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Invariant measures for the strong-facilitated exclusion process

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Abstract. Consider a generalized model of the facilitated exclusion process, which is a one-dimensional exclusion process with a dynamical constraint that prevents the particle at site x from jumping to $x+1$ (or $x-1$) if the sites $x-1, x-2$ (or $x+1, x+2$) are empty. It is non-gradient and lacks invariant measures of product form. The purpose of this paper is to identify the invariant measures and to show that they satisfy both exponential decay of correlations and equivalence of ensembles. These properties will play a pivotal role in deriving the hydrodynamic limit.

§1 Introduction

In the 1970s, Dobrushin and Spitzer initiated the idea of obtaining a mathematically precise understanding of the emergence of macroscopic behavior in gases or fluids from the microscopic interaction of a large number of identical particles with stochastic dynamics. This approach has been proven to be extremely fruitful in both probability theory and statistical physics, and it continues to garner attention today. One of the models introduced at the time is the exclusion process. It is a lattice model with a maximum of one particle per site that jumps after an exponential time. Despite the fact that the structure is simple, numerous results have been obtained (for example, see the books [3, 5]).

In the study of the hydrodynamic limit, exclusion processes with dynamical constraints have recently received a lot of attention. The models with hydrodynamic limits of a typical non-linear diffusion equation of the following form are of interest to us.

$$\partial_t \rho = \partial_u \Big(\rho^{m-1} \partial_u \rho \Big). \tag{1}$$

The equation (1) with a positive integer m arises in the models discussed in [2], whereas only the case $m = -1$ appears in the literature for the equation with m negative, see [1, 4].

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The model discussed in [2] has the dynamical constraint that the particle at site x (resp. $x+1$) is only allowed to jump to $x+1$ (resp. x) if the range $\{x-m+1, x-m+2\cdots, x-1, x+2, x+3, \cdots, x+m\}$ has at least one $m-1$-polymer, it should be noted that these $m-1$-polymers in the connected range $\{x-m+1, x-m+2, \cdots, x+m-1, x+m\}$ may not be connected. This dynamical constraint assures that these exclusion processes are gradient and have product invariant measures. While the model discussed in [1] has the dynamical constraint that allows the particle at site x to jump to $x+1$ (resp. $x-1$) only if the site $x-1$ (resp. $x+1$) is occupied. The FEP is irreversible and gradient, but it lacks invariant measures of product form, and therefore the particle distributions in two disjoint areas are not independent, which need more delicate verifications in order to derive the hydrodynamic limit.

According to Lukyanov's recent research [6, 7], the equation (1) with m negative (resp. positive) arises in the liquids in porous media, such as sand, considered at low (resp. high) saturation levels with liquid pathways at pore dimensions. In fact, different forces affect liquids in porous media ranging from low to high saturation. At low saturation levels, liquids are affected by capillary force and evolve as the equation (1) with $m \approx -1/2$. As a result, there is most likely an exclusion process with a specific dynamical constraint that has a hydrodynamic limit of the type (1) with some negative integer m other than -1. We consider a one-dimensional exclusion process with the dynamical constraint that prevents the particle at site x from jumping to $x+1$ (resp. $x-1$) if the sites $x-1, x-2$ (resp. $x+1, x+2$) are empty. Because this dynamical constraint can be viewed as a rather strong facilitated particle force, we refer to this exclusion process as the strong-facilitated exclusion process (s-FEP) and expect s-FEP to have a hydrodynamic limit of the type (1) with some negative integer m other than -1. This s-FEP is non-gradient and lacks invariant measures of product form.

In the study of hydrodynamic limit, a commonly used method is to split the torus of finite N sites into disjoint microscopic blocks, and to prove that for some observable (local function) on these blocks, the expected value of space averages under the canonical measure (CM) converges to the expected value of a certain function under the grand canonical measure (GCM) as N goes to infinity. To this, we need the so-called *equivalence of ensembles*. Further, to apply the Law of Large Number for the space averages, we need *exponential decay of correlations*. We aim to derive the hydrodynamic limit for s-FEP. As a foundation, the exponential decay of correlations and equivalence of ensembles should be verified. Hence in the present paper, we identify s-FEP's invariant measures and show the two properties mentioned before for them.

We begin by considering the canonical measures of the model with k particles in a torus of finite size N, and then let k, N tend to infinity such that k/N tends to some ρ, from which we can derive the grand canonical (invariant) measures using the ratio limit theorem. The exponential decay of correlations holds by a recurrence relation. As for equivalence of ensembles, we note that the canonical measures and grand canonical measures, which are limited to some range B_l (l is the length of B_l, are determined by the permutations of the particles on the boundary. To avoid boundary affection, we consider a local function with support away from the extremities of B_l. Letting l goes to infinity, the concavity of the trinomial coefficients allows us to show that the canonical measure concentrates on the configurations in which particles are evenly distributed on the two ranges separated by the local range, and thus equivalence of ensembles is valid according to the ratio limit theorem.

§2 The model and results

We first consider the periodic one-dimensional s-FEP. Let $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z} = \{1, \ldots, N\}$ be the discrete torus of size N. Denote by $\{0,1\}^{\mathbb{T}_N}$ and $\eta \in \{0,1\}^{\mathbb{T}_N}$ the state space and configuration of s-FEP respectively. For any measurable function $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$, and $x \in \mathbb{T}_N$, we denote by $\tau_x f$ the function obtained by translation as follows: $\tau_x f(\eta) := f(\tau_x \eta)$, where $(\tau_x \eta)(y) = \eta(x+y)$, for $y \in \mathbb{T}_N$. Define the configuration $\eta|_S$ as the configuration η restricted on the set $S \subset \mathbb{T}_N$.

We denote by $\mathcal{L}_N$ the infinitesimal generator ruling the evolution in time of s-FEP. It acts on functions $f : \{0,1\}^{\mathbb{T}_N} \to \mathbb{R}$ as

$$\mathcal{L}_N f(\eta) := \sum_{x \in \mathbb{T}_N} \big(c_{x,x+1}(\eta) + c_{x+1,x}(\eta)\big)\big(f(\eta^{x,x+1}) - f(\eta)\big), \tag{2}$$

where the constraint and the exclusion rule are encoded in the rates $c_{x,x+1}, c_{x+1,x}$ as

$$\begin{aligned} c_{x,x+1}(\eta) &= \big(\eta(x-2) + \eta(x-1) - \eta(x-2)\eta(x-1)\big)\eta(x)\big(1 - \eta(x+1)\big), \\ c_{x+1,x}(\eta) &= \big(\eta(x+3) + \eta(x+2) - \eta(x+3)\eta(x+2)\big)\eta(x+1)\big(1 - \eta(x)\big), \end{aligned} \tag{3}$$

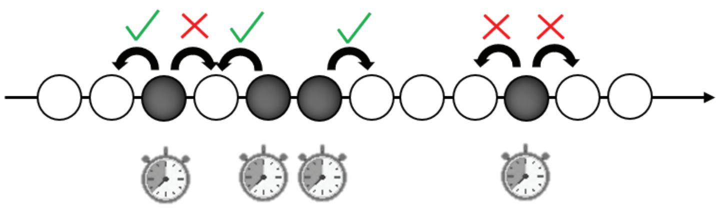

Figure 1. Allowed (resp. Forbidden) jumps are denoted by ✓ (resp. X).

and $\eta^{x,y}$ denotes the configuration obtained from η by exchanging the states of sites x, y, namely $\eta^{x,y}(x) = \eta(y), \eta^{x,y}(y) = \eta(x)$ and $\eta^{x,y}(z) = \eta(z)$ if $z \neq x, y$. Figure 1 above shows the dynamical constraint. Note that the dynamics conserves the total number of particles $\sum_{x \in \mathbb{T}_N} \eta(x)$.

The s-FEP is reducible under the state space $\{0,1\}^{\mathbb{T}_N}$ since the dynamic is degenerate. We therefore consider the ergodic component of $\{0,1\}^{\mathbb{T}_N}$ defined below.

Definition 2.1 (Ergodic component)**.** We denote by $\mathcal{E}_N \subset \{0,1\}^{\mathbb{T}_N}$ the set of ergodic configurations on $\mathbb{T}_N$, namely

$$\mathcal{E}_N = \big\{\eta \in \{0,1\}^{\mathbb{T}_N} : \phi(\eta) \leq 2 \text{ and } \sum_{x \in \mathbb{T}_N} \eta(x) > \frac{1}{3} N\big\}, \tag{4}$$

where $\phi(\eta)$ is the largest number of consecutive holes.

Denote by $\widehat{\mathcal{E}}_\Lambda$ the set of ergodic local configurations (which are actually restrictions of ergodic configurations) on a (finite) connected set $\Lambda \subset \mathbb{Z}$, namely

$$\widehat{\mathcal{E}}_\Lambda = \big\{\sigma \in \{0,1\}^\Lambda : \phi(\sigma) \le 2\big\}. \tag{5}$$

Let $\mathcal{H}_N^k$ be the hyperplane of configurations with k particles, with $k \in \{0,\dots,N\}$, namely

$$\mathcal{H}_N^k := \big\{\eta \in \{0,1\}^{\mathbb{T}_N} : \sum_{x\in\mathbb{T}_N} \eta(x) = k\big\}.$$

For $k > \frac{1}{3}N$, we also let $\Omega_N^k = \mathcal{H}_N^k \cap \mathcal{E}_N$ be the set of ergodic configurations on $\mathbb{T}_N$ which contain exactly k particles.

Define the instantaneous current

$$\begin{aligned} W(\eta) &= c_{0,1}(\eta) - c_{1,0}(\eta) \\ &= \big(\eta(-2)+\eta(-1)-\eta(-2)\eta(-1)\big)\eta(0)(1-\eta(1)) - \big(\eta(3)+\eta(2)-\eta(3)\eta(2)\big)\eta(1) \qquad (6)\\ &\qquad (1-\eta(0)). \qquad (7)\end{aligned}$$

In particular,

$$\mathcal{L}_N\eta(x) = \tau_{x-1}W(\eta) - \tau_x W(\eta).$$

One can check that the s-FEP is non-gradient on $\mathcal{E}_N$, i.e. there is no local function h such that the instantaneous current W can be written as

$$W(\eta) = h(\eta) - \tau_1 h(\eta),$$

by considering the mirror-symmetric items $\eta(3)\eta(1)\eta(0)$ and $\eta(-2)\eta(0)\eta(1)$ in $W(\eta)$.

Set $\Lambda_l = \{1,2,\cdots,l\}$. Considering the canonical (uniform) measure on Ω_N^k and letting $\frac{k}{N} \to \rho$ as N goes to infinity, we have the following result.

