left[\frac{1-p^k}{1-p}+\frac{p^k-p^{\ell+k}}{1-p}\right]\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^c;p)_\ell(q^{d+1};q)_k(p;p)_k(q;q)_\ell}x^\ell y^k\\ &=\frac{1}{1-p^d}\left[yD_{y,p}\Xi_1(p^{d+1})+\frac{1}{1-p}\Xi_1(q^{d+1},py)-\frac{1}{1-p}\Xi_1(q^{d+1},px,py)\right]. \end{aligned}$$

Proceeding along the same lines as above, formulas (28)-(29) can be proved. □

Theorem 2.7. *For $0<\Re(\alpha)<\Re(\gamma)$, the q-integral representations for Ξ_1 is true*

$$\begin{aligned} &\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\ &=\frac{1}{\Gamma_p(c)}\int_0^{\frac{1}{1-p}}E_p(-pu)u^{c-1}\Xi_2(p^a,q^b;p^d;p,q,x,(1-p)yu)d_pu. \end{aligned}\tag{30}$$

Proof. Using the p-shifted factorials $(p^c;p)_k$ and the p-Gamma functions are defined as follows

$$(p^c;p)_k=\frac{(1-p)^k\Gamma_p(c+k)}{\Gamma_p(c)}$$

and

$$\Gamma_p(c)=\int_0^{\frac{1}{1-p}}E_p(-pu)u^{c-1}d_pu,$$

we obtain (30). □

Theorem 2.8. *For $s\in\mathbb{N}$, the differentiation formulas for Ξ_1 and Ξ_2 hold true*

$$\mathbb{D}^s_{y,p}\Big[y^{a+s-1}\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\Big]=\frac{(p^a;p)_s}{(1-p)^s}y^{a-1}\Xi_1(p^{a+s},q^b,p^c;p^d;q,p,x,y),\tag{31}$$

$$\begin{aligned}\mathbb{D}^s_{x,q}\left[x^{b+s-1}\Xi_1(p^a,q^b,p^c;p^d;q,p,x,y)\right] &= \frac{(q^b;q)_s}{(1-q)^s}x^{b-1}\Xi_1(q^a,q^{b+s},p^c;p^d;q,p,x,y),\\ \mathbb{D}^s_{y,p}\left[x^{c+s-1}\Xi_1(p^a,q^b,p^c;p^d;q,p,x,y)\right] &= \frac{(p^c;p)_s}{(1-p)^s}y^{c-1}\Xi_1(p^a,q^b,p^{c+s};p^d;q,p,x,y)\end{aligned} \tag{32}$$

and

$$\begin{aligned}\mathbb{D}^s_{y,p}\left[y^{a+s-1}\Xi_2(p^a,q^b;p^c;p,q,x,y)\right] &= \frac{(p^a;p)_s}{(1-p)^s}y^{a-1}\Xi_2(p^{a+s},q^b;p^c;q,p,x,y),\\ \mathbb{D}^s_{x,q}\left[x^{b+s-1}\Xi_2(p^a,q^b;p^c;q,p,x,y)\right] &= \frac{(q^b;q)_s}{(1-q)^s}x^{b-1}\Xi_2(q^a,q^{b+s};p^c;q,p,x,y).\end{aligned} \tag{33}$$

Proof. Using

$$\mathbb{D}^s_{y,p}\left[y^{a+k+s-1}\right] = \frac{(p^{a+k};p)_s}{(1-p)^s}y^{a+k-1},$$

