*Appl. Math. J. Chinese Univ.*
2021, 36(2): 278-286

# On a subclass of bi-univalent functions defined by convex combination of order $\alpha$ with the Faber polynomial expansion

Wurenqiqige[1,2]  LI Shu-hai[1]  Tsedenbayar Dashdondog[3]

**Abstract**. In this paper ,we define new subclasses of bi-univalent functions involving a differential operator in the open unit disc

$$\triangle = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}.$$

Moreover ,we use the Faber polynomial expansion to obtain the bounds of the coefficients for functions belong to the subclasses.

## §1 Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \cdots\} = \mathbb{N}_0 \backslash \{0\}$$

be the set of positive integers.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk

$$\triangle = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}.$$

By $\mathcal{B}$ we denot the subclass of $\mathcal{A}$ consisting of functions the form (1.1) which are also univalent in $\triangle$. Further,$\mathcal{T}$ be the class of functions consisting of $\varphi$, such that

$$\varphi(z) = 1 + \sum_{n=1}^{\infty} \varphi_n z^n$$

which are regular in the open unit disc $\triangle$ and satisfy the condition $\Re(\varphi(z)) > 0$ in $\triangle$.

In 1970, Robertson[1]defined the concept of quasi-subordination , which was shown as fol-

Received: 2020-3-10.  Revised: 2020-10-31.
MR Subject Classification: 30C45.
Keywords: bi-univalent functions, convex combination, Faber polynomial expansion, coefficient bounds, subordination, combination of order $\alpha$.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-021-4060-7.

lows:

**Definition 1.1** Let $f$ and $g$ be two analytic functions in $\triangle$,the function $f$ is said to be quasi-subordinate to the function $g$ in $\triangle$, written as:

$$f(z) \prec_q g(z) \quad (z \in \triangle),$$

If there exists an analytic function $\varphi(|\varphi(z)| \leq 1)$such that the function$\frac{f(z)}{\varphi(z)}$ analytic in $\triangle$ and

$$\frac{f(z)}{\varphi(z)} \prec g(z) \quad (z \in \triangle),$$

that is, there exists the above-mentioned Schwarz function $\omega(z)$ such that

$$f(z) = \varphi(z)g(\omega(z)).$$

One observes that, in the special case when $\varphi(z) = 1$, the quasi-subordination coincides with the usual subordination .

**Definition 1.2** Let $f$ and $g$ be two analytic functions in $\triangle$, we say that $f$ is subordinate to the function $g$ written as:

$$f(z) \prec g(z) \quad (z \in \triangle),$$

if there exists a Schwarz function $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ analytic in $\triangle$, with

$$\omega(0) = 0 \;\; and \;\; |\omega(z)| < 1 \;\; (z \in \triangle)$$

such that

$$f(z) = g(\omega(z)) \;\; (z \in \triangle).$$

It is known that $f(z) \prec g(z) \quad (z \in \triangle) \Longrightarrow f(0) = g(0)$ and $f(\triangle) \subset g(\triangle)$.

Furthermore, if the function $g$ is univalent in $\triangle$, then we have the following equivalence

$$f(z) \prec g(z) \;\; (z \in \triangle) \Leftrightarrow f(0) = g(0) \quad and \quad f(\triangle) \subset g(\triangle).$$

The Koebe One-Quarter theorem [2] stated that the image of $\triangle$ under every function $f$ in the normalized univalent function class $\mathcal{B}$ contains a disc of radius $\frac{1}{4}$. Every such univalent function has an inverse $f^{-1}$ which satisfies the following condition:

$$f^{-1}(f(z)) = z \;\; (z \in \triangle)$$

and

$$f(f^{-1}(\omega)) = \omega \;\; (|\omega| < r_0(f); r_0(f) \geq \frac{1}{4}).$$

In fact,the inverse function $g = f^{-1}$ is given by

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\triangle$ if both $f$and $f^{-1}$are univalent in $\triangle$. let $\Sigma$ denote the class of bi-univalent functions in $\triangle$given by (1.1). For a history and examples of functions which are (or which are not)in the class . Recently,Srivastava ,Altinkaya and Yalcin [6]made an effort to introduce various subclasses of the bi-univalent function class $\Sigma$ and found non-sharp coefficient estimates on the initial coefficients$|a_2|$and $|a_3|$. But the coefficient problem for each one of the following Taylor-Maclaurin coefficients

$$|a_n|, n \in \mathbb{N}\backslash\{1, 2\} : \mathbb{N} = \{1, 2, 3, \cdots\}$$

is still an open problem.