Proposition 2.1 (Invariant measure). *For $1 > \rho > \frac{1}{3}, l \ge 1$, let*

$$\bar\rho = \frac{1}{\rho} - 2, \alpha(\rho) = \frac{\bar\rho + \sqrt{4-3\bar\rho^2}}{2(1-\bar\rho)}, \beta(\rho) = (\alpha(\rho)+1+\alpha(\rho)^{-1})^{-1}.$$

The translation invariant measure π_ρ of s-FEP on $\{0,1\}^{\mathbb{Z}}$ is

$$\pi_\rho\Big\{\xi_{|\Lambda_l} = \sigma\Big\} = \mathbf{1}_{\{\sigma\in\widehat{\mathcal{E}}_{\Lambda_l}\}}\rho\alpha(\rho)^l\big(\frac{\beta(\rho)}{\alpha^2(\rho)}\big)^p\big(\frac{\beta(\rho)}{\alpha(\rho)}\big)^{1-\sigma(1)-\sigma(l)}(1+\alpha(\rho))^{\sigma(2)(1-\sigma(1))+\sigma(l-1)(1-\sigma(l))} \tag{8}$$

where $\sigma \in \{0,1\}^{\Lambda_l}$ and $p = \sum_{x\in\Lambda_l}\sigma(x) \in \{0,\dots,l\}$ is the number of particles in σ.

Remark 2.1. Since $\alpha > 0$ if $\rho < 1$, we have $\beta \le \frac{1}{3}$ and $\alpha\beta = \frac{\alpha^2}{\alpha^2+\alpha+1} < 1$, then there exist constants C_ρ and $0 < a < 1$ such that for any $\sigma \in \Lambda_l$, $\pi_\rho(\sigma) \le C_\rho a^l$.

Let $\pi_\rho(f)$ be the expectation of function f under the measure π_ρ.

Theorem 2.2 (Exponential decay of correlations). *For local functions f, g, denote by S_f, S_g their supports respectively and let $d = dist(S_f, S_g)$. Then for any $\rho \in (\frac{1}{3}, 1)$, there exist constants C_1, C_2 depending on ρ such that*

$$\pi_\rho(f)\pi_\rho(g) - \pi_\rho(fg) \le C_1\|f\|_\infty\|g\|_\infty e^{-C_2 d}. \tag{9}$$

Before stating the equivalence of ensembles, we should define $B_k(x) = \{x-k,...,x+k\}$ (in particular we simplify $B_k = B_k(0)$) and the projections of the measure π_ρ on a finite box B_l.

Let the probability measure $\hat{\pi}_\rho^l$ on $\hat{\mathcal{E}}_{B_l}$ be as

$$\hat{\pi}_\rho^l(\sigma) := \pi_\rho\big(\xi_{|B_l} = \sigma\big) = \rho\big(\frac{\beta(\rho)}{\alpha(\rho)}\big)^{p+1-\sigma(-l)-\sigma(l)}\alpha(\rho)^{l-p}(1+\alpha(\rho))^{\sigma(-l+1)(1-\sigma(-l))+\sigma(l-1)(1-\sigma(l))} \tag{10}$$

where $p = \sum_{x\in B_l}\sigma(x)$. Denote by $\lfloor\cdot\rfloor, \lceil\cdot\rceil$ the floor function and ceiling function on $\mathbb{R}$ respectively. Then, for any integer $j \in \{\lceil(2l+1)/3\rceil, ..., 2l+1\}$, we denote by $\hat{\pi}_\rho^{l,j}$ the measure $\hat{\pi}_\rho^l$ conditioned on having j particles. Namely, if we denote by $\hat{\Omega}_l^j$ the hyperplane

$$\hat{\Omega}_l^j := \big\{\sigma \in \hat{\mathcal{E}}_{B_l} : \sum_{x\in B_l}\sigma(x) = j\big\},$$

then $\hat{\pi}_\rho^{l,j}$ is the probability measure on $\hat{\Omega}_l^j$ such that

$$\hat{\pi}_\rho^{l,j}(\sigma) = \frac{\hat{\pi}_\rho^l(\sigma)}{\hat{\pi}_\rho^l(\hat{\Omega}_l^j)}, \qquad \text{for any } \sigma \in \hat{\Omega}_l^j.$$

Finally, let $k \in \mathbb{N}$ be fixed and $\rho_l = \rho_l(j) := \frac{j}{2l+1}$ the density in B_l under $\hat{\pi}_\rho^{l,j}$. Furthermore, we introduce $E_l(\delta) = \{\lceil(2l+1)(1+\delta)/3\rceil, ..., \lfloor(1-\delta)(2l+1)\rfloor\}$, i.e. the set of possible particle numbers j in B_l satisfying that the density $\rho_l(j)$ is at a distance larger than δ to the critical densities $\frac{1}{3}$ and 1. If $\delta = 0$, we shorten $E_l(0)$ as E_l for simplicity. Then we can show that the CM $\hat{\pi}_\rho^{l,j}$ is locally close to the GCM $\pi_{\rho_l(j)}$ with parameter equals to the empirical density $\rho_l(j)$.

Theorem 2.3 (Equivalence of ensembles)**.** *For any local function $f : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, any $\rho \in (\frac{1}{3}, 1)$, and any $\delta > 0$, we have*

$$\lim_{l\to\infty} \max_{\substack{j\in E_l \\ x\in B_{(1-\delta)l}}} \Big|\hat{\pi}_\rho^{l,j}(\tau_x f) - \pi_{\rho_l(j)}(f)\Big| = 0. \tag{11}$$

The rest of the article is organized as follows. In the following section, we give an explicit deduction of the invariant measures on $\{0,1\}^{\mathbb{Z}}$. In Section 4 we will demonstrate exponential decay of correlations and equivalence of ensembles. In Appendix A we will discuss some of the properties of trinomial coefficients, which will be used throughout the article. In Appendix B we will discuss a ratio limit theorem which will be used in the derivation of invariant measures and the proof of equivalence of ensembles.

§3 Invariant measures

3.1 Canonical measures for s-FEP on $\mathbb{T}_N$

For $k > \frac{1}{3}N$, let π_N^k be the uniform measure on Ω_N^k. π_N^k is known as the invariant measure for the exclusion process on $\mathbb{T}_N$ with k particles. Also, it is translation invariant and satisfies the detailed balance condition for s-FEP: for any $x \in \mathbb{T}_N$ and $\eta \in \Omega_N^k$,

$$\begin{aligned}
&\pi_N^k(\eta)\big(\eta(x-2)+\eta(x-1)-\eta(x-2)\eta(x-1)\big)\eta(x)(1-\eta(x+1))\\
&=\pi_N^k(\eta)\big(\eta(x-2)+\eta(x-1)-\eta(x-2)\eta(x-1)\big)\eta(x)(1-\eta(x+1))\big(\eta(x+2)+\eta(x+3)-\eta(x+2)\eta(x+3)\big)\\
&=\pi_N^k(\eta^{x,x+1})\big(\eta(x-2)+\eta(x-1)-\eta(x-2)\eta(x-1)\big)\eta^{x,x+1}(x+1)(1-\eta^{x,x+1}(x))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\big(\eta(x+2)+\eta(x+3)-\eta(x+2)\eta(x+3)\big)\\
&=\pi_N^k(\eta^{x,x+1})(1-\eta^{x,x+1}(x))\eta^{x,x+1}(x+1)\big(\eta^{x,x+1}(x+2)+\eta^{x,x+1}(x+3)-\eta^{x,x+1}(x+2)\eta^{x,x+1}(x+3)\big),
\end{aligned}$$

where we used the fact that the distances between neighboring particles are at most 2 on Ω_N^k in the first and last equalities. Before characterizing the marginals of π_N^k, we should introduce the trinomial coefficient.

Denote by $\binom{k}{m}_2$ the trinomial coefficient of x^m in the polynomial $(1+x+\frac{1}{x})^k$. It is easy to see that for integer $|m| \leq k$,

$$\binom{k}{m}_2 = \sum_{r=-\infty}^{\infty} \binom{k}{m+r}\binom{k-m-r}{r}, \tag{1}$$

where $\binom{k}{m+r}$ is the binomial coefficient and by convention $\binom{a}{b} = 0$ if $a < b$ or $b < 0$. From the symmetry of x and $\frac{1}{x}$, we have $\binom{k}{m}_2 = \binom{k}{-m}_2$.

Lemma 3.1. *There are exactly $\binom{k}{m-k}_2$ choices of putting $k-1$ particles into $k-1+m$ sites such that there are no consecutive 3 holes.*

Proof. First, concatenate a new particle to the rightmost site and call it the k-th particle. Then the number of ways is the same as the number of solutions of $x_1+\cdots+x_k = m$ where x_j denotes the number of holes before the j-th particle and $x_j \leq 2$, which is equal to the coefficient of x^m in the polynomial $(1+x+x^2)^k$, i.e. $\binom{k}{m-k}_2$ since $(1+x+x^2)^k = x^k(\frac{1}{x}+1+x)^k$. □

Recall that Ω_N^k is the set of ergodic configurations on $\mathbb{T}_N$ which contain exactly k particles and $\Lambda_l = \{1, 2, \cdots, l\}$.

Lemma 3.2. *Let $k \in \{1, \ldots, N-1\}$ and $m = N-k$.*
(i) We have the identity

$$|\Omega_N^k| = \frac{N}{k}\binom{k}{m-k}_2.$$

(ii) Furthermore, fix $l \leq N$ and a local ergodic configuration $\sigma \in \widehat{\mathcal{E}}_{\Lambda_l}$. We define
$p := \sum_{x\in\Lambda_l} \sigma(x) \in \{1, \ldots, l\}$ its number of particles,
$z := l - p$ its number of holes,
and, assume $p \leq k$ and $z \leq m$. Then we have the following formula:
(1). $\sigma = \bullet\cdots\bullet$:

$$\left|\left\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\right\}\right| = \binom{k-p+1}{m-z-(k-p+1)}_2.$$

(2). $\sigma = \bullet\cdots\bullet\circ$ or $\circ\bullet\cdots\bullet$:

$$\left|\left\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\right\}\right| = \binom{k-p}{m-z-(k-p)}_2 + \binom{k-p}{m-z-1-(k-p)}_2.$$

(3). $\sigma = \circ\bullet\cdots\bullet\circ$:

$$\left|\left\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\right\}\right| = \binom{k-p}{m-z-(k-p)}_2 + \binom{k-p-1}{m-z-(k-p)}_2.$$

(4). $\sigma = \circ\bullet\cdots\circ\circ$ or $\circ\circ\cdots\bullet\circ$:

$$\left|\left\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\right\}\right| = \binom{k-p-1}{m-z-(k-p-1)}_2 + \binom{k-p-1}{m-z-(k-p)}_2.$$

(5). $\sigma = \bullet\cdots\circ\circ$ or $\circ\circ\cdots\bullet$:

$$\left|\left\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\right\}\right| = \binom{k-p}{m-z-(k-p)}_2.$$

(6). $\sigma = \circ\circ\cdots\circ\circ$:

$$\Big|\Big\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\Big\}\Big| = \binom{k-p-1}{m-z-(k-p-1)}_2.$$

Proof. (i) Ω_N^k consists of the three types of configurations, those with a particle at 1, those with an isolated hole at 1, and those with two holes at $\{1,2\}$ or $\{1, N-1\}$. The number of the first type of configurations is equal to the number of choices to put $k-1$ particles into $k-1+m$ sites where there are no consecutive 3 holes, which is $\binom{k}{m-k}_2$ by Lemma 3.1. The second type of configurations has an isolated hole at 1, thus the two particles adjacent to this isolated hole are locked, which is $\binom{k-1}{m-k}_2$ (the number of choices to put $k-2$ particles into $k+m-3$ sites) by the same argument above. The number of the third type of configurations can be determined as $2\binom{k-1}{m-k-1}_2$ by the same way. So by the identity (32),

$$|\Omega_N^k| = \binom{k}{m-k}_2 + \binom{k-1}{m-k}_2 + 2\binom{k-1}{m-k-1}_2 = \frac{N}{k}\binom{k}{m-k}_2.$$

Now turn to (ii). We will only compute the cardinality of the set in case (3) since others are similar and simpler. Note that after fixing the configuration σ on Λ_l, it remains to put $k-p$ particles into $k-p+m-z$ sites.