and

$$(p^a;p)_k(p^{a+k};p)_s = (p^a;p)_{k+s} = (p^a;p)_s(p^{a+s};p)_k,$$

we get

$$\begin{aligned}&\mathbb{D}^s_{y,p}\left[y^{a+s-1}\Xi_1(p^a,q^b,p^c;p^d;q,p,x,y)\right]\\ &= \sum_{\ell,k=0}^{\infty}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell\mathbb{D}^s_{y,p}\left[y^{a+k+s-1}\right]\\ &= \sum_{\ell,k=0}^{\infty}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell\frac{(p^{a+k};q)_s}{(1-p)^s}y^{a+k-1}\\ &= \frac{1}{(1-p)^s}y^{a-1}\sum_{\ell,k=0}^{\infty}\frac{(q^b;q)_\ell(p^a;p)_{k+s}(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k\\ &= \frac{(p^a;p)_s}{(1-p)^s}y^{a-1}\sum_{\ell,k=0}^{\infty}\frac{(p^{a+s};p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k\\ &= \frac{(p^a;p)_s}{(1-p)^s}y^{a-1}\Xi_1(p^{a+s},q^b,p^c;p^d;q,p,x,y).\end{aligned}$$

The proof Eqs. (32)-(33) are on the same lines as of Eq. (31). □

Theorem 2.9. *The summation formula for Ξ_1 and Ξ_2 holds true:*

$$\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y) = \sum_{\ell=0}^{\infty}\frac{(q^b;q)_\ell}{(p^d;p)_\ell(q;q)_\ell}x^\ell\ {}_2\phi_1(p^a,p^c;p^{d+\ell};p,y) \tag{34}$$

and

$$\Xi_2(p^a,q^b;p^c;p,q,x,y) = \sum_{\ell=0}^{\infty}\frac{(q^b;q)_\ell}{(p^c;p)_\ell(q;q)_\ell}x^\ell\ {}_2\phi_1(p^a,0;p^{c+\ell};p,y). \tag{35}$$

Proof. From the definition of Ξ_1 and (3), we have

$$\begin{aligned}&\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\ &= \sum_{\ell,k=0}^{\infty}\frac{(p^a;p)_k(q^b;q)_\ell(p^c;p)_k}{(p^d;p)_{\ell+k}(p;p)_k(q;q)_\ell}x^\ell y^k\end{aligned}$$

$$= \sum_{\ell=0}^{\infty} \frac{(q^b;q)_\ell}{(p^d;p)_\ell (q;q)_\ell} x^\ell \sum_{k=0}^{\infty} \frac{(p^a;p)_k (p^c;p)_k}{(p^{d+\ell};p)_k (p;p)_k} y^k$$
$$= \sum_{\ell=0}^{\infty} \frac{(q^b;q)_\ell}{(p^d;p)_\ell (q;q)_\ell} x^\ell \ {}_2\phi_1(p^a, p^c; p^{d+\ell}; p, y).$$

□

Theorem 2.10. *The summation formulas for* Ξ_1 *and* Ξ_1 *holds true:*

$$\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) = \frac{(p^a;p)_\infty}{(p^d;p)_\infty} \sum_{\ell,k=0}^{\infty} \frac{(q^b;q)_\ell (p^c;p)_k (p^{d+k};p)_\infty}{(p^{d+k};p)_\ell (p^{a+k};p)_\infty (p;p)_k (q;q)_\ell} x^\ell y^k, \tag{36}$$

$$\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) = \frac{(p^c;p)_\infty}{(p^d;p)_\infty} \sum_{\ell,k=0}^{\infty} \frac{(p^a;p)_k (q^b;q)_\ell (p^{d+k};p)_\infty}{(p^{d+k};p)_\ell (p^{c+k};p)_\infty (p;p)_k (q;q)_\ell} x^\ell y^k \tag{37}$$

and

$$\Xi_2(p^a, q^b; p^c; p, q, x, y) = \frac{(p^a;p)_\infty}{(p^c;p)_\infty} \sum_{\ell,k=0}^{\infty} \frac{(q^b;q)_\ell (p^{c+k};p)_\infty}{(p^{c+k};p)_\ell (p^{a+k};p)_\infty (p;p)_k (q;q)_\ell} x^\ell y^k. \tag{38}$$

