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A novel similarity measure between intuitionistic fuzzy sets based on the mid points of transformed triangular fuzzy numbers with applications to pattern recognition and medical diagnosis

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Abstract. Similarity measure is an essential tool to compare and determine the degree of similarity between intuitionistic fuzzy sets (IFSs). In this paper, a new similarity measure between intuitionistic fuzzy sets based on the mid points of transformed triangular fuzzy numbers is proposed. The proposed similarity measure provides reasonable results not only for the sets available in the literature but also gives very reasonable results, especially for fuzzy sets as well as for most intuitionistic fuzzy sets. To provide supportive evidence, the proposed similarity measure is tested on certain sets available in literature and is also applied to pattern recognition and medical diagnosis problems. It is observed that the proposed similarity measure provides a very intuitive quantification.

§1 Introduction

Fuzzy sets (FSs) were introduced by Zadeh[24], as a generalization of crisp sets. Latter as a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets (IFSs) was introduced by Atanassov[1]. Intuitionistic fuzzy sets are characterized by two functions namely the degree of membership and the degree of non-membership. IFSs have extensive applications to pattern recognition [2, 4-10, 15, 20, 23], medical diagnosis [2, 4-10] and decision making [7, 14, 20]. In particular the similarity measure of IFSs plays an important role in such fields.

Many similarity measures [2-5,8,12,14-18,20,23,25] have been presented in literature to deal with similarity measure between fuzzy and IFSs. Chen and Randyanto[5] presented a similarity measure between IFSs based on the medians of intervals, the Hausdorff distance, and the

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ratio of the uncertainty degrees of intuitionistic fuzzy values. Boran and Akay [2] presented a biparametric similarity measure between IFSs. Chen and Chang [4] presented the similarity between IFSs based on transformation technique with applications to pattern recognition. Chen, Cheng and Lan[8] presented the similarity measure between IFSs based on the centroid points of the transformed fuzzy numbers and applied the measure to the pattern recognition problems. Hoang Nguyen[13] presented knowledge-based similarity/dissimilarity measure between IFSs and apply the measure to the pattern recognition problem. The existing similarity measures [2-5,8,12,14-18,20,23,25] between IFSs adopts different concept with its own strengths and weakness. For categorizing undemanding pairs of fuzzy numbers as either similar or dissimilar all the measures work well. But in most demanding situations, most of the exisitng measures fail to calculate the similarity measure correctly.

In this paper, we propose a new method to measure the degree of similarity between IFSs based on the mid points of the transformed triangular fuzzy numbers and apply the proposed similarity measure to deal with pattern recognition and medical diagnosis problems. The paper is organized as follows. In Section 2, we briefly review the basic concepts of IFSs and properties of similarity measures between IFSs. In Section 3, we briefly review the existing similarity measures in literature. In Section 4, we analyse the drawback of the S_{CCL} [8] similarity measure. In Section 5, we propose a new similarity measure between IFSs and also proved its basic properties. In Section 6, we study the performance of the proposed measure with the existing similarity measures and use some examples to illustrate that the proposed similarity measure between IFSs can overcome the drawbacks of the existing similarity measures. In Section 7, the applications of the similarity measures in pattern recognition and medical diagnosis are discussed. Finally, the conclusions are discussed in Section 8.

§2 Preliminaries

This section briefly reviews some concepts of intuitionistic fuzzy sets [1] and basic properties of similarity measures between IFSs.

Definition 2.1. *[24] Let A be a* ***fuzzy set*** *in the universe of discourse* $X = \{x_1, x_2, ..., x_n\}$ *defined as* $A = \{< x_i, \mu_A(x_i) > /x_i \epsilon X\}$, *where the membership function* $\mu_A : X \to [0,1]$, $\mu_A(x_i)\epsilon[0,1]$ *and* $1 \leq i \leq n$.

Definition 2.2. *[1]Let A be an* ***intuitionistic fuzzy set*** *in the universe of discourse X, where* $X = \{x_1, x_2, ..., x_n\}$, $A = \{< x_i, \mu_A(x_i), \nu_A(x_i) > /x_i \epsilon X\}$, $\mu_A : X \to [0,1]$,$\nu_A : X \to [0,1]$, $\mu_A(x_i)$*and* $\nu_A(x_i)$ *denote the degree of membership and the degree of non-membership of element* x_i *belonging to the intuitionistic fuzzy set A, respectively,* $\mu_A(x_i)\epsilon[0,1]$, $\nu_A(x_i)\epsilon[0,1]$, $0 \leq \mu_A(x_i) + \nu_A(x_i) \leq 1$ *and* $1 \leq i \leq n$. *The degree of indeterminacy of element* x_i *belonging to the intuitionistic fuzzy set A is denoted by* $\pi_A(x_i)$ *where* $\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i)$, $\pi_A(x_i)\epsilon[0,1]$ *and* $1 \leq i \leq n$.

Based on [11], the intuitionistic fuzzy value of element x_i belonging to the intuitionistic fuzzy set, $A = \{x_i, \mu_A(x_i), \nu_A(x_i) : x_i \in X\}$ can be represented by $(\mu_A(x_i), \nu_A(x_i))$,where $1 \leq i \leq n$.

Definition 2.3. *[1]Let A and B be two IFSs in the universe of discourse X, where* $X = \{x_1, x_2, ..., x_n\}$. *The IFS A is contained in the IFS B, denoted by* $A \subseteq B$*, if and only if* $\forall x_i \in X, \mu_A(x_i) \leq \mu_B(x_i)$ *and* $\nu_A(x_i) \geq \nu_B(x_i)$*, where* $1 \leq i \leq n$.

Definition 2.4. *[13] IFS A is said to be most intuitionistic fuzzy set(MIFS) when* $\mu_A(x_i)$=$\nu_A(x_i)$ *=0 for* $x_i \epsilon X$.

Definition 2.5. *[12]Let A, B, C and D be IFSs defined in the universe of discourse X and let* $S(A, B)$ *denote the similarity measure between A and B. The following are the basic properties to be satisfied by* $S(A, B)$*:*
(1) $S(A, B) \in [0, 1]$.
(2) $S(A, B) = S(B, A)$.
(3) S(A,B)=1 if and only if A=B.
(4) $If A \subseteq B \subseteq C$*, then* $S(A, C) \leq S(A, B)$ *and* $S(A, C) \leq S(B, C)$.
The prime idea of defining an IFS as an extension of FS is to give equal importance to both membership values and non-membership values. Accordingly the similarity measure defined for the IFSs should also give equal priority for both membership values and the non-membership values. Based on this a similarity measure should satisfy property (5).
(5) If $\mid \mu_A - \mu_B \mid \geq \mid \mu_C - \mu_D \mid$ *and* $\mid \nu_A - \nu_B \mid \geq \mid \nu_C - \nu_D \mid$ *then* $S(A, B) \leq S(C, D)$.
(6)If A^r, B^r *are two new IFS obtained by interchanging simultaneously both* μ *and* ν *of A and B respectively, then for a similarity measure to be consistent* $S(A, B) = S(A^r, B^r)$.

§3 Existing similarity measures between IFSs

In this section, some of the existing measures that are relevant to the study of the proposed similarity measure between IFSs are provided. Let $X = \{x_1, x_2, ..., x_n\}$ be the universe of discourse. Let A and B be two IFSs in the universe of discourse X, where $A = \{(x_i, \mu_A(x_i), \nu_A(x_i))/1 \leq i \leq n\}$ and $B = \{(x_i, \mu_B(x_i), \nu_B(x_i))/1 \leq i \leq n\}$. Let $\pi_A(x_i)$ and $\pi_B(x_i)$ be the degrees of indeterminacy or hesitation of element x_i belonging to the IFSs A and B respectively, where $\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i)$, $\pi_B(x_i) = 1 - \mu_B(x_i) - \nu_B(x_i)$ and $1 \leq i \leq n$.

Chen's similarity measure S_C [3]

$$S_C(A, B) = 1 - \frac{\sum_{i=1}^{n} \mid \mu_A(x_i) - \nu_A(x_i) - (\mu_B(x_i) - \nu_B(x_i)) \mid}{2n}.$$

Hong and Kim's similarity measure S_{HK} [14]

$$S_{HK}(A, B) = 1 - \frac{\sum_{i=1}^{n} \mid \mu_A(x_i) - \mu_B(x_i) \mid + \mid \nu_A(x_i) - \nu_B(x_i) \mid}{2n}.$$

Li and Xu's similarity measure S_{LX} [18]

$$S_{LX}(A, B) = 1 - \left(\frac{\sum_{i=1}^{n} \mid \mu_A(x_i) - \nu_A(x_i) - (\mu_B(x_i) - \nu_B(x_i)) \mid}{4n} + \frac{\sum_{i=1}^{n} \mid \mu_A(x_i) - \mu_B(x_i) \mid + \mid (\nu_A(x_i) - \nu_B(x_i)) \mid}{4n} \right).$$

Li similarity measure S_{LO} [17]

$S_{LO}(A,B) = 1 - \sqrt{\dfrac{\sum_{i=1}^{n}(\mu_A(x_i) - \mu_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2}{2n}}.$

Dengfeng and chuntain's similarity measure S_{DC} [12]

$S_{DC}(A,B) = 1 - \sqrt[p]{\dfrac{\sum_{i=1}^{n} \mid \psi_A(x_i) - \psi_B(x_i) \mid}{n}}$, where $\psi_A(x_i) = \dfrac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2}$.