When $\sigma(-1) = \circ$ and $\sigma(l+1) = \circ$ (it means the left and right adjacent site of σ is empty), $\sigma(-2)$ and $\sigma(l+2)$ must have a particle, thus it remains to put $k-p-2$ particles into $k-p+m-z-4$ sites. By the same argument, when $\sigma(-1) = \bullet$ and $\sigma(l+1) = \circ$(or $\sigma(-1) = \circ$ and $\sigma(l+1) = \bullet$), it remains to put $k-p-2$ particles into $k-p+m-z-3$ sites; when $\sigma(-1) = \bullet$ and $\sigma(l+1) = \bullet$, it remains to put $k-p-2$ particles into $k-p+m-z-2$ sites. So

$$\begin{aligned}\Big|\Big\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\Big\}\Big| &= \binom{k-p-1}{m-z-2-(k-p-1)}_2 + 2\binom{k-p-1}{m-z-1-(k-p-1)}_2 + \binom{k-p-1}{m-z-(k-p-1)}_2 \\ &= \binom{k-1}{m-z-(k-p)}_2 + \binom{k-p-1}{m-z-(k-p)}_2\end{aligned}$$

by Lemma 3.1 and the identity (32). □

3.2 Gibbs measure for s-FEP on $\mathbb{Z}$

Note that, by periodizing the configurations, we can regard the measures π_N^k as measures on $\{0,1\}^{\mathbb{Z}}$. In that case, since the state space is compact, the sequence $(\pi_N^k)_N$ is uniformly tight. If $k/N \to \rho > \frac{1}{3}$, using Lemma 3.2, one can check that there is a unique limit point. This limit point is the measure π_ρ in (8).

Indeed, let $\{\xi_n\}$ be i.i.d random variables with probability P s.t. $P(\xi_1 = -1) = P(\xi_1 = 0) = P(\xi_1 = 1) = \frac{1}{3}$. We can compute the probability of the sum of these random variables

$$P\Big(\sum_{i=1}^{k} \xi_i = m\Big) = \frac{\binom{k}{m}_2}{3^k}.$$

Then thanks to the ratio limit lemma below, one can easily deduce by (8) that

$$\begin{aligned}\lim_{N\to\infty} \frac{\Big|\Big\{\eta \in \Omega_N^k : \eta_{|\Lambda_l} = \sigma\Big\}\Big|}{|\Omega_N^k|} =&\mathbf{1}_{\{\sigma\in\widehat{\mathcal{E}}_{\Lambda_l}\}}\rho\Big(\frac{\beta(\rho)}{\alpha(\rho)}\Big)^{p+1-\sigma(1)-\sigma(l)}\alpha(\rho)^{l-p}(1+\alpha(\rho))^{\sigma(2)(1-\sigma(1))+\sigma(l-1)(1-\sigma(l))} \\ =&\pi_\rho\{\eta_{|\Lambda_l} = \sigma\}.\end{aligned}$$

Lemma 3.3 (Ratio limit lemma). *For $p, z \in \mathbb{Z}$, $\varrho \in (-1,1)$, and any sequences of m, k such that $m/k \to \varrho$ as $k \to \infty$,*

$$\frac{\binom{k-p}{m-z}_2}{\binom{k}{m}_2} = \frac{P(\sum_{i=1}^{k-p} \xi_i = m-z)}{3^p P(\sum_{i=1}^{k} \xi_i = m)} \to b(\varrho)^p a(\varrho)^z,$$

where $a(\varrho) = \frac{\varrho + \sqrt{4-3\varrho^2}}{2(1-\varrho)}$, $b(\varrho) = (a(\varrho) + 1 + \frac{1}{a(\varrho)})^{-1}$.

Proof. The proof will be given at Appendix B. □

§4 Properties of π_ρ

In this section, we are going to prove exponential decay of correlations and equivalence of ensembles for the GCM π_ρ.

4.1 Exponential decay of correlations

$\alpha(\rho)$ and $\beta(\rho)$ will be shortened as α and β if no ambiguity arises. Let $P_l := \pi_\rho\big(\xi(0) = \xi(l) = 1\big)$. We first show that the two-point correlations under the invariant measure π_ρ decay exponentially.

Lemma 4.1. *For any $\rho \in (\frac{1}{3}, 1)$, there exist two constants $C = C(\rho) > 0$ and $q = q(\rho) > 0$ such that*

$$|P_l - \rho^2| \le q e^{-Cl}. \tag{1}$$

Proof. Rewrite

$$\begin{aligned}
&\pi_\rho\big(\xi(0) = \xi(l) = 1\big)\\
=&\pi_\rho\big(\xi(0) = 1, \xi(l-1) = 0, \xi(l) = 1\big) + \pi_\rho\big(\xi(0) = 1, \xi(l-1) = 1, \xi(l) = 1\big)\\
=&\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 0, \xi(l-1) = 0, \xi(l) = 1\big) + \pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 0, \xi(l) = 1\big)\\
&+ \pi_\rho\big(\xi(0) = 1, \xi(l-2) = 0, \xi(l-1) = 1, \xi(l) = 1\big) + \pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 1, \xi(l) = 1\big).
\end{aligned} \tag{2}$$

By the explicit formula (8) and some direct calculations, we have

$$\begin{aligned}
&\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 0, \xi(l-1) = 0, \xi(l) = 1\big) + \pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 0, \xi(l) = 1\big)\\
= &\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 0, \xi(l-1) = 0\big) + \frac{1}{\alpha+1}\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 0\big)\\
= &\pi_\rho\big(\xi(0) = 1, \xi(l-1) = 0\big) - \pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 0\big)\\
&\qquad + \frac{1}{\alpha+1}\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 0\big)\\
= &\rho - \pi_\rho\big(\xi(0) = 1, \xi(l-1) = 1\big) + \frac{\alpha}{\alpha+1}\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 0\big).\\
= &\rho - P_{l-1} - \alpha\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 0, \xi(l) = 1\big)\\
= &\rho - P_{l-1} - \alpha\beta P_{l-2}.
\end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}
&\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 0, \xi(l-1) = 1, \xi(l) = 1\big) + \pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 1, \xi(l) = 1\big)\\
= &\frac{\beta}{\alpha}\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 0, \xi(l-1) = 1\big) + \frac{\beta}{\alpha}\pi_\rho\big(\xi(0) = 1, \xi(l-2) = 1, \xi(l-1) = 1\big)\\
= &\frac{\beta}{\alpha}\pi_\rho\big(\xi(0) = 1, \xi(l-1) = 1\big) = \frac{\beta}{\alpha}P_{l-1}.
\end{aligned} \tag{4}$$

Combining (2), (3), and (4) together, we obtain the recurrence relation for P_l:

$$P_l = \rho - \frac{\alpha-\beta}{\alpha}P_{l-1} - \alpha\beta P_{l-2}. \tag{5}$$

By subtracting ρ^2 from both sides of the equation (5),

$$\begin{aligned} P_l - \rho^2 &= \rho - \rho^2 - \frac{\alpha-\beta}{\alpha}P_{l-1} - \alpha\beta P_{l-2} \\ &= -\frac{\alpha-\beta}{\alpha}(P_{l-1}-\rho^2) - \alpha\beta(P_{l-2}-\rho^2). \end{aligned}$$

The quadratic equation $q^2 + \frac{\alpha-\beta}{\alpha}q + \alpha\beta = 0$ has 2 distinct complex roots

$$z_1, z_2 = \frac{-\frac{\alpha-\beta}{\alpha} \pm \sqrt{(\frac{\alpha-\beta}{\alpha})^2 - 4\alpha\beta}}{2}.$$

Also,

$$0 < \alpha\beta = \frac{\alpha^2}{\alpha^2+\alpha+1} < \frac{\alpha^2+\alpha}{\alpha^2+\alpha+1} = \frac{\alpha-\beta}{\alpha} < 1,$$

then

$$|z_1| = |z_2| = \sqrt{\frac{(\frac{\alpha-\beta}{\alpha})^2 + |(\frac{\alpha-\beta}{\alpha})^2 - 4\alpha\beta|}{4}} < 1.$$

By Eq. (14) of Miles [8], $P_l - \rho^2 = c_1 z_1^l + c_2 z_2^l$ where c_1, c_2 are some constants depend on ρ. Then $|P_l - \rho^2| \le c_1|z_1|^l + c_2|z_2|^l$, the desired result easily holds. □

Note by the explicit formula (8)

$$\begin{aligned} &\pi_\rho\big(\xi(0)=1,\xi(1)=0,\xi(l)=1\big) \\ =\ &\pi_\rho\big(\xi(0)=1,\xi(l)=1\big) - \pi_\rho\big(\xi(0)=1,\xi(1)=1,\xi(l)=1\big) \\ =\ &P_l - \frac{\beta}{\alpha}\pi_\rho\big(\xi(1)=1,\xi(l)=1\big) = P_l - \frac{\beta}{\alpha}P_{l-1}. \end{aligned}$$

Using (1) and the identity $(\alpha+1+\frac{1}{\alpha})\beta = 1$, we have the following corollary.

Corollary 4.2. *There exist two constants $C' = C'(\rho) > 0$ and $q' = q'(\rho) > 0$ such that*

$$\big|\pi_\rho\big(\xi(0)=1,\xi(1)=0,\xi(l)=1\big) - \rho^2\beta(1+\alpha)\big| \le q'e^{-C'l},$$

and

$$\big|\pi_\rho\big(\xi(0)=1,\xi(1)=0,\xi(l-1)=0,\xi(l)=1\big) - \rho^2\beta^2(1+\alpha)^2\big| \le q^{'}e^{-C^{'}l}.$$

We now use Lemma 4.1 and Corollary 4.2 to show that the correlations under π_ρ of two boxes at distance l decay as e^{-Cl}, which immediately yields Theorem 2.2.