Proof. Using the identities

$$(p^a;p)_k = \frac{(p^a;p)_\infty}{(p^{a+k};p)_\infty},$$
$$(p^c;p)_k = \frac{(p^c;p)_\infty}{(p^{c+k};p)_\infty},$$
$$(p^d;p)_k = \frac{(p^d;p)_\infty}{(p^{d+k};p)_\infty},$$

applying the q-binomial theorem

$$\frac{(p^{d+k};p)_\infty}{(p^{a+k};p)_\infty} = \sum_{r=0}^{\infty} \frac{(p^{d-a};p)_s}{(p;p)_r} \left(p^{a+k}\right)^r,$$
$$\frac{(p^{d+k};p)_\infty}{(p^{c+k};p)_\infty} = \sum_{r=0}^{\infty} \frac{(p^{d-c};p)_s}{(p;p)_r} \left(p^{c+k}\right)^r,$$

the familiar (Gaussian polynomials) q-binomial coefficient defined by

$${}_1\mathbf{\Phi}_0(p^a; -; p, x) = \sum_{n=0}^{\infty} \frac{(p^a;p)_n}{(p;p)_n} x^n = \frac{(p^a x;p)_\infty}{(x;p)_\infty}; |x| < 1$$

and

$${}_0\mathbf{\Phi}_0(-;-;p,y) = \sum_{k=0}^{\infty} \frac{1}{(p;p)_k} y^k = \frac{1}{(y;p)_\infty}, |y| < 1.$$

Substituting the above equations, we get

$$\Xi_1(p^a, q^b, p^c; p^d; p, q, x, y)$$
$$= \sum_{\ell,k=0}^{\infty} \frac{(p^a;p)_k (q^b;q)_\ell (p^c;p)_k}{(p^d;p)_k (p^{d+k};p)_\ell (p;p)_k (q;q)_\ell} x^\ell y^k$$
$$= \sum_{\ell,k=0}^{\infty} \frac{(q^b;q)_\ell (p^c;p)_k (p^a;p)_\infty (p^{d+k};p)_\infty}{(p^{d+k};p)_\ell (p^d;p)_\infty (p^{a+k};p)_\infty (p;p)_k (q;q)_\ell} x^\ell y^k$$

$$= \frac{(p^a;p)_\infty}{(p^d;p)_\infty}\sum_{\ell,k=0}^{\infty}\frac{(q^b;q)_\ell(p^c;p)_k(p^{d+k};p)_\infty}{(p^{a+k};p)_\infty(p^{d+k};p)_\ell(p;p)_k(q;q)_\ell}x^\ell y^k$$

$$= \frac{(p^a;p)_\infty}{(p^d;p)_\infty}\sum_{\ell,k,r=0}^{\infty}\frac{(q^b;q)_\ell(p^c;p)_k(p^{d-a};p)_r}{(p^{d+k};p)_\ell(p;p)_k(q;q)_\ell(p;p)_r}\left(p^{a+k}\right)^r x^\ell y^k$$

$$= \frac{(p^a;p)_\infty}{(p^d;p)_\infty}\sum_{\ell,k=0}^{\infty}\frac{(q^b;q)_\ell(p^c;p)_k}{(p^{d+k};p)_\ell(p;p)_k(q;q)_\ell}x^\ell y^k\sum_{r=0}^{\infty}\frac{(p^{d-a};p)_r p^{(a+k)r}}{(p;p)_r}$$

$$= \frac{(p^a;p)_\infty}{(p^d;p)_\infty}\sum_{\ell,k=0}^{\infty}\frac{(q^b;q)_\ell(p^c;p)_k}{(p^{d+k};p)_\ell(p;p)_k(q;q)_\ell}x^\ell y^k\ {}_1\mathbf{\Phi}_0(p^{d-a};-;p,p^{a+k}).$$