In 1903, Faber [3] introduced The Faber polynomials and it played an important role in various areas of mathematical sciences, especially in Geometric Function Theory, using the Faber polynomial expansion of functions $f \in \mathcal{A}$ of the form (1.1), the coefficients of its inverse map $g = f^{-1}$ may be expressed as follows(see[4]):

$$g(\omega) = f^{-1}(\omega) = \omega + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots)\omega^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{[2(-n+1)]!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 \\ & + \frac{(-n)!}{[2(-n+2)]!(n-5)!} a_2^{n-5} [a_5 + (-n+2)a_3^2] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5)a_3 a_4] \\ & + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that $V_j (7 \leq j \leq n)$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$(see, for details[4]).

In 1908 , Jacson[5]defined the concept of q-derivative, which was shown as follows:

**Definition 1.3** The q-derivative of a function f is defined on a subset of $\mathbb{C}$ is given by

$$(D_q f)(z) = \frac{f(z) - f(zq)}{(1-q)z} (z \neq 0), \tag{1.3}$$

and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.

Note that

$$\lim_{q \to 1^-} (D_q f)(z) = \lim_{q \to 1^-} \frac{f(z) - f(qz)}{(1-q)z} = \frac{df(z)}{dz}$$

If f is differentiable, from (1.3)we deduce that

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{1.4}$$

where the symbol $[n]_q$denotes the so-called the twin-basic number is a natural generalization of the q-number , that is

$$[n]_q = \frac{1-q^n}{1-q} (q \neq 1).$$

In 2017, Serap Bulut [6] introduced the following function class $\mathfrak{B}_\Sigma(\lambda, t)$, and obtained initial coefficient bounds for functions belong to the subclass. The definition of the subclass is as follows: For $\lambda \geq 1$ and $t \in (\frac{1}{2}, 1)$, for all $z, \omega \in \triangle$, a function $f \in \Sigma$ given by (1.1) is satisfies the following conditions

$$(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2}$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega} + \lambda g'(\omega) \prec H(\omega,t) := \frac{1}{1-2t\omega+\omega^2}$$

where $g = f^{-1}$.

In 2018, Sahsene Altinkaya [7] introduced the following function class $T_\Sigma(q,\gamma)$, and obtained upper bounds for coefficients of functions belong to the subclass. The definition of the subclass is follows: Let the function $\psi \in \mathcal{T}$ be univalent in $\triangle$ and let $\psi(\triangle)$ be symmetrical about the real axis with $\psi'(0) > 0$. We say that a function $f \in \triangle$ is in the class $T_\Sigma(q,\gamma)(\gamma \geq 1)$, if the following quasi-subordinations hold true

$$(1-\lambda)\frac{f(z)}{z} + \lambda(D_q f)(z) \prec_q \psi(z) \quad (z \in \triangle)$$

and

$$(1-\lambda)\frac{g(\omega)}{\omega} + \lambda(D_q g)(\omega) \prec_q \psi(\omega) \quad (\omega \in \triangle)$$

where $g = f^{-1}$.

This study aims to introduce a new subclass of bi-univalent functions defined by using the Jackson q-derivative operator and use the Faber polynomial expansion techniques to derive bounds for the general Talor-Maclaurin coefficients $a_n$ for the functions in the subclass, also obtain the estimates for the initial coefficients $|a_n|,(n > 3)$ of the functions. This is an improvement of the conclusion in reference [7].

## §2 The general Taylor-Maclaurin Coefficients $|a_n|$

**Definition 2.1** Let the function $\Psi \in \mathcal{T}$ be univalent in $\triangle$ and let $\Psi(\triangle)$be symmetrical about the real axis with

$$\Psi'(0) > 0.$$

We say that a function $f \in \Sigma$ is in the class

$$T_\Sigma^\alpha(q;\lambda)(\lambda \geq 1, \alpha \in \mathbb{R})$$

If the following quasi-subordinations hold true:

$$(1-\lambda)[\frac{f(z)}{z}]^\alpha + \lambda[(D_q f)(z)]^\alpha \prec_q \Psi(z) \quad (z \in \triangle) \tag{2.1}$$

and

$$(1-\lambda)[\frac{g(\omega)}{\omega}]^\alpha + \lambda[(D_q g)(\omega)]^\alpha \prec_q \Psi(\omega) \quad (\omega \in \triangle), \tag{2.2}$$

where $g = f^{-1}$.