Corollary 4.3. *Fix $k, m \ge 1$, and let $A = A(k) := \tau_{-k}\Lambda_k$ and $B = B(l,m) := \tau_l\Lambda_m$. For any two configurations σ_1, σ_2 in $\{0,1\}^A, \{0,1\}^B$, for large l,x'*

$$\pi_\rho(\xi_{|A}=\sigma_1, \xi_{|B}=\sigma_2) = \pi_\rho(\xi_{|A}=\sigma_1)\pi_\rho(\xi_{|B}=\sigma_2)\big(1+\mathcal{O}(e^{-Cl})\big), \tag{6}$$

where the error $\mathcal{O}(e^{-Cl})$ depends on ρ, but can be bounded uniformly in k, m and σ_1, σ_2.

Proof. First, we consider two neighboring sets $A_1 = \{a, a+1, ..., b\}, A_2 = \{b+1, b+2, ..., c\}$ where $a < b-2 < c-4$. For any given two configurations σ_1, σ_2 on these sets separately, we can write, thanks to the explicit formula (8),

$$\pi_\rho(\xi_{|A_1}=\sigma_1, \xi_{|A_2}=\sigma_2) = \frac{1_{\{\phi_2(\sigma_1)+\phi_1(\sigma_2)\le 2\}}}{\big(\frac{\beta}{\alpha}\big)^{1-\sigma_1(b)-\sigma_2(b+1)}(1+\alpha)^{\phi(\sigma_1,\sigma_2)}}\pi_\rho(\xi_{|A_1}=\sigma_1)\pi_\rho(\xi_{|A_2}=\sigma_2) \tag{7}$$

where $\phi_1(\sigma)$ (resp. $\phi_2(\sigma)$) is the number of consecutive holes between the leftmost (resp. rightmost) particle and the left (resp. right) boundary, and $\phi(\sigma_1,\sigma_2)=\sigma_1(b-1)(1-\sigma_1(b))+\sigma_2(b+2)(1-\sigma_2(b+1))$. In particular, $\phi(\sigma_1,\sigma_2)=2-|\phi_2(\sigma_1)-1|-|\phi_1(\sigma_2)-1|$.

Without loss of generality, assume $l>4$. Summing over all the possible configurations σ in Λ_l, and using the identity (7) (since A,Λ_l and B are neighboring sets), we can write for any $\sigma_1\in\{0,1\}^A,\sigma_2\in\{0,1\}^B$,

$$
\begin{aligned}
&\pi_\rho(\xi_{|A}=\sigma_1,\xi_{|B}=\sigma_2)\\
=&\sum_{\sigma\in\{0,1\}^{\Lambda_l}}\pi_\rho(\xi_{|A}=\sigma_1,\xi_{|\Lambda_l}=\sigma,\xi_{|B}=\sigma_2)\\
=&\sum_{\sigma\in\{0,1\}^{\Lambda_l}}\pi_\rho(\xi_{|A}=\sigma_1)\frac{1_{\{\phi_2(\sigma_1)+\phi_1(\{\sigma,\sigma_2\})\le 2\}}}{\rho\big(\frac{\beta}{\alpha}\big)^{1-\sigma_1(0)-\sigma(1)}(1+\alpha)^{\phi(\sigma_1,\{\sigma,\sigma_2\})}}\pi_\rho(\xi_{|\Lambda_l}=\sigma,\xi_{|B}=\sigma_2)\\
=&\sum_{\sigma\in\{0,1\}^{\Lambda_l}}\pi_\rho(\xi_{|A}=\sigma_1)\frac{1_{\{\phi_2(\sigma_1)+\phi_1(\sigma)\le 2\}}}{\rho\big(\frac{\beta}{\alpha}\big)^{1-\sigma_1(0)-\sigma(1)}(1+\alpha)^{\phi(\sigma_1,\sigma)}}\pi_\rho(\xi_{|\Lambda_l}=\sigma)\\
&\qquad\times\frac{1_{\{\phi_2(\sigma)+\phi_1(\sigma_2)\le 2\}}}{\rho\big(\frac{\beta}{\alpha}\big)^{1-\sigma(l)-\sigma_2(l+1)}(1+\alpha)^{\phi(\sigma,\sigma_2)}}\pi_\rho(\xi_{|B}=\sigma_2)\\
=&\ \pi_\rho(\xi_{|A}=\sigma_1)\pi_\rho(\xi_{|B}=\sigma_2)\\
&\times\sum_{\sigma\in\{0,1\}^{\Lambda_l}}\frac{1_{\{\phi_2(\sigma_1)+\phi_1(\sigma)\le 2\}}1_{\{\phi_2(\sigma)+\phi_1(\sigma_2)\le 2\}}}{\rho^2\big(\frac{\beta}{\alpha}\big)^{2-\sigma_1(0)-\sigma(1)-\sigma(l)-\sigma_2(l+1)}(1+\alpha)^{\phi(\sigma_1,\sigma)+\phi(\sigma,\sigma_2)}}\pi_\rho(\xi_{|\Lambda_l}=\sigma).
\end{aligned}\tag{8}
$$

Turn to the sum in (8). We will focus on the case $\phi_2(\sigma_1)=\phi_1(\sigma_2)=1$, since other cases are treated similarly. In this case, the sum in (8) can be written as

$$\sum_{\sigma\in\{0,1\}^{\Lambda_l}}\frac{1_{\{\phi_1(\sigma)\le 1\}}1_{\{\phi_2(\sigma)\le 1\}}}{\rho^2\big(\frac{\beta}{\alpha}\big)^{2-\sigma(1)-\sigma(l)}(1+\alpha)^{2+\sigma(2)(1-\sigma(1))+\sigma(l-1)(1-\sigma(l))}}\pi_\rho(\xi_{|\Lambda_l}=\sigma).$$

To merge the items inside the sum above, we will construct a new configuration σ' by concatenating $\{\bullet\circ\}$ to the left boundary and a hole and $\{\circ\bullet\}$ to the right boundary. Its length is $l'=l+4$, and has particles $p'=p+2$. Again by the explicit formula (8),

$$
\begin{aligned}
&\pi_\rho(\xi_{|\Lambda_{l'}}=\sigma')\\
=&1_{\{\phi_1(\sigma)\le 1\}}1_{\{\phi_2(\sigma)\le 1\}}\rho\big(\frac{\beta}{\alpha}\big)^{p'-1}\alpha^{l'-p'}\\
=&1_{\{\phi_1(\sigma)\le 1\}}1_{\{\phi_2(\sigma)\le 1\}}\rho\big(\frac{\beta}{\alpha}\big)^{p+1}\alpha^{l-p+2}\\
=&1_{\{\phi_1(\sigma)\le 1\}}1_{\{\phi_2(\sigma)\le 1\}}\pi_\rho(\xi_{|\Lambda_l}=\sigma)\big(\frac{\beta}{\alpha}\big)^{\sigma(1)+\sigma(l)}\alpha^2(1+\alpha)^{-\sigma(2)(1-\sigma(1))-\sigma(l-1)(1-\sigma(l))}.
\end{aligned}
$$

Substitute the above expression in (8), we obtain

$$
\begin{aligned}
&\pi_\rho(\xi_{|A}=\sigma_1)\pi_\rho(\xi_{|B}=\sigma_2)\sum_{\sigma''\in\{0,1\}^{\Lambda_{l'-4}}}\frac{1}{\rho^2\beta^2(1+\alpha)^2}\pi_\rho(\xi(0)=1,\xi(1)=0,\xi_{\Lambda_{l'-4}}=\sigma'',\xi(l'-1)=0,\xi(l')=1)\\
=&\pi_\rho(\xi_{|A}=\sigma_1)\pi_\rho(\xi_{|B}=\sigma_2)\frac{\pi_\rho(\xi(0)=1,\xi(1)=0,\xi(l'-1)=0,\xi(l')=1)}{\rho^2\beta^2(1+\alpha)^2}.
\end{aligned}\tag{9}
$$

Therefore,

$$\pi_\rho(\xi_{|A}=\sigma_1,\xi_{|B}=\sigma_2)=\frac{\pi_\rho(\xi(0)=1,\xi(1)=0,\xi(l'-1)=0,\xi(l')=1)}{\rho^2\beta^2(1+\alpha)^2}=1+\mathcal{O}(e^{-Cl}),$$

the second equality holds by the Corollary 4.2. The proof is completed. □

4.2 Equivalence of ensembles

Recall that

$$E_l(\delta) = \big\{\lceil (2l+1)(1+\delta)/3\rceil, ..., \lfloor (1-\delta)(2l+1)\rfloor\big\}.$$

We begin by stating a proposition.

Proposition 4.4. *For any $\delta_2 > 0$, there exist some constant $C_1 := C_1(k, \delta_2)$, and a sufficiently small δ_1, such that*

$$\limsup_{l\to\infty} \max_{\substack{j\in E_l\backslash E_l(\delta_1)\\ x\in B_{(1-\delta_2)l-k}\\ \sigma\in\{0,1\}^{B_k}}} \Big|\hat{\pi}_\rho^{l,j}(\zeta_{|B_k(x)} = \sigma) - \pi_{\rho_l(j)}(\xi_{|B_k} = \sigma)\Big| \le C_1\delta_1. \tag{10}$$

For any $\delta_2, \delta_1, \varepsilon > 0$,

$$\limsup_{l\to\infty} \max_{\substack{j\in E_l(\delta_1)\\ x\in B_{(1-\delta_2)l-k}\\ \sigma\in\{0,1\}^{B_k}}} \Big|\hat{\pi}_\rho^{l,j}(\zeta_{|B_k(x)} = \sigma) - \pi_{\rho_l(j)}(\xi_{|B_k} = \sigma)\Big| \le \varepsilon. \tag{11}$$

Before proceeding, we complete the proof of Theorem 2.3.

Proof of Theorem 2.3. Fix $\rho \in (\frac{1}{3}, 1)$ and a local function f. There exists an integer k such that f only depends on sites in B_k. Then, if we choose $\delta_2 = \delta/2$, for any $l > 2k/\delta$, we have $B_{(1-\delta)l} \subset B_{(1-\delta_2)l-k}$. Then for l large enough, we can use the triangle inequality and Proposition 4.4 to write for any $\delta_1 > 0, \varepsilon > 0$,

$$\begin{aligned}
&\max_{\substack{j\in E_l\\ x\in B_{(1-\delta)l}}} \big|\hat{\pi}_\rho^{l,j}(\tau_x f) - \pi_{\rho_l(j)}(f)\big|\\
\le& 2^{2k+1}||f||_\infty \max_{\substack{j\in E_l\\ x\in B_{(1-\delta)l}\\ \sigma\in\{0,1\}^{B_k}}} \Big|\hat{\pi}_\rho^{l,j}(\zeta_{|B_k(x)} = \sigma) - \pi_{\rho_l(j)}(\xi_{|B_k} = \sigma)\Big|\\
\le& 2^{2k+1}||f||_\infty \Big(\max_{\substack{j\in E_l(\delta_1)\\ x\in B_{(1-\delta_2)l-k}\\ \sigma\in\{0,1\}^{B_k}}} \Big|\hat{\pi}_\rho^{l,j}(\zeta_{|B_k(x)} = \sigma) - \pi_{\rho_l(j)}(\xi_{|B_k} = \sigma)\Big|\\
&\qquad + \max_{\substack{j\in E_l\backslash E_l(\delta_1)\\ x\in B_{(1-\delta_2)l-k}\\ \sigma\in\{0,1\}^{B_k}}} \Big|\hat{\pi}_\rho^{l,j}(\zeta_{|B_k(x)} = \sigma) - \pi_{\rho_l(j)}(\xi_{|B_k} = \sigma)\Big|\Big)\\
\le& 2^{2k+1}||f||_\infty \Big((1+o_l(1))(\varepsilon + \delta_1 C_1)\Big).
\end{aligned}$$

Since k is fixed, we let $l \to \infty$, then $\varepsilon \to 0$ and $\delta_1 \to 0$. This proves Theorem 2.3. □

Now we turn to the proof of Proposition 4.4. We start with the case where the fixed density $\rho_l(j)$ is close to the extreme values $\frac{1}{3}$ or 1 (i.e. $j \in E_l\backslash E_l(\delta_1)$).