Similarly, we can prove (37)-(38). □

Theorem 2.11. *The following recursion formula holds true for* Ξ_1

$$\begin{aligned}
&\Xi_1(p^a,q^b,p^c;p^d;p,q,x,y)\\
&=\sum_{\ell=0}^{\infty}\frac{(q^b;q)_\ell(yp^{a+c-d-\ell};p)_\infty}{(p^d;p)_\ell(q;q)_\ell(y;p)_\infty}x^\ell\ {}_2\phi_1(p^{d+\ell-a},p^{d+\ell-c};p^{d+\ell};p,yp^{a+c-d-\ell})\\
&=\sum_{\ell=0}^{\infty}\frac{(q^b;q)_\ell(p^c;p)_\infty(yp^a;p)_\infty}{(p^d;p)_\ell(q;q)_\ell(p^{d+\ell};p)_\infty(y;p)_\infty}x^\ell\ {}_2\phi_1(p^{d+\ell-c},y;yp^a;p,p^c)\\
&=\sum_{\ell=0}^{\infty}\frac{(q^b;q)_\ell(p^{d+\ell-c};p)_\infty(yp^c;p)_\infty}{(p^d;p)_\ell(q;q)_\ell(p^{d+\ell};p)_\infty(y;p)_\infty}x^\ell\ {}_2\phi_1(yp^{a+c-d-\ell},p^c;yp^c;p,p^{d+\ell-c})\\
&=\sum_{\ell=0}^{\infty}\frac{(q^b;q)_\ell(yp^a;p)_\infty}{(p^d;p)_\ell(q;q)_\ell(y;p)_\infty}x^\ell\ {}_2\phi_2(p^a,p^{d+\ell-c};p^{d+\ell},yp^a;p,yp^c).
\end{aligned}\tag{39}$$

Proof. On the other hand, we shall need the Heine's transformations formulas for p-hypergeometric functions:(see)

$$\begin{aligned}
{}_2\phi_1(p^a,p^b;p^c;p,y)&=\frac{(yp^{a+b-c};p)_\infty}{(y;p)_\infty}\ {}_2\phi_1(p^{c-a},p^{c-b};p^c;p,yp^{a+b-c})\\
&=\frac{(p^b;p)_\infty(yp^a;p)_\infty}{(p^c;p)_\infty(x;p)_\infty}\ {}_2\phi_1(p^{c-b},y;yp^a;p,p^b)\\
&=\frac{(p^{c-b};p)_\infty(yp^b;p)_\infty}{(p^c;p)_\infty(y;p)_\infty}\ {}_2\phi_1(yp^{a+b-c},p^b;yp^b;p,p^{c-b})\\
&=\frac{(yp^a;p)_\infty}{(y;p)_\infty}\ {}_2\phi_2(p^a,p^{c-b};p^c,yp^a;p,yp^b).
\end{aligned}\tag{40}$$

In order to derive a list of p-analogues of transformation formulas for bibasic Humbert hypergeometric functions. Replacing p^a, p^b and p^c in (40) by p^a, p^c and $p^{d+\ell}$, gives

$$\begin{aligned}
{}_2\phi_1(p^a,p^c;p^{d+\ell};p,y)&=\frac{(yp^{a+c-d-\ell};p)_\infty}{(y;p)_\infty}\ {}_2\phi_1(p^{d+\ell-a},p^{d+\ell-c};p^{d+\ell};p,yp^{a+c-d-\ell})\\
&=\frac{(p^c;p)_\infty(yp^a;p)_\infty}{(p^{d+\ell};p)_\infty(y;p)_\infty}\ {}_2\phi_1(p^{d+\ell-c},y;yp^a;p,p^c)\\
&=\frac{(p^{d+\ell-c};p)_\infty(yp^c;p)_\infty}{(p^{d+\ell};p)_\infty(y;p)_\infty}\ {}_2\phi_1(yp^{a+c-d-\ell},p^c;yp^c;p,p^{d+\ell-c})
\end{aligned}$$

$$= \frac{(yp^a;p)_\infty}{(x;p)_\infty} {}_2\phi_2(p^a, p^{d+\ell-c}; p^{d+\ell}, yp^a; p, yp^c). \tag{41}$$

□

Remark 2.12. *The basic functions Ξ_1 and Ξ_2 are a q-analogue of the Humbert confluent hypergeometric functions Ξ_1 and Ξ_2 defined by ([7], page 225, Equations (20)-(22)):*

$$\lim_{q\longrightarrow 1} \Xi_1(p^a, q^b, p^c; p^d; q, p, \frac{1-p}{1-q}x, \frac{1}{1-p}y) = \Xi_1(a, b; c, d; x, y) \tag{42}$$

and

$$\lim_{q\longrightarrow 1} \Xi_2(p^a; q^b, q^c; q, p, \frac{1-p}{1-q}x, y) = \Xi_2(a, b; c, d; x, y). \tag{43}$$