It is clear from Definition 2.1 that $f \in T_\Sigma^\alpha(q;\lambda)$ if and only if there exists a function $h(|h(z)| \leq 1)$ such that

$$\frac{(1-\lambda)[\frac{f(z)}{z}]^\alpha + \lambda[(D_q f)(z)]^\alpha}{h(z)} \prec \Psi(z) \quad (z \in \triangle)$$

and

$$\frac{(1-\lambda)[\frac{g(\omega)}{\omega}]^\alpha + \lambda[(D_q g)(\omega)]^\alpha}{h(\omega)} \prec \Psi(\omega) \quad (\omega \in \triangle),$$

Throughout this paper ,we suppose that the function $\Psi \in \mathcal{T}$ is of the form:

$$\Psi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 > 0, z \in \triangle).$$

The function $h$, analytic in $\triangle$ and have the form:

$$h(z) = H_0 + H_1 z + H_2 z^2 + \cdots (|h(z)| \leq 1, z \in \triangle).$$

Our main result is given by Theorem 2.2 below.

**Theorem 2.2** Let $f$ given by (1.1), be in the class $T_{\Sigma}^{\alpha}(q;\lambda)$. If $a_m = 0$ for $2 \leq m \leq n-1$, then

$$|a_n| \leq \frac{B_1 + |H_{n-1}|}{\alpha[1 + ([n]_q - 1)\lambda]} \quad (n > 3).$$

**Proof** For analytic functions $f$ of the form (1.1), we have

$$(\frac{f(z)}{z})^{\alpha} = [1 + \sum_{n=2}^{\infty} a_n z^{n-1}]^{\alpha}.$$

Denote

$$F(z) = (\frac{f(z)}{z})^{\alpha}, R(\omega) = (\frac{g(\omega)}{\omega})^{\alpha}$$

and

$$p(z) = [(D_q f)(z)]^{\alpha}, Q(\omega) = [(D_q g)(\omega)]^{\alpha}$$

Then

$$(1-\lambda)F(z) + \lambda p(z) = (1-\lambda)(\frac{f(z)}{z})^{\alpha} + \lambda[(D_q f)(z)]^{\alpha} \prec_q \Psi(z) \quad (z \in \triangle), \tag{2.3}$$

$$(1-\lambda)R(\omega) + \lambda Q(\omega) = (1-\lambda)(\frac{g(\omega)}{\omega})^{\alpha} + \lambda[(D_q g)(\omega)]^{\alpha} \prec_q \Psi(\omega) \quad (\omega \in \triangle), \tag{2.4}$$

Let $h_1(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}$,

$$h_1'(z) = h_2(z),$$
$$h_2'(z) = h_3(z),$$
$$h_3'(z) = h_4(z),$$
$$\vdots$$
$$h_n'(z) = h_{n+1}(z)$$
$$\vdots$$

Then by Talor expansion formula , we have

$$F(z) = (\frac{f(z)}{z})^{\alpha} = F(0) + F'(0)z + \frac{1}{2!}F''(0)z^2 + \frac{1}{3!}F^{(3)}(0)z^3 + \cdots \frac{1}{n!}F^{(n)}(0)z^n + \cdots$$

We can calculate

$$h_1(0) = 1,$$
$$h_1'(0) = h_2(0) = a_2,$$
$$h_2'(0) = h_3(0) = 2!a_3,$$
$$h_3'(0) = h_4(0) = 3!a_4,$$
$$\vdots$$
$$h_{n-1}'(0) = h_n(0) = (n-1)!a_n,$$
$$\vdots$$

Therefore

$$F(z) = 1 + \alpha a_2 z + \frac{1}{2!}[\alpha(\alpha-1)a_2^2 + 2\alpha a_3]z^2 +$$
$$\frac{1}{3!}[\alpha(\alpha-1)(\alpha-2)a_2^3 + 3!\alpha(\alpha-1)a_2 a_3 + 3!\alpha a_4]z^3 + \cdots$$

$$+\frac{1}{(n-1)!}[A(\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1})+\alpha(n-1)!a_n]z^{n-1}+\cdots$$