Proof of Proposition 4.4 (10). We will only detail the case $\rho_l(j) \ge 1-\delta_1$, since the case $\rho_l(j) \le (1+\delta_1)/3$ is treated in the same way. Denote by $\mathbf{1}_k$ the constant configuration on B_k with one particle at each site, and no empty site. To prove (10), it is sufficient to show that for some constant $C_2 := C_2(k, \delta_2)$, we have for any $j \in E_l\backslash E_l(\delta_1)$ satisfying $\rho_l(j) \ge 1-\delta_1$,

$$\pi_{\rho_l(j)}\big(\xi_{|B_k} = \mathbf{1}_k\big) \ge 1 - C_2\delta, \tag{12}$$

and for any $x \in B_{(1-\delta_2)l-k}$,

$$\hat{\pi}_{\rho}^{l,j}\big(\xi_{|B_k(x)} = \mathbf{1}_k\big) \geq 1 - C_2\delta, \tag{13}$$

and then let $C_1(k,\delta_2) := 2C_2(k,\delta_2)$.

(12) holds noting δ_1 sufficiently small, $\rho_l(j) \geq 1-\delta_1$, $\bar{\rho}_l(j) = \frac{1}{\rho_l(j)} - 2$, and the explicit formula (10):

$$\begin{aligned}\pi_{\rho_l(j)}(\xi_{|B_k} = \mathbf{1}_k) &= \rho_l(j)\Big(\frac{\beta(\rho_l(j))}{\alpha(\rho_l(j))}\Big)^{k-1}\\ &> (1-\delta_1)\Big(\frac{4(1-\bar{\rho}_l(j))^2}{8-6\bar{\rho}_l(j)+2\sqrt{4-3\bar{\rho}_l(j)^2}}\Big)^{k-1}\\ &> (1-\delta_1)\Big(\frac{4(2-\frac{\delta_1}{1-\delta_1})^2}{8-6\bar{\rho}_l(j)+2\sqrt{4-3\bar{\rho}_l(j)^2}}\Big)^{k-1}\\ &> (1-\delta_1)(1-\delta_1)^{2k-2} > 1-\frac{1}{2}\delta_1.\end{aligned}$$

From now on, for $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4 \in \{0,1\}$, we denote by

$$\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\cdot) = \hat{\pi}_{\rho}^{l,j}\big(\,\cdot\,|\zeta(-l)=\varepsilon_1, \zeta(-l+1)=\varepsilon_2, \zeta(l-1)=\varepsilon_3, \zeta(l)=\varepsilon_4\big)$$

the measure $\hat{\pi}_{\rho}^{l,j}$ conditioned to be in the boundary condition $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$. Then, to prove (13), it is clearly sufficient to prove for any $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4 \in \{0,1\}$

$$\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)} = \mathbf{1}_k) \geq 1 - C_2\delta_1.$$

Furthermore, to prove the latter, it is sufficient to prove that for some constant $C(k,\delta_2)$, for any $\sigma \neq \mathbf{1}_k \in \{0,1\}^{B_k}$

$$\frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)} = \sigma)}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)} = \mathbf{1}_k)} \leq C(k,\delta_2)\delta_1. \tag{14}$$

Indeed, summing this bound above over $\sigma \in \{0,1\}^{B_k}\backslash\{\mathbf{1}_k\}$, yields

$$\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)} = \mathbf{1}_k) \geq \frac{1}{1+2^{|B_k|}C(k,\delta_2)\delta_1} \geq 1 - C_2(k,\delta_2)\delta_1,$$

as wanted. It remains to prove (14). First, define

$$l_1 := l_1(x,l,k) = l-k+x \geq \delta_2 l, \qquad l_2 := l_2(x,l,k) = l-k-x \geq \delta_2 l,$$

which are the respective sizes of the clusters to the left and right of $B_k(x)$ in B_l. Denote by n the number of particles in the cluster $\{-l,\dots,-l+l_1-x\}$ to the left of $B_k(x)$ in B_l. Since $\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}$ is the uniform measure on the subspace of space $\hat{\Omega}_l^j$ with boundary condition $\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}$, it follows for $\sigma,\sigma' \in \widehat{\mathcal{E}}_{\Lambda_l}$

$$\frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)} = \sigma')}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)} = \sigma)} = \frac{\sum_{n=1}^{j} a_n(\sigma')b_n(\sigma')}{\sum_{n=1}^{j} a_n(\sigma)b_n(\sigma)}, \tag{15}$$

where

$$\begin{aligned}a_n(\sigma) =& a_n(\sigma,j,l_1,\varepsilon_1,\varepsilon_2)\\ :=& \binom{n^\sigma}{l_1^\sigma - 2n^\sigma}_2 + \binom{n^\sigma}{l_1^\sigma - 2n^\sigma - 1}_2 (1-\varepsilon_2)\varepsilon_1\\ &+ \binom{n^\sigma}{l_1^\sigma - 2n^\sigma - 1}_2 \sigma_{-k+1}(1-\sigma_{-k}) + \binom{n^\sigma}{l_1^\sigma - 2n^\sigma - 2}_2 \varepsilon_1(1-\varepsilon_2)\sigma_{-k+1}(1-\sigma_{-k}),\end{aligned}$$

$$
\begin{aligned}
b_n(\sigma) =& b_n(\sigma, j, l_2, \varepsilon_3, \varepsilon_4) \\
:=& \binom{j^\sigma - n^\sigma}{l_2^\sigma - 2(j^\sigma - n^\sigma)}_2 + \binom{j^\sigma - n^\sigma}{l_2^\sigma - 2(j^\sigma - n^\sigma) - 1}_2 (1-\varepsilon_3)\varepsilon_4 \\
&+ \binom{j^\sigma - n^\sigma}{l_2^\sigma - 2(j^\sigma - n^\sigma) - 1}_2 \sigma_{k-1}(1-\sigma_k) + \binom{j^\sigma - n^\sigma}{l_2^\sigma - 2(j^\sigma - n^\sigma) - 2}_2 \varepsilon_4(1-\varepsilon_3)\sigma_{k-1}(1-\sigma_k),
\end{aligned}
$$

and

$$
\begin{aligned}
n^\sigma &= n - \varepsilon_1 + \sigma_{-k} - 1, \qquad j^\sigma = j - p(\sigma) - \varepsilon_1 - \varepsilon_4 + \sigma_{-k} + \sigma_k - 2, \\
l_1^\sigma &= l_1 + \sigma_{-k} + \varepsilon_2 - 3, \qquad l_2^\sigma = l_2 + \sigma_k + \varepsilon_3 - 3.
\end{aligned}
$$

The quantity $a_n(\sigma)$ is the number of ergodic configurations on the left cluster with n particles and compatible with the left boundary of σ, whereas $b_n(\sigma)$ is the number of ergodic configurations on the right cluster compatible with the right boundary of σ. Then $a_n(\sigma)b_n(\sigma)$ is the number of ergodic configurations on B_l with j particles, such that the configuration in $B_k(x)$ is given by σ, and such that the number of particles in $\{-l, \ldots, -l+l_1-1\}$ (resp. $\{l-l_2+1, \ldots, l\}$) is n (resp. $j - p(\sigma) - n$).

Notice $\binom{a}{b} = 0$ if $b \notin \{0, \ldots, a\}$, applying (15) to σ and $\mathbf{1}_k$ yields

$$
\frac{\widehat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\big(\zeta_{|B_k(x)} = \sigma\big)}{\widehat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\big(\zeta_{|B_k(x)} = \mathbf{1}_k\big)} = \frac{\sum_{n=1}^{j} a_n(\sigma)b_n(\sigma)}{\sum_{n=1}^{j} a_n(\mathbf{1}_k)b_n(\mathbf{1}_k)}. \tag{16}
$$

We shall prove that there exists a constant $C := C(k, \delta_2)$ such that for any $1 \le n \le j$ and any $\sigma \ne \mathbf{1}_k$,

$$
a_n(\sigma)b_n(\sigma) \le C\delta_1 a_n(\mathbf{1}_k)b_n(\mathbf{1}_k). \tag{17}
$$

Note that by the identity (31), we can show that $a_n(\sigma) \le a_n(\mathbf{1}_k)$. To prove the inequality (17), we are enough to derive that

$$
b_n(\sigma) \le C\delta_1 b_n(\mathbf{1}_k). \tag{18}
$$

Now we use the inequality (36) to bounded $\frac{b_n(\sigma)}{b_n(\mathbf{1}_k)}$. It is enough to show that each trinomial coefficient in $b_n(\sigma)$ is less than the respective term in $b_n(\mathbf{1}_k)$ multiplied $C\delta_1$. Here we only deal with the first trinomial coefficient in b_n, the others can be derived similarly.

Note $j \ge (1-\delta_1)(2l+1)$ and $l_2 \ge \delta_2 l \ge \delta_2(2l+1)/4$, then

$$
j - n \ge (1-\delta_1)(2l+1) - l_1 = l_2 + (2k+1) - \delta_1(2l+1) \ge l_2(1 - 4\delta_1/\delta_2),
$$

thus $l_2 - 2(j-n)$ is negative for δ_1 sufficiently small.