Proof. Using the limit

$$\lim_{q\longrightarrow 1} \frac{(q^\beta;q)_n}{(1-q)^n} = (\beta)_n,$$

we obtain (42)-(43). □

Theorem 2.13. *The following relationship holds true between the basic functions Ξ_1 and Ξ_2:*

$$\lim_{c\longrightarrow\infty} \Xi_1(p^a, q^b, p^c; p^d; p, q, x, y) = \Xi_2(p^a, q^b; p^d; q, p, x, y). \tag{44}$$

Proof. Using the limit

$$\lim_{c\longrightarrow\infty} (p^c;q)_n = (0;q)_n = 1,$$

we obtain (44). □

§3 Some properties of new special cases

Here, we take $p = q$ in the above section and apply the techniques of the p-derivative operator for Humbert series Ξ_1 and Ξ_2, simplification gives the desired results.

Theorem 3.1. *The following equalities hold true:*

$$\left[\mathbf{\Theta}_{y,p}\right]_p \Xi_1 = \frac{[a]_p[c]_p}{[d]_p} y\Xi_1(p^{a+1}, p^{c+1}; p^{d+1}), \tag{45}$$

$$\left[\mathbf{\Theta}_{x,p}\right]_p \Xi_1 = \frac{[b]_p}{(1-p)[d]_p} x\Xi_1(p^{b+1}; p^{d+1}),$$

$$\left[\mathbf{\Theta}_{y,p}\right]_p \Xi_2 = \frac{[a]_p}{(1-p)[c]_p} y\Xi_2(p^{a+1}; p^{c+1}), \tag{46}$$

$$\left[\mathbf{\Theta}_{x,p}\right]_p \Xi_2 = \frac{[b]_p}{(1-p)[c]_p} x\Xi_2(p^{b+1}; p^{c+1}).$$

Proof. Using the relations of (7),(8) and (9) and simplifying, we get (45) and (46). □

Theorem 3.2. *The following equalities hold true:*

$$\begin{aligned}
&\left[\mathbf{\Theta}_{y,p} + a\right]_p \Xi_1 = [a]_p \Xi_1(p^{a+1}),\\
&\left[\mathbf{\Theta}_{x,p} + b\right]_p \Xi_1 = [b]_p \Xi_1(p^{b+1}),\\
&\left[\mathbf{\Theta}_{y,p} + c\right]_p \Xi_1 = [c]_p \Xi_1(p^{c+1}),\\
&\left[\mathbf{\Theta}_{x,p} + \mathbf{\Theta}_{y,p} + d - 1\right]_p \Xi_1 = [d-1]_p \Xi_1(p^{d-1})
\end{aligned} \tag{47}$$

and

$$\begin{aligned}&\big[\boldsymbol{\Theta}_{y,p}+a\big]_p\Xi_2=[a]_p\Xi_2(p^{a+1}),\\&\big[\boldsymbol{\Theta}_{x,p}+b\big]_p\Xi_2=[b]_p\Xi_2(p^{b+1}),\\&\big[\boldsymbol{\Theta}_{x,p}+\boldsymbol{\Theta}_{y,p}+c-1\big]_p\Xi_1=[c-1]_p\Xi_2(p^{c-1}).\end{aligned}\tag{48}$$