Mitchell's similarity measure S_M[20]

$S_M(A,B) = \dfrac{\rho_\mu(A,B) + \rho_\nu(A,B)}{2}$ where $\rho_\mu(A,B) = 1 - \sqrt[p]{\dfrac{\sum_{i=1}^{n} \mid \mu_A(x_i) - \mu_B(x_i) \mid^p}{n}}$,

$\rho_\nu(A,B) = 1 - \sqrt[p]{\dfrac{\sum_{i=1}^{n} \mid \nu_A(x_i) - \nu_B(x_i) \mid^p}{n}}$ and $1 \leq p < \infty$.

Hung and Yang similarity measure$S_{HY1}, S_{HY2}, S_{HY3}$ [15]

$S_{HY1}(A,B) = 1 - \dfrac{\sum_{i=1}^{n} \max(\mid \mu_A(x_i) - \mu_B(x_i) \mid), \mid \nu_A(x_i) - \nu_B(x_i) \mid)}{n}$,

$S_{HY2}(A,B) = \dfrac{e^{S_{HY1}(A,B)-1} - e^{-1}}{1 - e^{-1}}$, $S_{HY3}(A,B) = \dfrac{S_{HY1}(A,B)}{2 - S_{HY1}(A,B)}$.

Ye's similarity measure S_Y [23]

$S_Y(A,B) = \dfrac{1}{n}\sum_{i=1}^{n} \dfrac{\mu_A(x_i)\mu_B(x_i) + \nu_A(x_i)\nu_B(x_i)}{\sqrt{\mu_A(x_i)^2 + \nu_A(x_i)^2}\sqrt{\mu_B(x_i)^2 + \nu_B(x_i)^2}}$.

Liang and Shi's similarity measures $S_{LS1}, S_{LS2}, S_{LS3}$ [16]

$S_{LS1}(A,B) = 1 - \sqrt[p]{\dfrac{\sum_{i=1}^{n} \mid \phi_\mu(x_i) + \phi_\nu(x_i) \mid^p}{n}}$, $S_{LS2}(A,B) = 1 - \sqrt[p]{\dfrac{\sum_{i=1}^{n} \mid \phi_{S1}(x_i) + \phi_{S2}(x_i) \mid^p}{n}}$ and

$S_{LS3}(A,B) = 1 - \sqrt[p]{\sum_{i=1}^{n} W_i(\sum_{m=1}^{3} w_m \phi_m(x_i))^p}$,where $\phi_\mu(x_i) = \dfrac{\mid \mu_A(x_i) - \mu_B(x_i) \mid}{2}$,

$\phi_\nu(x_i) = \dfrac{\mid 1 - \nu_A(x_i) - (1 - \nu_B(x_i)) \mid}{2}$, $\phi_{S1}(x_i) = \dfrac{\mid m_{A1}(x_i) - m_{B1}(x_i) \mid}{2}$,

$m_{A1}(x_i) = \dfrac{\mid \mu_A(x_i) + m_A(x_i) \mid}{2}$, $m_{A2}(x_i) = \dfrac{\mid 1 - \nu_A(x_i) + m_A(x_i) \mid}{2}$,

$m_A(x_i) = \dfrac{\mid \mu_A(x_i) + 1 - \nu_A(x_i) \mid}{2}$, $\phi_{S2}(x_i), m_{B1}(x_i), m_{B2}(x_i)$ and $m_B(x_i)$ are defined similarly.

$\phi_1(x_i) = \dfrac{\mid \mu_A(x_i) - \mu_B(x_i) \mid + \mid (1 - \nu_A(x_i)) - (1 - \nu_B(x_i)) \mid}{2}$,

$\phi_2(x_i) = \dfrac{\mid \mu_A(x_i) - (1 - \nu_A(x_i)) - (\mu_B(x_i) + (1 - \nu_B(x_i)) \mid}{2}$,

$\phi_3(x_i) = \max\left(\dfrac{\pi_A(x_i)}{2}, \dfrac{\pi_B(x_i)}{2}\right) - \min\left(\dfrac{\pi_A(x_i)}{2}, \dfrac{\pi_B(x_i)}{2}\right)$.

Zhang and Yu's similarity measure S_{ZY} [25]

$S_{ZY}(A,B) = 1 - \sum_{i=1}^{n} W_i(U_i - I_i)$ where $W_i \in [0,1]$ denotes the weight of element $x_i \in X$, $\sum_{i=1}^{n} W_i = 1$,

$I_i = \int_0^1 \min(\mu_A(t), \mu_B(t))dt$, $U_i = \int_0^{m_A} \max(\mu_A(t), \mu_B(t))dt + \mid m_B - m_A \mid + \int_{m_B}^{1} \max(\mu_A(t), \mu_B(t))dt$,

$A = (\mu_A(x_i), m_A, 1 - \nu_A(x_i))$, $B = (\mu_B(x_i), m_B, 1 - \nu_B(x_i))$,

$m_A = \dfrac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2}$ and

$$\mu_A(t) = \begin{cases} \dfrac{t-\mu_A(x_i)}{m_A-\mu_A(x_i)} & , if \mu_A(x_i) \leq t \leq m_A, \\ \dfrac{1-\nu_A(x_i)-t}{1-\nu_A(x_i)-m_A}, & if m_A(x_i) \leq t \leq 1-\nu_A(x_i), \\ 0 & , otherwise. \end{cases}$$

Chen and Randyanto's similarity measure S_{CR} [5]

$S_{CR}(A,B) = \left(1 - \dfrac{\sum_{i=1}^{n}(M([\mu_A(x_i), 1-\nu_A(x_i)], [\mu_B(x_i), 1-\nu_B(x_i)]))}{n}\right) \times \overline{M},$, where

$\overline{M} = \dfrac{\sum_{i=1}^{n}(M([\mu_A(x_i), 1-\nu_A(x_i)], [\mu_B(x_i), 1-\nu_B(x_i)]))}{n}.$

$M([\mu_A(x_i), 1-\nu_A(x_i)], [\mu_B(x_i), 1-\nu_B(x_i)]) =$
$\left(\dfrac{\min(\pi_A(x_i), \pi_B(x_i))+1}{\max(\pi_A(x_i), \pi_B(x_i))+1}\right) \times \left(1 - \dfrac{\mid \psi_A(x_i) - \psi_B(x_i) \mid}{2}\right) + \left(1 - \dfrac{\min(\pi_A(x_i), \pi_B(x_i))+1}{\max(\pi_A(x_i), \pi_B(x_i))+1}\right)$
$(1 - \max(\mid \mu_A(x_i) - \mu_B(x_i) \mid, \mid \nu_A(x_i) - \nu_A(x_i) \mid), \psi_A(x_i) = \dfrac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2}.$

Boran Akay's similarity measure S_{BA} [2]

$S_{BA}(A,B) = 1 - \{\sum_{i=1}^{n}\{\frac{1}{2n(s+1)^p}(\mid s(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) \mid^p +$
$\mid s(\nu_A(x_i) - \nu_B(x_i)) - (\mu_A(x_i) - \mu_B(x_i)) \mid^p)\}^{\frac{1}{p}}.$

Chen and Chang similarity measure S_{CC} [4]

$$S_{CC}(A,B) = \sum_{i=1}^{n}[W_i \times S(A_{x_i}, B_{x_i})], where S(A_{x_i}, B_{x_i}) = rs - us, \tag{1}$$

where $rs = 1 - \mid \mu_A(x_i) - \mu_B(x_i) \mid \times [1 - \dfrac{\pi_A(x_i) + \pi_B(x_i)}{2}]$ and
$us = [\int_0^1 \mid \mu_A(Z) - \mu_B(Z) \mid dZ] \times [\dfrac{\pi_A(x_i) + \pi_B(x_i)}{2}]$, where the membership function

$$\mu_A(Z) = \begin{cases} 1 & , z = \mu_A(x_i) = 1 - \nu_A(x_i), \\ \dfrac{1-\nu_A(x_i)-z}{1-\mu_A(x_i)-\nu_A(x_i)}, & z \in [\mu_A(x_i), 1-\nu_A(x_i)], \\ 0 & , otherwise. \end{cases} \tag{2}$$

and $z \in Z = [0,1]$.

Chen, Cheng and Lan similarity measure S_{CCL} [8]

$S_{CCL}(A,B) = \sum_{i=1}^{n}[W_i \times S(A_{x_i}, B_{x_i})],$

$$\text{where } S(A_{x_i}, B_{x_i}) = 1 - \frac{\mid 2(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) \mid}{3}\left(1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right)$$
$$- \frac{\mid 2(\nu_A(x_i) - \nu_B(x_i)) - (\mu_A(x_i) - \mu_B(x_i)) \mid}{3}\left(\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right). \tag{3}$$

§4 Analyzing the drawbacks of the Chen, Cheng and Lan S_{CCL} [8] similarity measure

S_{CCL} [8] has analysed that the similarity measure S_{CC} [4] does not satisfy property (4) of definition 2.5. Though S_{CCL} [8] overcomes this drawback, it fails to satisfy property (5) and property (6) of definition 2.5, particularly when one of the set considered is MIFS.

For a similarity measure to be consistent, it should consider both the cardinality and the differences that exists between pattern sets. This approximate quantification is very crucial, when the similarity measure is to be applied to pattern matching in a ranking problem. Property (5) and (6) states this condition. It is found that S_{CCL} [8] fails to satisfy this condition which is illustrated in example 4.1 and example 4.2.