Where $A(\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1})$is the sum of the functions formed by the product of $\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1}$and at least one of the product factors is $a_i, 2\le i\le n-1$ so

$$(1-\lambda)F(z)=(1-\lambda)(\frac{f(z)}{z})^{\alpha}$$
$$=(1-\lambda)+(1-\lambda)\alpha a_2 z+\frac{(1-\lambda)}{2!}[\alpha(\alpha-1)a_2^2+2\alpha a_3]z^2$$
$$+\frac{(1-\lambda)}{3!}[\alpha(\alpha-1)(\alpha-2)a_2^3+3!\alpha(\alpha-1)a_2a_3+3!\alpha a_4]z^3+\cdots$$
$$+\frac{(1-\lambda)}{(n-1)!}[A(\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1})+\alpha(n-1)!a_n]z^{n-1}+\cdots$$

Since

$$(D_qf)(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1},$$

Where

$$[n]_q=\frac{1-q^n}{1-q}(q\neq 1)$$

Denote

$$P(z)=[(D_qf)(z)]^{\alpha}=[1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}]^{\alpha}$$
$$=[1+[2]_q a_2 z+[3]_q a_3 z^2+[4]_q a_4 z^3+\cdots+[n]_q a_n z^{n-1}+\cdots]^{\alpha}$$

Let

$$p_1(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}$$
$$=1+[2]_q a_2 z+[3]_q a_3 z^2+[4]_q a_4 z^3+\cdots+[n]_q a_n z^{n-1}+\cdots$$

Then

$$p_1(0)=1$$
$$p_1'(0)=[2]_q a_2$$
$$p_1''(0)=2![3]_q a_3$$
$$p_1'''(0)=3![4]_q a_4$$
$$\vdots$$

By Talor expansion formula,we have

$$P(z)=P(0)+P'(0)z+\frac{1}{2!}P''(0)z^2+\frac{1}{3!}P'''(0)z^3+\cdots\frac{1}{n!}P^n(0)z^n+\cdots$$
$$p(0)=1,\quad and\quad P'(z)=\alpha[p_1(z)]^{\alpha-1}P_1'(z)$$
$$P''(z)=\alpha(\alpha-1)[p_1(z)]^{\alpha-2}[P_1'(z)]^2+\alpha[p_1(z)]^{\alpha-1}P_1''(z)$$
$$P'''(z)=\alpha(\alpha-1)(\alpha-2)[p_1(z)]^{\alpha-3}[P_1'(z)]^3+\alpha(\alpha-1)[p_1(z)]^{\alpha-2}2[P_1'(z)]P_1''(z)+$$
$$\alpha(\alpha-1)[p_1(z)]^{\alpha-2}P_1'(z)P_1''(z)+\alpha[p_1(z)]^{\alpha-1}P_1'''(z)$$
$$\vdots$$

So

$$P^{(n)}(0)=H(\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1})+\alpha(n-1)![n]_q a_n$$

Where $H(\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1})$ is the sum of the functions formed by the product of $\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1}$ and at least one of the product factors is $a_i, 2\leq i\leq n-1$. So

$$P(z)=1+(\alpha[2]_q a_2)z+\frac{1}{2!}[\alpha(\alpha-1)([2]_q a_2)^2+2[3]_q\alpha a_3]z^2+$$
$$\frac{1}{3!}[\alpha(\alpha-1)(\alpha-2)([2]_q a_2)^3+6\alpha(\alpha-1)([2]_q a_2)[3]_q a_3+\alpha 3![4]_q a_4]z^3+\cdots$$
$$+\frac{1}{n!}[H(\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1})+\alpha(n-1)![n]_q a_n]z^{n-1}+\cdots$$

Then

$$\lambda P(z)=\lambda+\lambda(\alpha[2]_q a_2)z+\frac{\lambda}{2!}[\alpha(\alpha-1)([2]_q a_2)^2+2[3]_q\alpha a_3]z^2$$
$$+\frac{\lambda}{3!}[\alpha(\alpha-1)(\alpha-2)([2]_q a_2)^3+6\alpha(\alpha-1)([2]_q a_2)[3]_q a_3+\alpha 3![4]_q a_4]z^3$$
$$+\cdots+\frac{\lambda}{(n-1)!}[H(\alpha,(\alpha-1),(\alpha-2),\cdots(\alpha-n+1),a_2,a_3,\cdots a_{n-1})$$
$$+\alpha(n-1)![n]_q a_n]z^{n-1}+\cdots$$

If $a_m=0$ for $2\leq m\leq n-1$, then The coefficient of $z^{n-1}$in (2.3) is

$$[1+([n]_q-1)\lambda]\alpha a_n$$

Similarly , the coefficient of $\omega^{n-1}$in (2.4) is

$$[1+([n]_q-1)\lambda]\alpha b_n$$

Where $b_n=\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,\cdots a_n)$.