Proof. Using the relations (7) and (9), simplifying, we get (47)-(48). □

Theorem 3.3. *The following equalities hold true:*

$$\Big(\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}+\boldsymbol{\Theta}_{y,p}+d-1\big]_p-y\big[\boldsymbol{\Theta}_{y,p}+a\big]_p\big[\boldsymbol{\Theta}_{y,p}+c\big]_p\Big)\Xi_1=0,\tag{49}$$

$$\Big(\big[\boldsymbol{\Theta}_{x,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}+\boldsymbol{\Theta}_{y,p}+d-1\big]_p-\frac{x}{1-p}\big[\boldsymbol{\Theta}_{x,p}+b\big]_p\Big)\Xi_1=0\tag{50}$$

and

$$\begin{aligned}&\Big(\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}+\boldsymbol{\Theta}_{y,p}+c-1\big]_p-\frac{y}{1-p}\big[\boldsymbol{\Theta}_{y,p}+a\big]_p\Big)\Xi_2=0,\\&\Big(\big[\boldsymbol{\Theta}_{x,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}+\boldsymbol{\Theta}_{y,p}+c-1\big]_p-\frac{x}{1-p}\big[\boldsymbol{\Theta}_{x,p}+b\big]_p\Big)\Xi_2=0.\end{aligned}\tag{51}$$

Proof. Using the relations (7) and (9)and simplifying, we get (49)-(51). □

Corollary 3.4. *The following relations hold true:*

$$\begin{aligned}&\Big[\Big(p^{d-1}p^{\boldsymbol{\Theta}_{x,p}}-yp^{a+c}\Big)\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{y,p}\big]_p+p^{d-1}\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}\big]_p\\&-\Big(p^{d-1}-\big[d\big]_p+y\big(p^a\big[c\big]_p+p^c\big[a\big]_p\big)\Big)\big[\boldsymbol{\Theta}_{y,p}\big]_p-y[a]_p[c]_p\Big]\Xi_1=0,\\&\Big[\Big(p^{d-1}-yp^{a+c}\Big)\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{y,p}\big]_p+p^{d-1}p^{\boldsymbol{\Theta}_{y,p}}\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}\big]_p\\&-\Big(p^{d-1}-\big[d\big]_p+y\big(p^a\big[c\big]_p+p^c\big[a\big]_p\big)\Big)\big[\boldsymbol{\Theta}_{y,p}\big]_p-y[a]_p[c]_p\Big]\Xi_1=0,\\&\Big[p^{d-1}p^{\boldsymbol{\Theta}_{x,p}}\big[\boldsymbol{\Theta}_{x,p}\big]_p\big[\boldsymbol{\Theta}_{y,p}\big]_p+p^{d-1}\big[\boldsymbol{\Theta}_{x,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}\big]_p\\&-\Big(p^{d-1}-\big[d\big]_p+\frac{xp^b}{1-p}\Big)\big[\boldsymbol{\Theta}_{y,p}\big]_p-\frac{x}{1-p}[b]_p\Big]\Xi_1=0,\\&p^{d-1}\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}\big]_p+p^{d-1}p^{\boldsymbol{\Theta}_{y,p}}\big[\boldsymbol{\Theta}_{x,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}\big]_p\\&-\Big(p^{d-1}-\big[d\big]_p+\frac{xp^b}{1-p}\Big)\big[\boldsymbol{\Theta}_{x,p}\big]_p-\frac{x}{1-p}[b]_p\Big]\Xi_1=0\end{aligned}\tag{52}$$

and

$$\begin{aligned}&\Big[p^{c-1}p^{\boldsymbol{\Theta}_{x,p}}\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{y,p}\big]_p+p^{c-1}\big[\boldsymbol{\Theta}_{y,p}\big]_p\big[\boldsymbol{\Theta}_{x,p}\big]_p\\&-\Big(p^{c-1}-\big[c\big]_p+\frac{yp^a}{1-p}\Big)\big[\boldsymbol{\Theta}_{y,p}\big]_p-\frac{y}{1-p}[a]_p\Big]\Xi_2=0,\end{aligned}$$