Example 4.1. *Let A, B, C and D be four IFSs in the universe of discourse* $X = \{x_1\}$, *where* $A = \{< x_1, 0.6, 0.4 >\}$, $B = \{< x_1, 0, 0 >\}$, $C = \{< x_1, 0, 0.87 >\}$, $D = \{< x_1, 0.28, 0.55 >\}$.
$| \mu_A(x_1) - \mu_B(x_1) |= 0.6$, $| \mu_C(x_1) - \mu_D(x_1) |= 0.28$.
Therefore, $| \mu_A(x_1) - \mu_B(x_1) |>| \mu_C(x_1) - \mu_D(x_1) |$.
Also $| \nu_A(x_1) - \nu_B(x_1) |= 0.4$, $| \nu_C(x_1) - \nu_D(x_1) |= 0.32$.
Therefore, $| \nu_A(x_1) - \nu_B(x_1) |>| \nu_C(x_1) - \nu_D(x_1) |$.
Let w_1 *be the weight of element* x_1, *where* $w_1 = 1$.

$$S(A_{x_1}, B_{x_1}) = 1 - \frac{| 2(0.6-0) - (0.4-0) |}{3}\left(1 - \frac{0+1}{2}\right) - \frac{| 2(0.4-0) - (0.6-0) |}{3}\left(\frac{0+1}{2}\right).$$

$S_{CCL}(A, B) = w_1 \times S(A_{x_1}, B_{x_1}) = 0.8333$.

$$S(C_{x_1}, D_{x_1}) = 1 - \frac{| 2(0-0.28) - (0.87-0.55) |}{3}\left(1 - \frac{0.13+0.17}{2}\right)$$
$$- \frac{| 2(0.87-0.55) - (0-0.28) |}{3}\left(\frac{0.13+0.17}{2}\right).$$

$S_{CCL}(C, D) = w_1 \times S(C_{x_1}, D_{x_1}) = 0.7047$.

$S_{CCL}(A, B) > S_{CCL}(C, D)$, *which is contradictory to property (5) of definition 2.5.*

As a consequence, when the similarity measure S_{CCL} *[8] is applied in a ranking problem, it would result in incorrect ordering of patterns. Table 2 discusses about this drawback in detail.*

Example 4.2. *Let* A^r *and* B^r *be the two IFSs obtained by interchanging the membership and non-membership values of the IFSs A and B respectively.*
Consider, $A = \{< x_1, 0.1, 0 >\}$, $B = \{< x_1, 0.2, 0 >\}$ *then*
$A^r = \{< x_1, 0, 0.1 >\}$, $B^r = \{< x_1, 0, 0.2 >\}$.
$S_{CC}(A, B) = 0.9425$, $S_{CC}(A^r, B^r) = 0.9575 \Rightarrow S_{CC}(A, B) < S_{CC}(A^r, B^r)$.
$S_{CCL}(A, B) = 0.9617$, $S_{CCL}(A^r, B^r) = 0.9383 \Rightarrow S_{CCL}(A, B) > S_{CCL}(A^r, B^r)$.
which is contradictory to property (6) of definition 2.5.

Moreover, it is also found that S_{CCL} [8] fails to discriminate the minute difference between very similar IFSs. This is illustrated in example 4.3.

Example 4.3. *Consider* $A = \{< x_1, 0, 0 >\}$, $B = \{< x_1, 0.5, 0.5 >\}$, $C = \{< x_1, 0.499, 0.501 >\}$. *Here B and C are the two very similar IFSs.*
So, intuitively $S_{CCL}(A, B)$ *should be slightly different from* $S_{CCL}(A, C)$.

In table 3, it is found that S_{CCL} *[8] fails to identify this minute difference and provides* $S_{CCL}(A, B) = S_{CCL}(A, C) = \mathbf{0.8333}$, *and the similarity value obtained is also unreasonable. The values highlighted in the tables represent the incorrect results.*

To overcome the drawbacks of S_{CC} [4] and S_{CCL} [8], a new similarity measure between IFSs is proposed.

§5 A new similarity measure between Intuitionistic fuzzy sets

In this section, we propose a new similarity measure between IFSs A and B based on the mid points of transformed triangular fuzzy numbers. Let $A = \{< x_i, \mu_A(x_i), \nu_A(x_i) > /1 \le i \le n\}$ be an IFS in the universe of discourse X, where $X = \{x_1, x_2, ..., x_n\}$. First, we propose the transformation techniques between an intuitionistic fuzzy values $< \mu_A(x_i), 1 - \nu_A(x_i) >$ and a right-angled triangular fuzzy numbers A_{x_i}, where $1 \le i \le n$. Let $A_{x_i} = (\mu_A(x_i), \mu_A(x_i), 1 - \nu_A(x_i))$ be the transformed right-angled triangular fuzzy number in the universe of discourse Z=[0,1], where $0 \le \mu_A(x_i) \le 1 - \nu_A(x_i) \le 1$.

The similarity degree $S(A_{x_i}, B_{x_i})$ between the intuitionistic fuzzy values $(\mu_A(x_i), 1-\nu_A(x_i))$ and $(\mu_B(x_i), 1 - \nu_B(x_i))$ is calculated as follows:

$$S(A_{x_i}, B_{x_i}) = 1- \mid \psi_A(x_i)-\psi_B(x_i) \mid \left(1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right) - \mid \pi_A(x_i)-\pi_B(x_i) \mid \left(\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right), \tag{4}$$

where $S(A_{x_i}, B_{x_i}) \in [0, 1]$, $\psi_A(x_i) = \dfrac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2}$, $\psi_B(x_i) = \dfrac{\mu_B(x_i) + 1 - \nu_B(x_i)}{2}$.

$\psi_A(x_i)$and $\psi_B(x_i)$ represents the mid points of the transformed triangular fuzzy numbers A_{x_i} and B_{x_i} respectively.

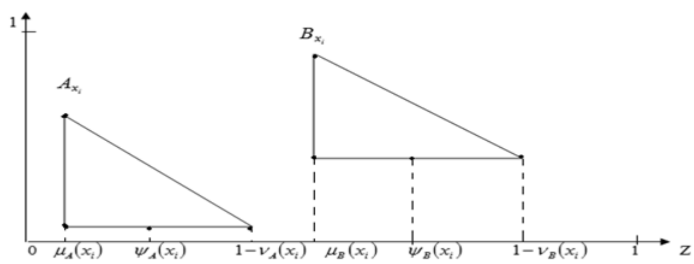

Figure 1: The transformed triangular fuzzy numbers A_{x_i} and B_{x_i} of the intuitionistic fuzzy values $(\mu_A(x_i), 1 - \nu_A(x_i))$ and $(\mu_B(x_i), 1 - \nu_B(x_i))$ respectively.

The proposed similarity measure $S_{sd}(A,B)$ between the intuitionistic fuzzy sets A and B is defined as $S_{sd}(A,B)=\sum_{i=1}^{n}(w_i \times S(A_{x_i},B_{x_i}))$, where $S_{sd}(A,B)\in[0,1]$, w_i is the weight of element $x_i\in X, w_i\in[0,1], \sum_{i=1}^{n} w_i=1$.

The similarity degree $S(A_{x_i},B_{x_i})$ satisfies the following properties.

Property 1: $0\leq S(A_{x_i},B_{x_i})\leq 1$.
Proof.
(i) Assume that $\mu_A(x_i)=\nu_B(x_i)=1$ and assume that $\mu_B(x_i)=\nu_A(x_i)=0$ then $\pi_A(x_i)=0$, $\pi_B(x_i)=0$, $\psi_A(x_i)=1$ and $\psi_B(x_i)=0$. Based on equation (4), we get $S(A_{x_i},B_{x_i})=0$.

(ii)Assume that $\mu_A(x_i)=\nu_B(x_i)=0$ and assume that $\mu_B(x_i)=\nu_A(x_i)=1$ then $\pi_A(x_i)=0$, $\pi_B(x_i)=0$, $\psi_A(x_i)=0$ and $\psi_B(x_i)=1$. Based on equation (4), we get $S(A_{x_i},B_{x_i})=0$.

(iii) Assume that $(\mu_A(x_i),\nu_A(x_i))=(\mu_B(x_i),\nu_B(x_i))$, then based on equation (4), we get $S(A_{x_i},B_{x_i})=1$.
From (i),(ii) and (iii), we get $0\leq S(A_{x_i},B_{x_i})\leq 1$.

Property 2: $S(A_{x_i},B_{x_i})=1$ if and only if $A_{x_i}=B_{x_i}$.
Proof.
(i)If $S(A_{x_i},B_{x_i})=1$, then based on equation (4), we get $\mid\psi_A(x_i)-\psi_B(x_i)\mid=0$ and $(\mid\psi_A(x_i)-\psi_B(x_i)\mid-\mid\pi_A(x_i)-\pi_B(x_i)\mid)=0$ or $\left(\frac{\pi_A(x_i)+\pi_B(x_i)}{2}\right)=0$.

Case1: When $\mid\psi_A(x_i)-\psi_B(x_i)\mid=0$ and $\left(\frac{\pi_A(x_i)+\pi_B(x_i)}{2}\right)=0$.
$\mid\frac{1+\mu_A(x_i)-\nu_A(x_i)}{2}-\frac{1+\mu_B(x_i)-\nu_B(x_i)}{2}\mid=0$.
Then we get $\mu_A(x_i)-\mu_B(x_i)=\nu_A(x_i)-\nu_B(x_i)$.
Also $\pi_A(x_i)=\pi_B(x_i)=0$,
we get $\mu_A(x_i)=\mu_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)\Rightarrow A_{x_i}=B_{x_i}$.

Case 2: When $\mid\psi_A(x_i)-\psi_B(x_i)\mid=0$ and $(\mid\psi_A(x_i)-\psi_B(x_i)\mid-\mid\pi_A(x_i)-\pi_B(x_i)\mid)=0$,
$\mid\frac{1+\mu_A(x_i)-\nu_A(x_i)}{2}-\frac{1+\mu_B(x_i)-\nu_B(x_i)}{2}\mid=0\Rightarrow\mu_A(x_i)-\mu_B(x_i)=\nu_A(x_i)-\nu_B(x_i)$.
Also, $\mid-\mu_A(x_i)-\nu_A(x_i)+\mu_B(x_i)+\nu_B(x_i)\mid-\mid\frac{\mu_A(x_i)-\nu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)}{2}\mid=0$
$\Rightarrow\mu_A(x_i)=\mu_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)\Rightarrow A_{x_i}=B_{x_i}$.