Set $\tilde{n} = j - n$. If $\sigma(-k) = 1$, since $k > p(\sigma)$, using the inequality (39), we have

$$
\frac{b_n(\sigma)}{b_n(\mathbf{1}_k)} = \frac{\binom{\tilde{n} - p(\sigma) - \varepsilon_4 + 1 - 1}{l_2 - 2(\tilde{n} - p(\sigma)) + 2\varepsilon_4 + \varepsilon_3 - 1 - 1}_2}{\binom{\tilde{n} - k - \varepsilon_4 + 1 - 1}{l_2 - 2(\tilde{n} - k) + 2\varepsilon_4 + \varepsilon_3 - 1 - 1}_2} \le \left(\frac{\binom{\tilde{n} - k - \varepsilon_4 + 1}{l_2 - 2\tilde{n} + 2k + 2\varepsilon_4 + \varepsilon_3 - 2 - 2}_2}{\binom{\tilde{n} - k - \varepsilon_4}{l_2 - 2\tilde{n} + 2k + 2\varepsilon_4 + \varepsilon_3 - 2}_2} \right)^{k - p(\sigma)}.
$$

Since $k, \varepsilon_3, \varepsilon_4$ are fixed, we can choose a δ_1 such that $l_2 - 2\tilde{n} + 2k + 2\varepsilon_4 + \varepsilon_3 - 2$ is negative. So by the inequality (36),

$$
\begin{aligned}
\frac{b_n(\sigma)}{b_n(\mathbf{1}_k)} &\le \Big(\frac{l_2 - \tilde{n} + k + \varepsilon_4 + \varepsilon_3 - 4}{2\tilde{n} - l_2 - 2k - 2\varepsilon_4 - \varepsilon_3 + 3} \Big)^{k - p(\sigma)} \\
&\le \Big(\frac{\frac{4\delta_1}{\delta_2} l_2 + k + \varepsilon_4 + \varepsilon_3 - 4}{(1 - \frac{8\delta_1}{\delta_2}) l_2 - 2k - 2\varepsilon_4 - \varepsilon_3 + 3} \Big)^{k - p(\sigma)} \le C(k, \delta_2)\delta_1
\end{aligned}
$$

for l large enough.

If $\sigma(-k)=0$, using the inequality (39) again, we have

$$\frac{b_n(\sigma)}{b_n(\mathbf{1}_k)}=\frac{\binom{\tilde{n}-p(\sigma)-\varepsilon_4-1}{l_2-2(\tilde{n}-p(\sigma))+2\varepsilon_4+\varepsilon_3-1}_2}{\binom{\tilde{n}-k-\varepsilon_4+1-1}{l_2-2(\tilde{n}-k)+2\varepsilon_4+\varepsilon_3-1-1}_2}\le\Big(\frac{\binom{\tilde{n}-k-\varepsilon_4}{l_2-2\tilde{n}+2k+2\varepsilon_4+\varepsilon_3-2-1}_2}{\binom{\tilde{n}-k-\varepsilon_4}{l_2-2\tilde{n}+2k+2\varepsilon_4+\varepsilon_3-2}_2}\Big)^{k-p(\sigma)}.$$

Thus by the inequality (35),

$$\frac{b_n(\sigma)}{b_n(\mathbf{1}_k)}\le\Big(\frac{l_2-\tilde{n}+k+\varepsilon_4+\varepsilon_3-2}{2\tilde{n}-l_2-2k-2\varepsilon_4-\varepsilon_3+3}\Big)^{k-p(\sigma)}\le\Big(\frac{\frac{4\delta_1}{\delta_2}l_2+k+\varepsilon_4+\varepsilon_3-2}{(1-\frac{8\delta_1}{\delta_2})l_2-2k-2\varepsilon_4-\varepsilon_3+3}\Big)^{k-p(\sigma)}\le C(k,\delta_2)\delta_1$$

for l large enough. The proof of (10) is concluded. □

Now we proceed to the proof of (11) in the case $j\in E_l(\delta_1)$.

Proof of Proposition 4.4 (11). First, for any $l>l_0$ (l_0 only depends on k), we claim for $\sigma\in\hat{\mathcal{E}}_{B_k}$

$$\pi_{\rho_l}\big(\xi_{|B_k(x)}=\sigma\big)>0\quad\text{and}\quad\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\big(\zeta_{|B_k(x)}=\sigma\big)>0.\tag{19}$$

Indeed, the first statement holds since ρ_l is neither $\frac{1}{3}$ nor 1, and the second statement holds since $\rho_l\in[\frac{1}{3}(1+\delta_1),1-\delta_1]$. Using (19), we have

$$\begin{aligned}
&\Big|\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)-\pi_{\rho_l}(\xi_{|B_k}=\sigma)\Big|\\
&\le\Big|\frac{1}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)}-\frac{1}{\pi_{\rho_l}(\xi_{|B_k}=\sigma)}\Big|\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\big(\zeta_{|B_k(x)}=\sigma\big)\\
&=\Big|\sum_{\sigma'\in\{0,1\}^{B_k}}\frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma')}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)}-\frac{\pi_{\rho_l}(\xi_{|B_k}=\sigma')}{\pi_{\rho_l}(\xi_{|B_k}=\sigma)}\Big|\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\big(\zeta_{|B_k(x)}=\sigma\big)\\
&\le 2^{2k+1}\max_{\sigma'\in\{0,1\}^{B_k}}\Big|\frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma')}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)}-\frac{\pi_{\rho_l}(\xi_{|B_k}=\sigma')}{\pi_{\rho_l}(\xi_{|B_k}=\sigma)}\Big|\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\big(\zeta_{|B_k(x)}=\sigma\big).
\end{aligned}\tag{20}$$

It is sufficient to prove that for any $\varepsilon>0$, for any two configurations $\sigma,\sigma'\in\{0,1\}^{B_k}$, there exists a constant $L:=L(k,\sigma,\sigma',\delta_1,\delta_2)$ such that for any $x\in B_{(1-\delta_2)l-k}$ and $j\in E_l(\delta_1)$, if $l>L$,

$$\max_{\substack{j\in E_l(\delta_1)\\ x\in B_{(1-\delta_2)l-k}}}\Big|\frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma')}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)}-\frac{\pi_{\rho_l}(\xi_{|B_k}=\sigma')}{\pi_{\rho_l}(\xi_{|B_k}=\sigma)}\Big|\le\frac{\varepsilon}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}\big(\zeta_{|B_k(x)}=\sigma\big)}.\tag{21}$$

Define

$$F(\rho,\sigma)=\frac{\beta(\rho)^{p(\sigma)+\varepsilon_1+\varepsilon_4+2}}{\alpha(\rho)^{2p(\sigma)+2\varepsilon_1+\varepsilon_2+2\varepsilon_4+\varepsilon_3-2}}\Big(\frac{\beta(\rho)}{\alpha(\rho)}\Big)^{-\sigma_{-k}-\sigma_k}(1+\alpha(\rho))^{(1-\varepsilon_2)\varepsilon_1+\sigma_{-k+1}(1-\sigma_{-k})+(1-\varepsilon_3)\varepsilon_4+\sigma_{k-1}(1-\sigma_k)}.\tag{22}$$

Then

$$\frac{F(\rho,\sigma')}{F(\rho,\sigma)}=\Big(\frac{\beta(\rho)}{\alpha(\rho)^2}\Big)^{p(\sigma')-p(\sigma)}\Big(\frac{\beta(\rho)}{\alpha(\rho)}\Big)^{\sigma_{-k}+\sigma_k-\sigma'_{-k}-\sigma'_k}(1+\alpha(\rho))^{\nu(\sigma')-\nu(\sigma)}=\frac{\pi_{\rho_l}(\xi_{|B_k}=\sigma')}{\pi_{\rho_l}(\xi_{|B_k}=\sigma)},\tag{23}$$

where $\nu(\sigma)=\sigma(2)(1-\sigma(1))+\sigma(l-1)(1-\sigma(l))$. Note that $F(\rho,\sigma)$ is uniformly continuous for $\rho\in[a,b]\subset(\frac{1}{3},1)$.

Set $S(\sigma)=\sum_{n=1}^{j}a_n(\sigma)b_n(\sigma)$ and $S^*=\sum_{n=1}^{j}a_n^*b_n^*$, where

$$a_n^*=\binom{n}{l_1-2n}_2,\quad b_n^*=\binom{(j-n)}{l_2-2(j-n)}_2.$$

Thus it easily follows

$$\begin{aligned} & \frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma')}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)} - \frac{\pi_{\rho_l}(\xi_{|B_k}=\sigma')}{\pi_{\rho_l}(\xi_{|B_k}=\sigma)} \\ = & \frac{S(\sigma')}{S(\sigma)} - \frac{F(\rho_l,\sigma')}{F(\rho_l,\sigma)} \\ = & \frac{F(\rho_l,\sigma)S^*S(\sigma') - F(\rho_l,\sigma')S^*S(\sigma)}{F(\rho_l,\sigma)S^*S(\sigma)} \end{aligned} \tag{24}$$

By the triangle inequality,

$|F(\rho_l,\sigma)S^*S(\sigma') - F(\rho_l,\sigma')S^*S(\sigma)| \le |(F(\rho_l,\sigma)S^* - S(\sigma))S(\sigma')| + |(F(\rho_l,\sigma')S^* - S(\sigma'))S(\sigma)|$.

Since $j \in E_l(\delta_1)$, there exist constants $c_F^1 := c_F(k,\delta_1) > 0, c_F^2 := c_F^2(k,\delta) > 0$ such that $c_F^1 < F(\rho_l,\sigma) < c_F^2$. We claim that for any configuration $\sigma \in \{0,1\}^{B_k}$, and any positive δ_1,δ_2, there exists a constant $L := L(k,\delta_1,\delta_2)$ such that for $l > L$, for all $j \in E_l(\delta_1), x \in B_{(1-\delta_2)l-k}$,

$$|S(\sigma) - F(\rho_l,\sigma)S^*| \le S^* c_F^1 \varepsilon/2. \tag{25}$$

Once it holds, we have

$$\Big|\frac{S(\sigma')}{S(\sigma)} - \frac{F(\rho_l,\sigma')}{F(\rho_l,\sigma)}\Big| \le \frac{S(\sigma')+S(\sigma)}{F(\rho_l,\sigma)S(\sigma)} c_F^1\varepsilon/2.$$

It follows for any $j \in E_l(\delta_1)$

$$\begin{aligned} \Big|\frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma')}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)} - \frac{\pi_{\rho_l}(\xi_{|B_k}=\sigma')}{\pi_{\rho_l}(\xi_{|B_k}=\sigma)}\Big| &\le \frac{c_F^1\varepsilon}{2c_F^1}\Big(1 + \frac{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma')}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)}\Big) \\ &\le \frac{\varepsilon}{\hat{\pi}^{l,j}_{\rho,\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4}(\zeta_{|B_k(x)}=\sigma)}, \end{aligned}$$

which proves (21). □

It remains to prove the claim (25). To this end, we need the following lemma.