$$\begin{aligned}
&p^{c-1}\big[\mathbf{\Theta}_{y,p}\big]_p\big[\mathbf{\Theta}_{y,p}\big]_p + p^{c-1}p^{\mathbf{\Theta}_{y,p}}\big[\mathbf{\Theta}_{y,p}\big]_p\big[\mathbf{\Theta}_{x,p}\big]_p \\
&-\left(p^{c-1}-\big[c\big]_p+\frac{yp^a}{1-p}\right)\big[\mathbf{\Theta}_{y,p}\big]_p-\frac{y}{1-p}[a]_p\Big]\Xi_2=0, \\
&\Big[p^{c-1}p^{\mathbf{\Theta}_{x,p}}\big[\mathbf{\Theta}_{x,p}\big]_p\big[\mathbf{\Theta}_{y,p}\big]_p + p^{c-1}\big[\mathbf{\Theta}_{x,p}\big]_p\big[\mathbf{\Theta}_{x,p}\big]_p \\
&-\left(p^{c-1}-\big[c\big]_p+\frac{xp^b}{1-p}\right)\big[\mathbf{\Theta}_{y,p}\big]_p-\frac{x}{1-p}[b]_p\Big]\Xi_2=0, \\
&p^{c-1}\big[\mathbf{\Theta}_{y,p}\big]_p\big[\mathbf{\Theta}_{x,p}\big]_p + p^{c-1}p^{\mathbf{\Theta}_{y,p}}\big[\mathbf{\Theta}_{x,p}\big]_p\big[\mathbf{\Theta}_{x,p}\big]_p \\
&-\left(p^{c-1}-\big[c\big]_p+\frac{xp^b}{1-p}\right)\big[\mathbf{\Theta}_{x,p}\big]_p-\frac{x}{1-p}[b]_p\Big]\Xi_2=0.
\end{aligned} \tag{53}$$

Proof. Using the relation

$$[\mu+\nu]_q=[\mu]_q+q^{\mu}[\nu]_q=[\nu]_q+q^{\nu}[\mu]_q \tag{54}$$

we obtain our relations (52) and (53). □

Theorem 3.5. *The following recursion formulas hold true for* Ξ_1 *and* Ξ_2

$$\begin{aligned}
\Xi_1(p^a,p^b,p^c;p^d;p,x,y)&=\sum_{\ell=0}^{\infty}\frac{(p^b;p)_\ell}{(p^d;p)_\ell(p;p)_\ell}x^\ell\ {}_2\phi_1(p^a,p^c;p^{d+\ell};p,y),\\
\Xi_1(p^a,p^b,p^c;p^d;p,x,y)&=\sum_{k=0}^{\infty}\frac{(p^a;p)_k(p^c;p)_k}{(p^d;p)_k(p;p)_k}y^k\ {}_2\phi_1(0,p^b;p^{d+k};p,x)
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
\Xi_2(p^a,p^b;p^c;p,x,y)&=\sum_{\ell=0}^{\infty}\frac{(p^b;p)_\ell}{(p^c;p)_\ell(p;p)_\ell}x^\ell\ {}_2\phi_1(p^a,0;p^{c+\ell};p,y),\\
\Xi_2(p^a,p^b;p^c;p,x,y)&=\sum_{k=0}^{\infty}\frac{(p^a;p)_k}{(p^c;p)_k(p;p)_k}y^k\ {}_2\phi_1(0,p^b;p^{c+k};p,x).
\end{aligned} \tag{56}$$

Proof. Using relations (7) and (9) and simplifying, we get (55) and (56). □

Corollary 3.6. *The following formulas hold true:*

$$\begin{aligned}
\Xi_1(p^a,p^b,p^c;p^d;p,x,0)&={}_2\phi_1(0,p^b;p^d;p,x),\\
\Xi_1(p^a;p^b,p^c;p^d;p,0,y)&={}_2\phi_1(p^a,p^c;p^d;p,y)
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
\Xi_2(p^a,p^b;p^c;p,x,0)&={}_2\phi_1(0,p^b;p^d;p,x),\\
\Xi_2(p^a;p^b;p^c;p,0,y)&={}_2\phi_1(p^a,0;p^c;p,y).
\end{aligned} \tag{58}$$

Proof. By using (6) and (7), and setting $x=0$ and $y=0$ in relations (55) respectively and simplifying we get (57), similarly by using (6) and (8), and setting $x=$ and $y=0$ in relations (56) respectively and simplifying, we get the relation (58). □

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Declarations

Conflict of interest The authors declare no conflict of interest.

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