(ii) Based on equation (4), we get

$$S(A_{x_i},B_{x_i})=1-\mid\psi_A(x_i)-\psi_A(x_i)\mid\left(1-\frac{\pi_A(x_i)+\pi_A(x_i)}{2}\right)-\mid\pi_A(x_i)-\pi_A(x_i)\mid\left(\frac{\pi_A(x_i)+\pi_A(x_i)}{2}\right)$$
$$=1.$$

From (i) and (ii) we conclude that $S(A_{x_i},B_{x_i})=1$ if and only if $A_{x_i}=B_{x_i}$.

Property 3: $S(A_{x_i}, B_{x_i}) = S(B_{x_i}, A_{x_i})$.
Proof.
From equation (4), we see that

$$\begin{aligned} S(A_{x_i}, B_{x_i}) &= 1- \mid \psi_A(x_i) - \psi_B(x_i) \mid \left(1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right) - \mid \pi_A(x_i) - \pi_B(x_i) \mid \left(\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right) \\ &= 1- \mid \psi_B(x_i) - \psi_A(x_i) \mid \left(1 - \frac{\pi_B(x_i) + \pi_A(x_i)}{2}\right) - \mid \pi_B(x_i) - \pi_A(x_i) \mid \left(\frac{\pi_B(x_i) + \pi_A(x_i)}{2}\right) \\ &= S(B_{x_i}, A_{x_i}). \end{aligned}$$

Property 4: If $A \subseteq B \subseteq C$, then $S(A_{x_i}, C_{x_i}) \leq S(A_{x_i}, B_{x_i})$ and $S(A_{x_i}, C_{x_i}) \leq S(B_{x_i}, C_{x_i})$.
Proof.
By definition 2.3, if $A \subseteq B \subseteq C$ then $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$.
Hence $[\psi_A(x_i) - \psi_B(x_i)] = \frac{1}{2}[(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i))] \leq 0$.
Similarly $[\psi_A(x_i) - \psi_C(x_i)] \leq 0$ and $[\psi_B(x_i) - \psi_C(x_i)] \leq 0$.
whereas $(\pi_A(x_i) - \pi_B(x_i))$ and $(\pi_A(x_i) - \pi_C(x_i))$ are either ≤ 0 or ≥ 0.
Hence four different cases occurs.
Case (i): When $(\pi_A(x_i) - \pi_B(x_i)) \leq 0$ and $(\pi_A(x_i) - \pi_C(x_i)) \leq 0 \Rightarrow (\pi_B(x_i) - \pi_C(x_i)) \leq 0$ or ≥ 0,
To prove $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$.
From equation (4),

$$S(A_{x_i}, B_{x_i}) = 1 + [\psi_A(x_i) - \psi_B(x_i)] \left[1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right] + [\pi_A(x_i) - \pi_B(x_i)] \left[\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right]$$

$$= 1+\left[\frac{1}{2}(\mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i))\right] \left[\frac{\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i)}{2}\right]+\left[\frac{\pi_A^2(x_i) - \pi_B^2(x_i)}{2}\right]$$

$$= 1 + \frac{1}{4}[\mu_A^2(x_i) - \mu_B^2(x_i) + \nu_B^2(x_i) - \nu_A^2(x_i) + (\mu_A(x_i) - \mu_B(x_i))(\nu_A(x_i) + \nu_B(x_i))$$
$$+(\nu_B(x_i) - \nu_A(x_i))(\mu_A(x_i) + \mu_B(x_i))] + \left[\frac{\pi_A^2(x_i) - \pi_B^2(x_i)}{2}\right].$$

Similarly

$$S(A_{x_i}, C_{x_i}) = 1 + \frac{1}{4}[\mu_A^2(x_i) - \mu_C^2(x_i) + \nu_C^2(x_i) - \nu_A^2(x_i) + (\mu_A(x_i) - \mu_C(x_i))(\nu_A(x_i) + \nu_C(x_i))$$
$$+(\nu_C(x_i) - \nu_A(x_i))(\mu_A(x_i) + \mu_C(x_i)))] + \left[\frac{\pi_A^2(x_i) - \pi_C^2(x_i)}{2}\right],$$

Now $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) = \frac{1}{4}\{[\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)]$
$+2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)]\} + \frac{1}{2}[\pi_C^2(x_i) - \pi_B^2(x_i)]$
$= \frac{1}{4}X + \frac{1}{2}Y_1$,
where $X = [\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)]$
and $Y = [\pi_C^2(x_i) - \pi_B^2(x_i)]$.
When $(\pi_B(x_i) - \pi_C(x_i)) \leq 0$ and by definition 2.3, the expressions as X and Y are all positive.
Therefore $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$ for $(\pi_B(x_i) - \pi_C(x_i)) \leq 0$.
Now we prove $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$ for $(\pi_B(x_i) - \pi_C(x_i)) \geq 0$.
i.e to prove $\frac{1}{4}X + \frac{1}{2}Y_1 \geq 0 \Rightarrow X + 2Y_1 \geq 0$.
$X+2Y_1 = [\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)] +$
$2\{(\mu_C^2(x_i) - \mu_B^2(x_i)) + (\nu_C^2(x_i) - \nu_B^2(x_i)) + 2(\mu_B(x_i) - \mu_C(x_i)) + 2(\nu_B(x_i) - \nu_C(x_i)) + 2(\mu_C(x_i)\nu_C(x_i) -$

$\mu_B(x_i)\nu_B(x_i))\}$
$= 3[\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_C^2(x_i) - \nu_B^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)] + 4(\mu_B(x_i) - \mu_C(x_i)) + 4(\nu_B(x_i) - \nu_C(x_i)) + 4(\mu_C(x_i)\nu_C(x_i) - \mu_B(x_i)\nu_B(x_i))$
$= (\mu_C(x_i)-\mu_B(x_i))[3\mu_C(x_i)+3\mu_B(x_i)+2\nu_A(x_i)-4]+(\nu_C(x_i)+\nu_B(x_i))(\nu_C(x_i)-\nu_B(x_i))+2\mu_A(x_i)(\nu_B(x_i) - \nu_C(x_i)) + 4(\nu_B(x_i) - \nu_C(x_i)) + 4(\mu_C(x_i)\nu_C(x_i) - \mu_B(x_i)\nu_B(x_i))$.
When $(\pi_B(x_i) - \pi_C(x_i)) \geq 0 \Rightarrow (\mu_C(x_i) - \mu_B(x_i)) \geq (\nu_B(x_i) - \nu_C(x_i))$.
$X + 2Y_1 \geq (\nu_B(x_i) - \nu_C(x_i))[3\mu_C(x_i) + 3\mu_B(x_i) + 2\nu_A(x_i) - 4] + (\nu_C(x_i) + \nu_B(x_i))(\nu_C(x_i) - \nu_B(x_i)) + 2\mu_A(x_i)(\nu_B(x_i) - \nu_C(x_i)) + 4(\nu_B(x_i) - \nu_C(x_i)) + 4(\mu_C(x_i)\nu_C(x_i) - \mu_B(x_i)\nu_B(x_i))$
$= (\nu_B(x_i) - \nu_C(x_i))[2\mu_A(x_i) - \mu_B(x_i) + 3\mu_C(x_i) + 2\nu_A(x_i) - \nu_C(x_i) - \nu_B(x_i)]$
≥ 0 [by definition 2.3].
Therefore $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$ for $(\pi_B(x_i) - \pi_C(x_i)) \geq 0$.

Case (ii): When $(\pi_A(x_i) - \pi_B(x_i)) \leq 0$ and $(\pi_A(x_i) - \pi_C(x_i)) \geq 0 \Rightarrow (\pi_B(x_i) - \pi_C(x_i)) \geq 0$,
From equation (4),

$$S(A_{x_i}, B_{x_i}) = 1 + [\psi_A(x_i) - \psi_B(x_i)]\left[1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right] + [\pi_A(x_i) - \pi_B(x_i)]\left[\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right]$$
$$= 1 + \frac{1}{4}[\mu_A^2(x_i) - \mu_B^2(x_i) + \nu_B^2(x_i) - \nu_A^2(x_i) + (\mu_A(x_i) - \mu_B(x_i))(\nu_A(x_i) + \nu_B(x_i))$$
$$+(\nu_B(x_i) - \nu_A(x_i))(\mu_A(x_i) + \mu_B(x_i))] + \left[\frac{\pi_A^2(x_i) - \pi_B^2(x_i)}{2}\right].$$