Lemma 4.5. *Let $h = j/(l_1+l_2), n_0 = h * l_1$. There exist $L > 0$,$\delta_3 > 0$, for $l > L$,*

$$\max_{\substack{j\in E_l(\delta_1) \\ x\in B_{(1-\delta_2)l-k}}} \max_{n\in I_l(\sigma)} \Big|\frac{a_n(\sigma)b_n(\sigma)}{a_n^* b_n^*} - F(\rho_l,\sigma)\Big| \le c_F^1\varepsilon/4, \tag{26}$$

where $I_l(\sigma)$ is the set of n's in $\{0,...,j\}$ such that $n \in (n_0 - \delta_3 l, n_0 + \delta_3 l)$. Moreover,

$$\max_{\substack{j\in E_l(\delta_1) \\ x\in B_{(1-\delta_2)l-k}}} \frac{\sum_{n\notin I_l(\sigma)}(a_n(\sigma)b_n(\sigma) + a_n^* b_n^*)}{a_{n_0}b_{n_0}} \le \frac{c_F^1\varepsilon}{(1+F(\rho_l,\sigma))4}. \tag{27}$$

Proof. Since $j \in E_l(\delta_1)$ and $l_1 + l_2 + k = 2l+1$, $\frac{j}{l_1+l_2} \in [\frac{1}{3}(1+\delta_1), 1-\frac{1}{2}\delta_1]$ for l large enough. Then by Corollary B.2, for $\varepsilon > 0$, for each $\rho \in [\frac{1}{3}(1+\delta_1), 1-\frac{1}{2}\delta_1]$, there exist $L_\rho, \delta_\rho > 0$ such that for $l > L_\rho$, for each j,n satisfying $|n - \rho l_1| < \delta_\rho l_1$, $|j - n - \rho l_2| \le \delta_\rho l_2$,

$$\Big|\frac{a_n(\sigma)b_n(\sigma)}{a_n^* b_n^*} - F(\rho,\sigma)\Big| \le \frac{\varepsilon}{2}.$$

By the uniform continuity of F in the range $[\frac{1}{3}(1+\delta_1), 1-\frac{1}{2}\delta_1]$, we can find a $\delta_3 > 0$ such that for $\rho_1, \rho_2 \in [\frac{1}{3}(1+\delta_1), 1-\frac{1}{2}\delta_1]$ satisfying $|\rho_1 - \rho_2| < \delta_3$,

$$|F(\rho_1,\sigma) - F(\rho_2,\sigma)| \le \frac{\varepsilon}{2}. \tag{28}$$

Let the open interval sets $\{(\rho-\min(\delta_\rho,\delta_3)/2,\rho+\min(\delta_\rho,\delta_3)/2)\}$ cover $[\frac{1}{3}(1+\delta_1),1-\frac{1}{2}\delta_1]$. By the Finite Covering Theorem, there exist $L',\delta_4>0$, for $l>L', j\in E_l(\delta_1)$, for each n where $|n-n_0|<\delta_4\min\{l_1,l_2\}$, there exists a $\rho\in[\frac{1}{3}(1+\delta_1),1-\frac{1}{2}\delta_1]$ satisfying $|\rho-\rho_l|\le\delta_3$ such that

$$\Big|\frac{a_n(\sigma)b_n(\sigma)}{a_n^*b_n^*}-F(\rho,\sigma)\Big|\le\frac{\varepsilon}{2}. \tag{29}$$

Thus

$$\Big|\frac{a_n(\sigma)b_n(\sigma)}{a_n^*b_n^*}-F(\rho_l,\sigma)\Big|\le\Big|\frac{a_n(\sigma)b_n(\sigma)}{a_n^*b_n^*}-F(\rho,\sigma)\Big|+\big|F(\rho,\sigma)-F(\rho_l,\sigma)\big|\le\varepsilon.$$

(26) holds by choosing a suitable ε.

As for (27), we only handle the case $n\le n_0$ since the case $n\ge n_0$ is treated similarly. Define $f(\rho)=\frac{\beta(\rho)}{\alpha^2(\rho)}, g(\rho)=\frac{j-\rho l_1}{l_1+l_2}$. Again by Corollary B.2, set $\varepsilon'=\frac{1}{2}|1-\frac{f(h-\frac{\delta_3}{2})}{f(g(h-\frac{\delta_3}{2}))}|$, one can check that there exist $L,\delta'>0$ such that for $l>L$, $|n-(n_0-\frac{\delta_3}{2}(l_1+l_2))|\le\delta'\min\{l_1,l_2\}$,

$$\Big|\frac{a_nb_n}{a_{n+1}b_{n+1}}-\frac{f(h-\frac{\delta_3}{2})}{f(g(h-\frac{\delta_3}{2}))}\Big|\le\varepsilon'.$$

It is easy to see that $f(\rho),1/f(g(\rho))$ are monotone increasing and $h=g(h)$. So $f(h)/f(g(h))=f(h)/f(h)=1$ and $f(\rho)/f(g(\rho))<1$ for $\rho<h$. Then

$$\frac{a_nb_n}{a_{n+1}b_{n+1}}\le\frac{1+\frac{f(h-\frac{\delta_3}{2})}{f(g(h-\frac{\delta_3}{2}))}}{2}<1.$$

Since $\frac{a_nb_na_{n+2}b_{n+2}}{(a_{n+1}b_{n+1})^2}<1$, there exists an $L>0$ such that for $l>L$,

$$\frac{a_{n_0-\frac{\delta_3}{2}-\sqrt{l}}b_{n_0-\frac{\delta_3}{2}-\sqrt{l}}}{a_{n_0-\frac{\delta_3}{2}}b_{n_0-\frac{\delta_3}{2}}}\le\Big(\frac{1+\frac{f(h-\frac{\delta_3}{2})}{f(g(h-\frac{\delta_3}{2}))}}{2}\Big)^{\sqrt{l}}\le\varepsilon.$$

Therefore,

$$\begin{aligned}\sum_{n\le n_0-\delta_3 l}a_nb_n&\le\frac{1}{1-\frac{1+\frac{f(h-\frac{\delta_3}{2})}{f(g(h-\frac{\delta_3}{2}))}}{2}}a_{n_0-\frac{\delta_3}{2}-\sqrt{l}}b_{n_0-\frac{\delta_3}{2}-\sqrt{l}}\\&\le a_{n_0-\frac{\delta_3}{2}}b_{n_0-\frac{\delta_3}{2}}\varepsilon\le a_{n_0}b_{n_0}\varepsilon,\end{aligned}$$

and (27) holds by choosing a suitable ε. □

Finally, turn back to the proof of (25). By the triangle inequality,

$$\begin{aligned}|S(\sigma)-F(\rho_l,\sigma)S^*|&=\Big|\sum_{n=1}^{j}\big(a_n(\sigma)b_n(\sigma)-F(\rho_l,\sigma)a_n^*b_n^*\big)\Big|\\&\le\sum_{n\in I_l(\sigma)}\Big|\frac{a_n(\sigma)b_n(\sigma)}{a_n^*b_n^*}-F(\rho_l,\sigma)\Big|a_n^*b_n^*\\&\quad+\sum_{n\notin I_l(\sigma)}(a_n(\sigma)b_n(\sigma)+F(\rho_l,\sigma)a_n^*b_n^*).\end{aligned} \tag{30}$$

Using Lemma 4.5, (30) becomes

$$|S(\sigma)-F(\rho_l,\sigma)S^*|\le\sum_{n\in I_l(\sigma)}a_n^*b_n^*c_F^1\varepsilon/4+(1+F(\rho_l,\sigma))\frac{c_F^1\varepsilon}{(1+F(\rho_l,\sigma))4}a_{n_0}^*b_{n_0}^*\le S^*c_F^1\varepsilon/2,$$

which proves (25). □

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Appendix A Properties of trinomial coefficients

Proposition A.1. *(i)*

$$\binom{k}{m}_2 = \binom{k-1}{m+1}_2 + \binom{k-1}{m}_2 + \binom{k-1}{m-1}_2. \tag{31}$$

(ii)

$$\frac{k+m}{k}\binom{k}{m}_2 = \binom{k-1}{m}_2 + 2\binom{k-1}{m-1}_2. \tag{32}$$

(iii)

$$\frac{m}{k}\binom{k}{m}_2 = \binom{k-1}{m-1}_2 - \binom{k-1}{m+1}_2. \tag{33}$$

(iv)

$$\frac{k-m}{k}\binom{k}{m}_2 = \binom{k-1}{m}_2 + 2\binom{k-1}{m+1}_2. \tag{34}$$

(v) For $m \le 0$,

$$\frac{\binom{k}{m-1}_2}{\binom{k}{m}_2} \le \frac{k+m}{-m+1}, \tag{35}$$

$$\frac{\binom{k}{m}_2}{\binom{k-1}{m+2}_2} \le \frac{k+m+1}{-m-1}. \tag{36}$$

For $m \ge 0$,

$$\frac{\binom{k-1}{m+2}_2}{\binom{k}{m}_2} \le \frac{k-m-1}{k+m+1}. \tag{37}$$

(vi) For $|m| \le k-1$,

$$\frac{\binom{k}{m-1}_2}{\binom{k}{m}_2} \le \frac{\binom{k}{m}_2}{\binom{k}{m+1}_2}. \tag{38}$$

(vii) For $|m| \le k-3$,

$$\frac{\binom{k-1}{m+2}_2}{\binom{k}{m}_2} \le \frac{\binom{k}{m}_2}{\binom{k+1}{m-2}_2}. \tag{39}$$

Proof. (i) It is easy to derive the identity (31) from the equation $(1+x+\frac{1}{x})^{k-1}(1+x+\frac{1}{x}) = (1+x+\frac{1}{x})^k$ and the definition of trinomial coefficient.

(ii) Identity (32) is satisfied by the expansion formula (1),

$$\begin{aligned}\binom{k-1}{m}_2 + 2\binom{k-1}{m-1}_2 &= \sum_{r=-\infty}^{\infty}\binom{k-1}{m+r}\binom{k-m-r-1}{r} + \sum_{r=-\infty}^{\infty}\binom{k-1}{m+r-1}\binom{k-m-r}{r}\\ &= \sum_{r=-\infty}^{\infty}\left(\frac{k-m-2r}{k}\binom{k}{m+r}\binom{k-m-r}{r} + \frac{2(m+r)}{k}\binom{k}{m+r}\binom{k-m-r}{r}\right)\\ &= \frac{k+m}{k}\binom{k}{m}_2.\end{aligned}$$

(iii) and (iv) (33) can be deduced from the identities (31) and (32), and (34) can be derived by the identity (32) and $\binom{k}{m}_2 = \binom{k}{-m}_2$.