On the other hand , the inequalities (2.1) and (2.2) imply the existence of two Schwarz functions $u(z)=\sum_{n=1}^{\infty}c_n z^n$and $v(\omega)=\sum_{n=1}^{\infty}d_n\omega^n$so that

$$(1-\lambda)[\frac{f(z)}{z}]^{\alpha}+\lambda[(D_q f)(z)]^{\alpha}=h(z)\Psi(u(z)) \tag{2.5}$$

and

$$(1-\lambda)[(\frac{g(\omega)}{\omega})]^{\alpha}+\lambda[(D_q g)(\omega)]^{\alpha}=h(\omega)\Psi(\nu(\omega)) \tag{2.6}$$

Thus, from (2.3) and (2.5)yields

$$[1+([n]_q-1)\lambda]\alpha a_n=H_{n-1}+\sum_{t=1}^{n-1}\sum_{k=1}^{t}B_k E_n^k(c_1,c_2,\cdots,c_n)H_{n-(t+1)}(H_0=1) \tag{2.7}$$

Similarly , by using (2.4) and (2.6), we find that

$$[1+([n]_q-1)\lambda]\alpha b_n=H_{n-1}+\sum_{t=1}^{n-1}\sum_{k=1}^{t}B_k E_n^k(d_1,d_2,\cdots,d_n)H_{n-(t+1)} \tag{2.8}$$

We note that, for $a_m=0(2\leq m\leq n-1)$, we have $b_n=-a_n$ and so

$$\alpha[1+([n]_q-1)\lambda]a_n=\alpha a_n+\alpha\lambda([n]_q-1)a_n=B_1 C_{n-1}+H_{n-1}$$
$$\alpha[1+([n]_q-1)\lambda]b_n=\alpha b_n+\alpha\lambda([n]_q-1)b_n=B_1 d_{n-1}+H_{n-1}$$

Now taking the absolute Values of the above two equations and using the facts that $|C_{n-1}|\leq 1$

and $|d_{n-1}| \le 1$, we obtain

$$|a_n| = \frac{|B_1 C_{n-1} + H_{n-1}|}{|\alpha[1 + ([n]_q - 1)\lambda]|} = \frac{|B_1 d_{n-1} + H_{n-1}|}{|\alpha[1 + ([n]_q - 1)\lambda]|} \le \frac{B_1 + |H_{n-1}|}{\alpha[1 + ([n]_q - 1)\lambda]} \quad . \tag{2.9}$$

Which evidently completes the proof of Theorem 2.2.

**Corollary 2.3**[7] If we take $\alpha = 1$, then obtain

$$|a_n| \le \frac{B_1 + |H_{n-1}|}{1 + ([n]_q - 1)\lambda} \quad (n > 3).$$

this is the conclusion of Reference[7].

**Corollary 2.4** If we take $h(z) = 1$ and $\Psi(z) = (\frac{1+z}{1-z})^\xi (0 < \xi \le 1)$ which gives $B_1 = 2\xi$, in Theorem 2.2, then we obtain

$$|a_n| \le \frac{2\xi}{\alpha[1 + ([n]_q - 1)\lambda]} \quad (n > 3).$$

**Corollary 2.5** If we take $h(z) = 1$ and $\Psi(z) = \frac{1+(1-2\xi)z}{1-z} (0 < \xi \le 1)$ which gives $B_1 = 2(1-\xi)$,in Theorem 2.2, then we obtain

$$|a_n| \le \frac{2(1-\xi)}{\alpha[1 + ([n]_q - 1)\lambda]} \quad (n > 3).$$

## §3 Conclusions

In this study, we investigate the problems of coefficients of the classes $T_\Sigma^\alpha(q;\lambda)$ defined by subordination relation , by the Faber polynomials expansion to obtain upper bounds for $|a_n|(n > 3)$coefficients of functions belong to a subclass of bi-univalent functions defined by convex combination of order $\alpha$.The results obtained generalize and unify the theory of coefficients in function theory. In addition, if we put $\alpha = 1$ in Theorem 2.2, then we easily get the results were mentioned in the reference [7].

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[1]Normal College, Chifeng University, Chifeng 024000, China.
Email: wurenqiqige@126.com, lishms66@163.com

[2]Department of Mathematics, Mongolian National University of Education, Ulaanbaatar 210646, Mongolia.

[3]Department of Applied Mathematics, Mongolian University of Science and Technology, Ulaanbaatar 210646, Mongolia.
Email: cdnbr@yahoo.com