Similarly,

$$S(A_{x_i}, C_{x_i}) = 1 + \frac{1}{4}[\mu_A^2(x_i) - \mu_C^2(x_i) + \nu_C^2(x_i) - \nu_A^2(x_i) + (\mu_A(x_i) - \mu_C(x_i))(\nu_A(x_i) + \nu_C(x_i))$$
$$+(\nu_C(x_i) - \nu_A(x_i))(\mu_A(x_i) + \mu_C(x_i)))] - \left[\frac{\pi_A^2(x_i) - \pi_C^2(x_i)}{2}\right],$$
$$S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) = \frac{1}{4}\{[\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)]\} + \frac{1}{2}[2\pi_A^2(x_i) - \pi_B^2(x_i) - \pi_C^2(x_i)]$$
$$= \frac{1}{4}X + \frac{1}{2}Y_2,$$

where $X = [\mu_C^2(x_i)-\mu_B^2(x_i)]+[\nu_B^2(x_i)-\nu_C^2(x_i)]+2\mu_A(x_i)[\nu_B(x_i)-\nu_C(x_i)]+2\nu_A(x_i)[\mu_C(x_i)-\mu_B(x_i)]$and $Y = [2\pi_A^2(x_i) - \pi_B^2(x_i) - \pi_C^2(x_i)]$.
To prove $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$,
That is to prove $\frac{1}{4}X + \frac{1}{2}Y_2 \geq 0 \Rightarrow X + 2Y_2 \geq 0$.
$X + 2Y_2 = [\mu_C^2(x_i)-\mu_B^2(x_i)]+[\nu_B^2(x_i)-\nu_C^2(x_i)]+2\mu_A(x_i)[\nu_B(x_i)-\nu_C(x_i)]+2\nu_A(x_i)[\mu_C(x_i)-\mu_B(x_i)]+ 2[2\pi_A^2(x_i) - \pi_B^2(x_i) - \pi_C^2(x_i)]$
$\geq \{[\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)]\} + 2[\pi_C^2(x_i) - \pi_B^2(x_i)]$
$= \{[\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)]\} + 2\{(\mu_C^2(x_i) - \mu_B^2(x_i)) + (\nu_C^2(x_i) - \nu_B^2(x_i)) + 2(\mu_B(x_i) - \mu_C(x_i)) + 2(\mu_C(x_i)\nu_C(x_i) - \mu_B(x_i)\nu_B(x_i)) + 2(\nu_B(x_i) - \nu_C(x_i))\}$
$= 3[\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_C^2(x_i) - \nu_B^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)] + 4(\mu_B(x_i) - \mu_C(x_i)) + 4(\nu_B(x_i) - \nu_C(x_i)) + 4(\mu_C(x_i)\nu_C(x_i) - \mu_B(x_i)\nu_B(x_i))$
$\geq (\mu_C(x_i)-\mu_B(x_i))[3\mu_C(x_i)+3\mu_B(x_i)+2\nu_A(x_i)-4]+(\nu_C(x_i)+\nu_B(x_i))(\nu_C(x_i)-\nu_B(x_i))+2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 4(\nu_B(x_i) - \nu_C(x_i)) + 4\mu_B(x_i)(\nu_C(x_i) - \nu_B(x_i))$.
When $(\pi_B(x_i) - \pi_C(x_i)) \geq 0 \Rightarrow \mu_C(x_i) - \mu_B(x_i) \geq \nu_B(x_i) - \nu_C(x_i)$,
$X + 2Y_2 \geq (\nu_B(x_i) - \nu_C(x_i))[3\mu_C(x_i) + 3\mu_B(x_i) + 2\nu_A(x_i) - 4] + (\nu_C(x_i) + \nu_B(x_i))(\nu_C(x_i) - \nu_B(x_i)) + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 4(\nu_B(x_i) - \nu_C(x_i)) + 4\mu_B(x_i)(\nu_C(x_i) - \nu_B(x_i))$

$= (\nu_B(x_i) - \nu_C(x_i))[2\mu_A(x_i) - \mu_B(x_i) + 3\mu_C(x_i) + 2\nu_A(x_i) - \nu_C(x_i) - \nu_B(x_i)]$.
By definition 2.3, $X + 2Y_2 \geq 0$.
Hence $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$.
Case (iii): When $(\pi_A(x_i) - \pi_B(x_i)) \geq 0$ and $(\pi_A(x_i) - \pi_C(x_i)) \leq 0 \Rightarrow (\pi_B(x_i) - \pi_C(x_i)) \leq 0$,
From equation (4),
$S(A_{x_i}, B_{x_i}) = 1 + [\psi_A(x_i) - \psi_B(x_i)]\left[1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right] - [\pi_A(x_i) - \pi_B(x_i)]\left[\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right]$.
$= 1 + \frac{1}{4}[\mu_A^2(x_i) - \mu_B^2(x_i) + \nu_B^2(x_i) - \nu_A^2(x_i) + (\mu_A(x_i) - \mu_B(x_i))(\nu_A(x_i) + \nu_B(x_i))$
$+(\nu_B(x_i) - \nu_A(x_i))(\mu_A(x_i) + \mu_B(x_i))] - \left[\frac{\pi_A^2(x_i) - \pi_B^2(x_i)}{2}\right]$.
Similarly,
$S(A_{x_i}, C_{x_i}) = 1 + \frac{1}{4}[\mu_A^2(x_i) - \mu_C^2(x_i) + \nu_C^2(x_i) - \nu_A^2(x_i) + (\mu_A(x_i) - \mu_C(x_i))(\nu_A(x_i) + \nu_C(x_i))$
$+(\nu_C(x_i) - \nu_A(x_i))(\mu_A(x_i) + \mu_C(x_i)))] + \left[\frac{\pi_A^2(x_i) - \pi_C^2(x_i)}{2}\right]$,
$S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) = \frac{1}{4}[\mu_C^2(x_i) - \mu_B^2(x_i) + \nu_B^2(x_i) - \nu_C^2(x_i) + 2\mu_A(x_i)(\nu_B(x_i) - \nu_C(x_i))+$
$2\nu_A(x_i)(\mu_C(x_i) - \mu_B(x_i))] + \frac{1}{2}[\pi_B^2(x_i) + \pi_C^2(x_i) - 2\pi_A^2(x_i)]$
$= \frac{1}{4}X + \frac{1}{2}Y_3$,
where $X = \mu_C^2(x_i) - \mu_B^2(x_i) + \nu_B^2(x_i) - \nu_C^2(x_i) + 2\mu_A(x_i)(\nu_B(x_i) - \nu_C(x_i)) + 2\nu_A(x_i)(\mu_C(x_i) - \mu_B(x_i))$
and $Y = [\pi_B^2(x_i) + \pi_C^2(x_i) - 2\pi_A^2(x_i)]$.
To prove $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$,
That is to prove $\frac{1}{4}X + \frac{1}{2}Y_3 \geq 0 \Rightarrow X + 2Y_3 \geq 0$,
$X + 2Y_3 = [\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)] +$
$2[\pi_B^2(x_i) + \pi_C^2(x_i) - 2\pi_A^2(x_i)]$
$\geq \{[\mu_C^2(x_i) - \mu_B^2(x_i)] + [\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)]\} +$
$2[\pi_B^2(x_i) - \pi_C^2(x_i)]$
$= (\mu_C^2(x_i) - \mu_B^2(x_i)) + (\nu_B^2(x_i) - \nu_C^2(x_i)) + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)] +$
$2[(\mu_B^2(x_i) - \mu_C^2(x_i)) + (\nu_B^2(x_i) - \nu_C^2(x_i)) - 2\mu_B(x_i) + 2\mu_C(x_i) + 2\mu_B(x_i)\nu_B(x_i) - 2\nu_B(x_i) + 2\nu_C(x_i) -$
$2\mu_C(x_i)\nu_C(x_i)$
$= [\mu_B^2(x_i) - \mu_C^2(x_i)] + 3[\nu_B^2(x_i) - \nu_C^2(x_i)] + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 2\nu_A(x_i)[\mu_C(x_i) - \mu_B(x_i)] +$
$4(\mu_C(x_i) - \mu_B(x_i)) + 4(\mu_B(x_i)\nu_B(x_i) - \mu_C(x_i)\nu_C(x_i)) + 4(\nu_C(x_i) - \nu_B(x_i))$
$= (\mu_B(x_i) - \mu_C(x_i))[\mu_B(x_i) + \mu_C(x_i) - 2\nu_A(x_i) + 4\nu_C(x_i) - 4] + 3(\nu_B(x_i) + \nu_C(x_i))(\nu_B(x_i) - \nu_C(x_i)) +$
$2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 4(\nu_C(x_i) - \nu_B(x_i))$.
When $(\pi_B(x_i) - \pi_C(x_i)) \leq 0 \Rightarrow \mu_B(x_i) - \mu_C(x_i) \geq \nu_C(x_i) - \nu_B(x_i)$,
$X + 2Y_3 \geq (\nu_C(x_i) - \nu_B(x_i))[\mu_B(x_i) + \mu_C(x_i) - 2\nu_A(x_i) + 4\nu_C(x_i) - 4] + 3(\nu_B(x_i) + \nu_C(x_i))(\nu_B(x_i) -$
$\nu_C(x_i)) + 2\mu_A(x_i)[\nu_B(x_i) - \nu_C(x_i)] + 4(\nu_C(x_i) - \nu_B(x_i))$
$= (\nu_B(x_i) - \nu_C(x_i))[2\mu_A(x_i) - \mu_B(x_i) - \mu_C(x_i) + 2\nu_A(x_i) - \nu_C(x_i) + 3\nu_B(x_i)]$.
When $(\pi_A(x_i) - \pi_C(x_i)) \leq 0 \Rightarrow \mu_A(x_i) + \nu_A(x_i) \geq \mu_C(x_i) + \nu_C(x_i)$,
$X + 2Y_3 \geq (\nu_B(x_i) - \nu_C(x_i))[2\mu_C(x_i) - \mu_B(x_i) - \mu_C(x_i) + 2\nu_C(x_i) - \nu_C(x_i) + 3\nu_B(x_i)]$
$= (\nu_B(x_i) - \nu_C(x_i))[\mu_C(x_i) - \mu_B(x_i) + \nu_C(x_i) + 3\nu_B(x_i)]$.
By definition 2.3, $X + 2Y_3 \geq 0$.
Hence $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$.
Case (iv): When $(\pi_A(x_i) - \pi_B(x_i)) \geq 0$ and $(\pi_A(x_i) - \pi_C(x_i)) \geq 0 \Rightarrow (\pi_B(x_i) - \pi_C(x_i)) \leq 0$ or ≥ 0,
To prove $S(A_{x_i}, B_{x_i}) - S(A_{x_i}, C_{x_i}) \geq 0$.