(v) Suppose $m \le 0$. One can easily prove the inequality (35) by knowing that $\binom{k}{m-1}_2 =$

$\binom{k}{-m+1}_2$, $\binom{k}{-m}_2 = \binom{k}{m}_2$, and

$$\begin{aligned}\binom{k}{-m+1}_2 &= \sum_{r=0}^{k} \frac{k!}{(-m+1+r)!r!(k+m-1-2r)!} \\ &= \sum_{r=0}^{k} \frac{k+m-2r}{-m+r+1} \times \frac{k!}{(-m+r)!r!(k+m-2r)!} \\ &\le \sum_{r=0}^{k} \frac{k+m}{-m+1} \times \frac{k!}{(-m+r)!r!(k+m-2r)!} \\ &= \frac{k+m}{-m+1}\binom{k}{-m}_2.\end{aligned}$$

As for the inequality (36), we will introduce the following identity.

$$\begin{aligned}\binom{k}{m}_2 &= \binom{k-1}{m+1}_2 + \binom{k-1}{m}_2 + \binom{k-1}{m-1}_2 \\ &= \frac{k+m+1}{k}\binom{k}{m+1}_2 - 2\binom{k-1}{m} + \binom{k-1}{m}_2 + \binom{k-1}{m-1}_2 \\ &= \frac{k+m+1}{k}\left(\binom{k-1}{m+2}_2 + \binom{k-1}{m+1}_2 + \binom{k-1}{m}_2\right) - \binom{k-1}{m}_2 + \binom{k-1}{m-1}_2 \\ &= \frac{k+m+1}{k}\binom{k-1}{m+2}_2 + \frac{k+m+1}{k}\binom{k-1}{m+1}_2 + \frac{m+1}{k}\binom{k-1}{m}_2 + \binom{k-1}{m-1}_2, \end{aligned} \tag{40}$$

where we use the identity (32) to $\binom{k}{m+1}_2$ for the second equality. Then, by subtracting $\frac{k+m+1}{k}\binom{k}{m}_2$ on both sides of (40) and using the identity (31) to decompose $\binom{k}{m}_2$, we have

$$\begin{aligned}\frac{-m-1}{k}\binom{k}{m}_2 &= \frac{k+m+1}{k}\binom{k-1}{m+2}_2 - \binom{k-1}{m}_2 + \frac{-m-1}{k}\binom{k-1}{m-1}_2 \\ &\le \frac{k+m+1}{k}\binom{k-1}{m+2}_2\end{aligned}$$

since $\binom{k-1}{m}_2$ is greater than $\binom{k-1}{m-1}_2$ when $m \le 0$. Thus (36) holds.

Turn to the inequality (37). For $m \ge 0$, by subtracting $\frac{m+1}{k}\binom{k}{m}_2$ on both sides of (40) and using the identity (31) to decompose $\binom{k}{m}_2$, we have

$$\begin{aligned}\frac{k-m-1}{k}\binom{k}{m}_2 &= \frac{k+m+1}{k}\binom{k-1}{m+2}_2 + \binom{k-1}{m+1}_2 + \frac{k-m-1}{k}\binom{k-1}{m-1}_2 \\ &\ge \frac{k+m+1}{k}\binom{k-1}{m+2}_2.\end{aligned}$$

Thus (37) holds.

(vi): We shall use the induction argument below. The statement (38) is trivial for $k=1$. Assume the statement is valid for $k \ge 1$. Then, for $|m| \le k-1$, by the identity (31), one can check that

$$\frac{\binom{k+1}{m-1}_2}{\binom{k+1}{m}_2} = \frac{\binom{k}{m-2}_2 + \binom{k}{m-1}_2 + \binom{k}{m}_2}{\binom{k}{m-1}_2 + \binom{k}{m}_2 + \binom{k}{m+1}_2} \le \frac{\binom{k}{m-1}_2 + \binom{k}{m}_2 + \binom{k}{m+1}_2}{\binom{k}{m}_2 + \binom{k}{m+1}_2 + \binom{k}{m+2}_2} = \frac{\binom{k+1}{m}_2}{\binom{k+1}{m+1}_2}.$$

Also, note $\binom{k+1}{k+1}_2 = 1$, $\binom{k+1}{k}_2 = k+1$, $\binom{k+1}{k-1}_2 = \frac{k^2+3k+2}{2}$, then $\frac{\binom{k+1}{k-1}_2}{\binom{k+1}{k}_2} \le \frac{\binom{k+1}{k}_2}{\binom{k+1}{k+1}_2}$. Thus we have proved the statement is valid for $k+1$.

(vii): We first use the identity (31) to decompose $\binom{k+1}{m-2}_2$ and $\binom{k}{m}_2$, so the inequality (39) is equivalent to the following inequality

$$\frac{\binom{k}{m-3}_2 + \binom{k}{m-2}_2 + \binom{k}{m-1}_2}{\binom{k}{m}_2} \le \frac{\binom{k-1}{m-1}_2 + \binom{k-1}{m}_2 + \binom{k}{m+1}_2}{\binom{k-1}{m+2}_2},$$

which can in turn be written as

$$\sum_{i=1}^{3} \Big\{ \binom{k}{m-i}\binom{k-1}{m+2} - \binom{k}{m}\binom{k-1}{m+2-i} \Big\} \le 0.$$

The term inside the bracket above can be further written as

$$\sum_{j=-1}^{1} \Big\{ \binom{k-1}{m-i+j}\binom{k-1}{m+2} - \binom{k-1}{m+j}\binom{k-1}{m+2-i} \Big\}.$$

Each term inside the bracket above is negative by the inequality (38), thus the inequality (39) holds. □

Remark A.1 (Concavity). Actually, for each $i, j > 0$, the inequality

$$\binom{k-i}{m+j}_2 \binom{k+i}{m-j}_2 \le \binom{k}{m}_2^2$$

holds by the same method as we used in 38. But this inequality is not true for $j = 0, i > 0$.

Appendix B Ratio limit theorem for one-dimensional probability

Let μ denote a probability measure on the Borel subsets of $\mathbb{R}$ with characteristic function f defined by

$$f(\theta) = \int_{\mathbb{R}} e^{ix\theta}\mu(dx), \theta \in \mathbb{R}.$$

We assume that μ is normalized in the following sense. There exists a real number α such that $f(2\pi n) = e^{2\pi n i\alpha}$ for integers n and $|f(\theta)| < 1$ for other values of θ.

Let $\mu^{(n)}$ denote the n-fold convolution of μ with itself. It is clear that $\mu^{(n)}$ is supported by $D_n = \{x \in \mathbb{R} | \ x - n\alpha \text{ is an integer}\}$. Suppose μ satisfies Cramér's condition: for some constant $c > 0$

$$\int_{\mathbb{R}} e^{c|x|}\mu(dx) < \infty.$$

Let g denote the moment generating function of μ, defined for all $s \in \mathbb{R}$ by

$$g(s) = \int_{\mathbb{R}} e^{sx}\mu(dx).$$

Under Cramér's condition, g is continuously differentiable any number of times for $|s| < c$, and in particular

$$g'(0) = \int_{\mathbb{R}} x\mu(dx) = m.$$

Let us say that μ is not one-sided if $\mu\{x \mid \theta x > 0\} > 0$ for all non-zero $\theta \in \mathbb{R}$. It is exactly in this case that there is an (necessarily unique) $s_0 \in \mathbb{R}$ such that $\inf_{s\in\mathbb{R}} g(s) = g(s_0)$. Then we have the following theorem:

Theorem B.1. *[Stone [9],Theorem 5] Suppose μ is normalized and not one-sided, and let s_0 be defined as above. Then for every integer n_0 and $\epsilon > 0$, there is a $\delta > 0$ such that if $n \ge \delta^{-1}$,*

$x \in D_n, y \in D_{n+n_0}, |x-y| \le \epsilon^{-1}$ *and* $|x| \le \delta n$*, then*

$$\Big|\frac{\mu^{(n+n_0)}(y)}{\mu^{(n)}(x)} - (g(s_0))^{n_0} e^{s_0(x-y)}\Big| \le \epsilon.$$

Remark B.1. The original theorem considers a small area $\bar{I}_h$ around x and y. But we only consider the one-dimensional lattice measure which means that $\bar{I}_h = \{0\}$.

Let ν be normalized and not one-sided with the moment generating function f, then there is an (necessarily unique) $s_0' \in \mathbb{R}$ such that $\inf_{s\in\mathbb{R}} f(s) = f(s_0')$, and then we can immediately have

Corollary B.2. *For every integer* n_0, n_0', x_0, y_0 *and* $\varepsilon > 0$*, there is a* $\delta > 0$ *such that if* $n > \delta^{-1}$,$x \in D_n, x + x_0 \in D_{n+n_0}, y \in D_n', y + y_0 \in D'_{n+n_0'}$ *and* $x, y \le \delta n$*, then*

$$\Big|\frac{\mu^{(n+n_0)}(x+x_0)}{\mu^{(n)}(x)}\frac{\nu^{(n+n_0')}(y+y_0)}{\nu^{(n)}(y)} - (g(s_0))^{n_0} e^{s_0 x_0} (f(s_0'))^{n_0'} e^{s_0' y_0}\Big| \le \epsilon.$$

Now we are ready to give the proof of Lemma 3.3.

Proof of Lemma 3.3. Recall that $\{\xi_n\}$ are i.i.d random variables with probability P such that $P(\xi_1 = -1) = P(\xi_1 = 0) = P(\xi_1 = 1) = \frac{1}{3}$. Let $\{\zeta_n\}$ be the i.i.d random variables with probability Q defined as $Q(\zeta_1 = -1-\rho) = Q(\zeta_1 = -\rho) = Q(\zeta_1 = 1-\rho) = \frac{1}{3}$, then

$$P(\sum_{i=1}^{k} \xi_i = m) = Q(\sum_{i=1}^{k} \zeta_i = m - \rho k).$$

Notice that Q is normalized with $\alpha = -\rho$ and not one-sided, and p, z are constants. For any $\delta > 0$, we can find k large enough such that $m - \rho k < \delta k$ since $\frac{m}{k} \to \rho$ as k goes to infinity. Then there exists a s_0 such that

$$\lim_{k\to\infty} \frac{P(\sum_{i=1}^{k-p} \xi_i = m - z)}{3^p P(\sum_{i=1}^{k} \xi_i = m)} = \lim_{k\to\infty} \frac{Q(\sum_{i=1}^{k-p} \zeta_i = m - z - \rho(k-p))}{3^p P(\sum_{i=1}^{k} \zeta_i = m - \rho k)} = \frac{1}{3^p}(g(s_0))^{-p} e^{s_0(z+p\rho)}$$

by Corollary B.2. It remains to determine s_0. Notice that $g(s) = Ee^{s\zeta_1}$, $g''(s) = E\zeta_1^2 e^{s\zeta_1} > 0$, and $\inf_{s\in\mathbb{R}} g(s) = g(s_0')$, we have $g'(s_0) = 0$. So

$$E\zeta_1 e^{s_0\zeta_1} = \frac{1}{3}((1-\rho)e^{s_0(1-\rho)} - \rho e^{-s_0\rho} - (1+\rho)e^{-s_0(1+\rho)}) = 0.$$

The formula above can be simplified as

$$(1-\rho)e^{2s_0} - \rho e^{s_0} - (1+\rho) = 0.$$

Thus $e^{s_0} = \frac{\rho + \sqrt{4-3\rho^2}}{2(1-\rho)}$ since $|\rho| < 1$ and $g(s_0) = Ee^{s_0\zeta_1} = \frac{1}{3}e^{-\rho s_0}(e^{s_0} + 1 + e^{-s_0})$. Hence,

$$\lim_{k\to\infty} \frac{P(\sum_{i=1}^{k-p} \xi_i = m - z)}{3^p P(\sum_{i=1}^{k} \xi_i = m)} = \frac{1}{3^p}(g(s_0))^{-p} e^{s_0(z+p)} = \frac{e^{s_0 z}}{(e^{s_0} + 1 + e^{-s_0})^p}.$$

The proof of Lemma 3.3 ends. □

Declarations

Conflict of interest The authors declare no conflict of interest.

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