From equation (4),
$S(A_{x_i},B_{x_i}) = 1+[\psi_A(x_i)-\psi_B(x_i)]\left[1-\frac{\pi_A(x_i)+\pi_B(x_i)}{2}\right]-[\pi_A(x_i)-\pi_B(x_i)]\left[\frac{\pi_A(x_i)+\pi_B(x_i)}{2}\right]$
$=1+\frac{1}{4}[\mu_A^2(x_i)-\mu_B^2(x_i)+\nu_B^2(x_i)-\nu_A^2(x_i)+(\mu_A(x_i)-\mu_B(x_i))(\nu_A(x_i)+\nu_B(x_i))$
$+(\nu_B(x_i)-\nu_A(x_i))(\mu_A(x_i)+\mu_B(x_i))]-\left[\frac{\pi_A^2(x_i)-\pi_B^2(x_i)}{2}\right].$
Similarly, we have
$S(A_{x_i},C_{x_i}) = 1+\frac{1}{4}[\mu_A^2(x_i)-\mu_C^2(x_i)+\nu_C^2(x_i)-\nu_A^2(x_i)+(\mu_A(x_i)-\mu_C(x_i))(\nu_A(x_i)+\nu_C(x_i))$
$+(\nu_C(x_i)-\nu_A(x_i))(\mu_A(x_i)+\mu_C(x_i)))]-\left[\frac{\pi_A^2(x_i)-\pi_C^2(x_i)}{2}\right],$
$S(A_{x_i},B_{x_i})-S(A_{x_i},C_{x_i})$
$=\frac{1}{4}\{[\mu_C^2(x_i)-\mu_B^2(x_i)]+[\nu_B^2(x_i)-\nu_C^2(x_i)]+2\mu_A(x_i)[\nu_B(x_i)-\nu_C(x_i)]+2\nu_A(x_i)[\mu_C(x_i)-\mu_B(x_i)]\}+$
$\frac{1}{2}[\pi_B^2(x_i)-\pi_C^2(x_i)]$
$=\frac{1}{4}X+\frac{1}{2}Y_4,$
where $X=[\mu_C^2(x_i)-\mu_B^2(x_i)]+[\nu_B^2(x_i)-\nu_C^2(x_i)]+2\mu_A(x_i)[\nu_B(x_i)-\nu_C(x_i)]+2\nu_A(x_i)[\mu_C(x_i)-\mu_B(x_i)]$ and $Y=[\pi_B^2(x_i)-\pi_C^2(x_i)]$.
When $(\pi_B(x_i)-\pi_C(x_i))\geq 0$ and by definition 2.3, the expressions as X and Y are all positive.
Therefore $S(A_{x_i},B_{x_i})-S(A_{x_i},C_{x_i})\geq 0$ for $(\pi_B(x_i)-\pi_C(x_i))\geq 0$.
Now we prove $S(A_{x_i},B_{x_i})-S(A_{x_i},C_{x_i})\geq 0$ for $(\pi_B(x_i)-\pi_C(x_i))\leq 0$,
i.e to prove $\frac{1}{4}X+\frac{1}{2}Y_4\geq 0\Rightarrow X+2Y_4\geq 0$.
$X+2Y_4=[\mu_C^2(x_i)-\mu_B^2(x_i)]+[\nu_B^2(x_i)-\nu_C^2(x_i)]+2\mu_A(x_i)[\nu_B(x_i)-\nu_C(x_i)]+2\nu_A(x_i)[\mu_C(x_i)-\mu_B(x_i)]+$
$2((\mu_B^2(x_i)-\mu_C^2(x_i))+(\nu_B^2(x_i)-\nu_C^2(x_i))+2(\mu_C(x_i)-\mu_B(x_i))+2(\nu_C(x_i)-\nu_B(x_i))+2(\mu_B(x_i)\nu_B(x_i)-$
$\mu_C(x_i)\nu_C(x_i)))$
$=[\mu_C^2(x_i)-\mu_B^2(x_i)]+[\nu_B^2(x_i)-\nu_C^2(x_i)]+2\mu_A(x_i)[\nu_B(x_i)-\nu_C(x_i)]+2\nu_A(x_i)[\mu_C(x_i)-\mu_B(x_i)]+4(\mu_C(x_i)-$
$\mu_B(x_i))+4(\nu_C(x_i)-\nu_B(x_i))+2[(\mu_B^2(x_i)-\mu_C^2(x_i))+(\nu_B^2(x_i)-\nu_C^2(x_i))+2(\mu_B(x_i)\nu_B(x_i)-\mu_C(x_i)\nu_C(x_i))]$
$=[\mu_C^2(x_i)-\mu_B^2(x_i)]+[\nu_B^2(x_i)-\nu_C^2(x_i)]+2\mu_A(x_i)[\nu_B(x_i)-\nu_C(x_i)]+2\nu_A(x_i)[\mu_C(x_i)-\mu_B(x_i)]+4(\mu_C(x_i)-$
$\mu_B(x_i))+4(\nu_C(x_i)-\nu_B(x_i))+2[(\mu_B(x_i)+\nu_B(x_i))^2-(\mu_C(x_i)+\nu_C(x_i))^2]$
$=(\nu_B(x_i)-\nu_C(x_i))[\nu_B(x_i)+\nu_C(x_i)+2\mu_A(x_i)-4]+(\mu_C(x_i)+\mu_B(x_i))(\mu_C(x_i)-\mu_B(x_i))+2\nu_A(x_i)[\mu_C(x_i)-$
$\mu_B(x_i)]+4(\mu_C(x_i)-\mu_B(x_i))+2[(\mu_B(x_i)+\nu_B(x_i))^2-(\mu_C(x_i)+\nu_C(x_i))^2]$.
When $(\pi_B(x_i)-\pi_C(x_i))\leq 0\Rightarrow \nu_B(x_i)-\nu_C(x_i)\geq \mu_C(x_i)-\mu_B(x_i)$,
$X+2Y_4\geq(\mu_C(x_i)-\mu_B(x_i))[\nu_B(x_i)+\nu_C(x_i)+2\mu_A(x_i)+\mu_C(x_i)+\mu_B(x_i)+2\nu_A(x_i)]$.
By definition 2.3, $X+2Y_4\geq 0$.
Hence $S(A_{x_i},B_{x_i})-S(A_{x_i},C_{x_i})\geq 0$ Similarily, it could also be prove that $S(A_{x_i},C_{x_i})\leq S(B_{x_i},C_{x_i})$.

Corollary 1: If $|\mu_A(x_i)-\mu_B(x_i)|\geq|\mu_C(x_i)-\mu_D(x_i)|$ and $|\nu_A(x_i)-\nu_B(x_i)|\geq|\nu_C(x_i)-\nu_D(x_i)|$ then

$$|\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)|\geq|\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)| \quad (5)$$

and

$$|\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)|\geq|\mu_D(x_i)-\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)|. \quad (6)$$

Proof.
To prove equations (5) and (6), we use the method of contradiction,
Suppose if $|\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)|\leq|\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)|$ then

$- \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid$

$$\leq (\mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i)) \leq \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid . \tag{7}$$

and suppose if $\mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \leq \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid$ then
$- \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid$

$$\leq (\mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i)) \leq \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid . \tag{8}$$

Given $\mid \mu_A(x_i) - \mu_B(x_i) \mid \geq \mid \mu_C(x_i) - \mu_D(x_i) \mid$ and $\mid \nu_A(x_i) - \nu_B(x_i) \mid \geq \mid \nu_C(x_i) - \nu_D(x_i) \mid$
$\Rightarrow \mid \nu_B(x_i) - \nu_A(x_i) \mid \geq \mid \nu_D(x_i) - \nu_C(x_i) \mid$

$$\mu_A(x_i) - \mu_B(x_i) \geq \mid \mu_C(x_i) - \mu_D(x_i) \mid \tag{9}$$

or

$$\mu_A(x_i) - \mu_B(x_i) \leq - \mid \mu_C(x_i) - \mu_D(x_i) \mid \tag{10}$$

and

$$\nu_B(x_i) - \nu_A(x_i) \geq \mid \nu_D(x_i) - \nu_C(x_i) \mid \tag{11}$$

or

$$\nu_B(x_i) - \nu_A(x_i) \leq - \mid \nu_D(x_i) - \nu_C(x_i) \mid . \tag{12}$$

$$\begin{aligned} (9) + (11) \Rightarrow \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) &\geq \mid \mu_C(x_i) - \mu_D(x_i) \mid + \mid \nu_D(x_i) - \nu_C(x_i) \mid \\ &\geq \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid . \end{aligned}$$

which contradicts RHS of inequation (7),

$$\begin{aligned} (10) + (12) \Rightarrow \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) &\leq -[\mid \mu_C(x_i) - \mu_D(x_i) \mid + \mid \nu_D(x_i) - \nu_C(x_i) \mid] \\ &\leq - \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid . \end{aligned}$$

which contradicts LHS of inequation (7),
Therefore, $\mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \geq \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid$.
Similarly considering,

$$\begin{aligned} (9) + (12) \Rightarrow \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) &\leq -[\mid \mu_C(x_i) - \mu_D(x_i) \mid + \mid \nu_D(x_i) - \nu_C(x_i) \mid] \\ &= -[\mid \mu_D(x_i) - \mu_C(x_i) \mid + \mid \nu_D(x_i) - \nu_C(x_i) \mid] \\ &\leq - \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid . \end{aligned}$$

which contradicts LHS of inequation (8),
and

$$\begin{aligned} (10) + (11) \Rightarrow \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) &\geq \mid \mu_C(x_i) - \mu_D(x_i) \mid + \mid \nu_D(x_i) - \nu_C(x_i) \mid] \\ &= \mid \mu_D(x_i) - \mu_C(x_i) \mid + \mid \nu_D(x_i) - \nu_C(x_i) \mid] \\ &\geq \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid . \end{aligned}$$

which contradicts RHS of inequation (8),
Therefore $\mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \geq \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid$.

Property 5:
If $\mid \mu_A(x_i) - \mu_B(x_i) \mid \geq \mid \mu_C(x_i) - \mu_D(x_i) \mid$ and $\mid \nu_A(x_i) - \nu_B(x_i) \mid \geq \mid \nu_C(x_i) - \nu_D(x_i) \mid$ then $S(A_{x_i}, B_{x_i}) \leq S(C_{x_i}, D_{x_i})$.
Proof. $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i})$

$$= \left(1 - \mid \psi_A(x_i) - \psi_B(x_i) \mid \left(1 - \frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right) - \mid \pi_A(x_i) - \pi_B(x_i) \mid \left(\frac{\pi_A(x_i) + \pi_B(x_i)}{2}\right)\right)$$
$$- \left(1 - \mid \psi_C(x_i) - \psi_D(x_i) \mid \left(1 - \frac{\pi_C(x_i) + \pi_D(x_i)}{2}\right) - \mid \pi_C(x_i) - \pi_D(x_i) \mid \left(\frac{\pi_C(x_i) + \pi_D(x_i)}{2}\right)\right)$$
$$= - \mid \frac{\mu_A(x_i) + 1 - \nu_A(x_i)}{2} - \frac{\mu_B(x_i) + 1 - \nu_B(x_i)}{2} \mid \left(1 - \frac{1}{2}(1 - \mu_A(x_i) - \nu_A(x_i) + 1 - \mu_B(x_i) - \nu_B(x_i))\right) -$$
$$\mid 1 - \mu_A(x_i) - \nu_A(x_i) - (1 - \mu_B(x_i) - \nu_B(x_i)) \mid \left(\frac{1}{2}(1 - \mu_A(x_i) - \nu_A(x_i) + 1 - \mu_B(x_i) - \nu_B(x_i))\right) +$$
$$\mid \frac{\mu_C(x_i) + 1 - \nu_C(x_i)}{2} - \frac{\mu_D(x_i) + 1 - \nu_D(x_i)}{2} \mid \left(1 - \frac{1}{2}(1 - \mu_C(x_i) - \nu_C(x_i) + 1 - \mu_D(x_i) - \nu_D(x_i))\right) +$$
$$\mid 1 - \mu_C(x_i) - \nu_C(x_i) - (1 - \mu_D(x_i) - \nu_D(x_i)) \mid \left(\frac{1}{2}(1 - \mu_C(x_i) - \nu_C(x_i) + 1 - \mu_D(x_i) - \nu_D(x_i))\right)$$
$$= -\frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid (\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i)) - \frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid (2 - (\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i))) + \frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)) + \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid (2 - (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)))$$
$$= -\frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid (\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i)) - \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid (\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i)) + \frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)) + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid - \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))$$
$$= (- \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid) + (\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i))(\frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid) + (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid - \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid).$$

Here the absolute difference,
$\frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid$ is either ≤ 0 or ≥ 0
and
$\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid - \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid$ is either ≤ 0 or ≥ 0.

Also either
$\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \leq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$
(or)
$\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \geq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$.

Case (i):

Consider $\frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \leq 0$,
$\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid - \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid \leq 0$
and $\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \leq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$.
By using equation (6), $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq 0$,
Also for $\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \geq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$.
Again using equation (6), we have $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq 0$.
Hence $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq 0$.

Case (ii):
Consider $\frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \geq 0$,
$\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid - \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid \geq 0$
and $\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \leq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$.

$$S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq (- \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid) + (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(\frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid) + (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid - \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid)$$

$$= \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \left(-1 + \frac{1}{2}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)) \right) + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid \left(1 - \frac{1}{2}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)) \right) - \frac{1}{4}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(\mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid)$$

$$= \left(1 - \frac{1}{2}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)) \right) (- \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid) - \frac{1}{4}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(\mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid).$$

Since $0 \leq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i) \leq 2$, $1 - \frac{1}{2}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i) \geq 0$.

Hence by using equations (5) and (6), we have $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq 0$.

When $\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \geq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$.

$$S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq (- \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid) + (\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i))(\frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid) + (\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i))(\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid - \frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid)$$

$$= \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \left(-1 + \frac{1}{2}(\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i)) \right) + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid \left(1 - \frac{1}{2}(\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i)) \right) - \frac{1}{4}(\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i))(\mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid - \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid)$$

$$= \left(1 - \frac{1}{2}(\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i)) \right) (- \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_D(x_i) -$$

$\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid)-\frac{1}{4}(\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i))(\mid\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid$
$-\mid\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid)$.
Since $0\le\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)\le 2$, $1-\frac{1}{2}(\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)\ge 0$,
Hence from the equations (5) and (6), we have $S(A_{x_i},B_{x_i})-S(C_{x_i},D_{x_i})\le 0$.

Case (iii):
Consider $\frac{1}{2}\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid-\frac{1}{4}\mid\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid\le 0$,
$\frac{1}{4}\mid\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid-\frac{1}{2}\mid\mu_D(x_i)-\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid\ge 0$
and $\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)\ge\mu_C(x_i)+\mu_D(x_i)+\nu_C(x_i)+\nu_D(x_i)$.
$S(A_{x_i},B_{x_i})-S(C_{x_i},D_{x_i})$
$\le(-\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid+\mid\mu_D(x_i)-\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid)+(\mu_A(x_i)+\mu_B(x_i)$
$+\nu_A(x_i)+\nu_B(x_i))(-\frac{1}{4}\mid\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid+\frac{1}{2}\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid$
$)+(\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i))(\frac{1}{4}\mid\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid-\frac{1}{2}\mid\mu_D(x_i)-\mu_C(x_i)+$
$\nu_D(x_i)-\nu_C(x_i)\mid)$
$=\left(1-\frac{1}{2}(\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i))\right)(-\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid+\mid\mu_D(x_i)-$
$\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid)+\frac{1}{4}(\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i))(-\mid\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid$
$+\mid\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid$.
Since $0\le\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)\le 2$, $1-\frac{1}{2}(\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i))\ge 0$,
From equations (5) and (6), we get $S(A_{x_i},B_{x_i})-S(C_{x_i},D_{x_i})\le 0$.

When $\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)\le\mu_C(x_i)+\mu_D(x_i)+\nu_C(x_i)+\nu_D(x_i)$
$S(A_{x_i},B_{x_i})-S(C_{x_i},D_{x_i})$
$=(-\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid+\mid\mu_D(x_i)-\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid)+(\mu_A(x_i)+\mu_B(x_i)$
$+\nu_A(x_i)+\nu_B(x_i))(\frac{1}{2}\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid$
$-\frac{1}{4}\mid\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid)+(\mu_C(x_i)+\mu_D(x_i)+\nu_C(x_i)+\nu_D(x_i))$
$\left(\frac{1}{4}\mid\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid-\frac{1}{2}\mid\mu_D(x_i)-\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid\right)$
$=(T_1+xT_2)+yT_3$,
where,
$T_1=(-\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid+\mid\mu_D(x_i)-\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid)\le 0$.
By Corollary 1,$T_1\epsilon[-1,0]$,
$T_2=\frac{1}{2}\mid\mu_B(x_i)-\mu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid-\frac{1}{4}\mid\mu_A(x_i)-\mu_B(x_i)+\nu_B(x_i)-\nu_A(x_i)\mid$.
By the case considered, $T_2\le 0$, $T_2\epsilon[-0.25,0]$,
$T_3=\frac{1}{4}\mid\mu_C(x_i)-\mu_D(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid-\frac{1}{2}\mid\mu_D(x_i)-\mu_C(x_i)+\nu_D(x_i)-\nu_C(x_i)\mid$.
By the case considered, $T_3\ge 0$, $T_3\epsilon[0,0.25]$,
$x=\mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)\epsilon[0,2]$,
$y=\mu_C(x_i)+\mu_D(x_i)+\nu_C(x_i)+\nu_D(x_i)\epsilon[0,2]$.
To prove $\mid T_1+xT_2\mid\ge yT_3$ and $T_1+xT_2<0$,

$$\begin{aligned}T_1+xT_2&\epsilon[-1,0]+x[-0.25,0]\\&\epsilon[-1-0.25x,0],\end{aligned}$$

Therefore $T_1 + xT_2 \leq 0$.
Since by case considered $x \leq y \Rightarrow 0.25x \leq 0.25y \Rightarrow 1 + 0.25x \geq 0.25y \Rightarrow \mid T_1 + xT_2 \mid \geq yT_3$.
Hence $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq 0$.

Case (iv):
Consider $\frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid -\frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid \geq 0$, $\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid -\frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid \leq 0$ and $\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \leq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$.
$S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i})$
$\leq (- \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid) + (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(-\frac{1}{4} \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \frac{1}{2} \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid) + (\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(\frac{1}{4} \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid -\frac{1}{2} \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid)$
$= \left(1 - \frac{1}{2}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))\right) (- \mid \mu_B(x_i) - \mu_A(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_D(x_i) - \mu_C(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid) + \frac{1}{4}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i))(- \mid \mu_A(x_i) - \mu_B(x_i) + \nu_B(x_i) - \nu_A(x_i) \mid + \mid \mu_C(x_i) - \mu_D(x_i) + \nu_D(x_i) - \nu_C(x_i) \mid$.
Since $0 \leq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i) \leq 2$, $1 - \frac{1}{2}(\mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i) \geq 0$.
From equations (5) and (6), we get $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq 0$.

When $\mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) \geq \mu_C(x_i) + \mu_D(x_i) + \nu_C(x_i) + \nu_D(x_i)$.
As discussed in case (iii),

$$S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) = T_1 + xT_2 + yT_3.$$

By Corollary 1, $T_1 \epsilon [-1, 0]$.
By the case considered, $T_2 \geq 0$, $T_2 \epsilon [0, 0.5]$ and $T_3 \leq 0$, $T_3 \epsilon [-0.5, 0]$.
To prove $\mid T_1 + yT_3 \mid \geq xT_2$ and $T_1 + yT_3 < 0$.

$$\begin{aligned} T_1 + yT_3 &\epsilon [-1, 0] + y[-0.5, 0] \\ &\epsilon [-1 - 0.5y, 0]. \end{aligned}$$

Therefore $T_1 + yT_3 \leq 0$.
Since by case considered $x \geq y \Rightarrow 0.5x \geq 0.5y \Rightarrow 1 + 0.5y \geq 0.5x \Rightarrow \mid T_1 + yT_3 \mid \geq xT_2$.
Hence $S(A_{x_i}, B_{x_i}) - S(C_{x_i}, D_{x_i}) \leq 0$.
Property 6: If $A^r_{x_i}$, $B^r_{x_i}$ are two new IFS obtained by interchanging simultaneously both μ_{x_i} and ν_{x_i} of A and B, then for a similarity measure to be consistent $S(A_{x_i}, B_{x_i}) = S(A^r_{x_i}, B^r_{x_i})$.
Proof. Let $A = \{< x_i, \mu_A(x_i), \nu_A(x_i) > / x_i \epsilon X\}$ and $B = \{< x_i, \mu_B(x_i), \nu_B(x_i) > / x_i \epsilon X\}$. and let A^r, B^r are two new IFS obtained by interchanging simultaneously both μ_{x_i} and ν_{x_i} of A and B.
That is, $A^r = \{< x_i, \nu_A(x_i), \mu_A(x_i) > / x_i \epsilon X\}$ and $B^r = \{< x_i, \nu_B(x_i), \mu_B(x_i) > / x_i \epsilon X\}$.
Then, we get $\pi_A(x_i) = 1 - \mu_A(x_i) - \nu_A(x_i) = 1 - \nu_A(x_i) - \mu_A(x_i) = \pi_{A^r}(x_i)$ and
$\pi_B(x_i) = 1 - \mu_B(x_i) - \nu_B(x_i) = 1 - \nu_B(x_i) - \mu_B(x_i) = \pi_{B^r}(x_i)$.
Also $\mid \psi_A(x_i) - \psi_B(x_i) \mid = \mid \psi_{A^r}(x_i) - \psi_{B^r}(x_i) \mid$.

Based on equation (4), we have

$S(A_{x_i}, B_{x_i})$

$= 1- \mid \psi_A(x_i) - \psi_B(x_i) \mid \left(1 - \dfrac{\pi_A(x_i) + \pi_B(x_i)}{2}\right) - \mid \pi_A(x_i) - \pi_B(x_i) \mid \left(\dfrac{\pi_A(x_i) + \pi_B(x_i)}{2}\right)$

$= 1- \mid \psi_{A^r}(x_i) - \psi_{B^r}(x_i) \mid \left(1 - \dfrac{\pi_{A^r}(x_i) + \pi_{B^r}(x_i)}{2}\right) - \mid \pi_{A^r}(x_i) - \pi_{B^r}(x_i) \mid \left(\dfrac{\pi_{A^r}(x_i) + \pi_{B^r}(x_i)}{2}\right)$

$= S(A^r_{x_i}, B^r_{x_i})$,which completes the proof.

Similarly, it is also verified that $S_{sd}(A, B)$ satisfies the above six properties.

§6 Comparative study towards the existing similarity measures

In this section, to examine the performance of the proposed similarity measure on precision and discriminatory ability, a comparative study is conducted on different sets used in the literature. Let A, B and C be three intuitionistic fuzzy sets in the universe of discourse. Table 1 and table 4 adopted from [23,8] are used to compare the proposed similarity measure S_{sd} with the existing similarity measures [2-5,8,12,14-18,20,23,25]. In table 1, six different patterns, case 1 to case 6 are taken and the results obtained by [2-5,8,12,14-18,20,23,25] and S_{sd} are listed. It is seen that the proposed similarity measure S_{sd} can overcome the drawbacks of getting the unreasonable results of the existing measures S_C[3], S_{HK}[14], S_{LX}[18], S_{LO}[17], S_{DC}[12], S_M[20], S_{LS1}[16], S_{LS2}[16], S_{LS3}[16], S_{HY1}[15], S_{HY2}[15], S_{HY3}[15], S_Y[23], S_{ZY}[25], S_{CR}[5],S_{BA}[2], S_{CC}[4] and S_{CCL}[8]. The values highlighted in table 1 and table 4 denotes unreasonable results, "N/A" denotes it cannot calculate the degree of similarity due to "division by zero problems".

For table 1 to table 7, we take p=1 for S_{DC} [12], S_M [20], S_{LS1} [16], S_{LS2} [16], S_{LS3} [16] and $w_1 = 1$ for S_{LS3} [16], S_{ZY} [25], S_{CC} [4], S_{CCL} [8] and S_{sd}.

Table 1: A comparison of the results of the proposed similarity measure S_{sd} with the existing similarity measures for different sets of IFSs adopted from [23]

Similarity measures	Case 1: A={(x:0.3,0.3)} B={(x:0.4,0.4)}	Case 2: A={(x:0.3,0.4)} B={(x:0.4,0.3)}	Case 3: A={(x:0.6,0.4)} B={(x:0,0)}	Case 4: A={(x:0.5,0.5)} B={(x:0,0)}	Case 5: A={(x:0.4,0.2)} B={(x:0.5,0.3)}	Case 6: A={(x:0, 0.87)} B={(x:0.28,0.55)}
S_C[3]	**1**	**0.9**	**0.9**	**1**	**1**	0.7
S_{HK}[14]	**0.9**	**0.9**	**0.5**	**0.5**	**0.9**	0.698
S_{LX}[18]	0.95	0.9	**0.7**	**0.75**	0.95	0.6993
S_{LO}[17]	**0.9**	**0.9**	0.4901	0.5	**0.9**	0.7
S_{DC}[12]	**1**	**0.9**	**0.9**	**1**	**1**	0.7
S_M[20]	**0.9**	**0.9**	**0.5**	**0.5**	**0.9**	0.7
S_{LS1}[16]	**0.9**	**0.9**	**0.5**	**0.5**	**0.9**	0.7
S_{LS2}[16]	0.95	0.9	**0.75**	**0.75**	0.95	0.7
S_{LS3}[16]	**0.9333**	**0.9333**	0.6333	0.6667	**0.9333**	0.7933
S_{HY1}[15]	**0.9**	**0.9**	0.4	0.5	**0.9**	0.68
S_{HY2}[15]	**0.8495**	**0.8495**	0.2862	0.3775	**0.8495**	**0.5668**
S_{HY3}[15]	**0.8182**	**0.8182**	0.25	0.3333	**0.8182**	0.5152
S_Y[23]	**1**	0.96	**N/A**	**N/A**	0.9971	0.8912
S_{ZY}[25]	0.9	0.8167	**N/A**	**N/A**	0.9	0.626
S_{CR}[5]	0.9857	0.9	**0.65**	**0.75**	0.9857	0.6993
S_{BA}[2]	0.967	0.9	**0.8333**	**0.8333**	0.9667	0.7
S_{CC}[4]	0.9225	0.88	0.45	0.5	0.9225	0.7395
S_{CCL}[8]	0.9667	0.9	**0.8333**	**0.8333**	0.9667	0.7047
S_{sd}	0.94	0.93	0.45	0.5	0.94	0.739

From table 1, it is also seen that when one of the sets is MIFS as in case 3 and case 4, S_{BA}[2],S_{CCL}[8] gives unreasonable results. Few more such pattern sets are taken in table 5 to validate the above reasoning.

Due to failure of property (5) for S_{CCL} [8] as discussed in section 4, it is found from table 2, that the ranking provided by S_{CCL} [8] is incorrect. Whereas, S_{CC} [4] and S_{sd} gives correct ranking.

Table 2: Ranking of pattern sets taken from table 1.

Ranking	I	II	III	IV	V
S_{CC}[4]	case 1 and case 5	case 2	case 6	case 3	case 4
S_{CCL}[8]	case 1 and case 5	case 2	**case 3 and case 4**	**case 6**	
S_{sd}	case 1 and case 5	case 2	case 6	case 3	case 4

As discussed in example 4.3, table 3 deals with similarity measure for very similar IFSs. Here B and C are the two very similar IFSs. So, intuitively $S(A, B)$ should be slightly different from $S(A, C)$. It is found that S_{CCL} [8] fails to discriminate the minute difference between very similar IFSs.

Table 3: Comparison of similarity measures for very similar IFSs

Similarity measures	$A = \{(x, 0, 0\}$ $B = \{(x, 0.5, 0.5\}$	$A = \{(x, 0, 0\}$ $C = \{(x, 0.499, 0.501\}$	$B = \{(x, 0.5, 0.5\}$ $C = \{(x, 0.499, 0.501\}$
S_{CC} [4]	0.5	0.5005	0.999
S_{CCL} [8]	**0.8333**	**0.8333**	0.999
S_{sd}	0.5	0.4995	0.999

As discussed in example 4.2, it is noted from table 4, for the IFSs in case 3, case 4 and case 6, case 7; the similarity measures S_{CC}[4] and S_{CCL}[8] fails to satisfy the property (6) of definition 2.5. It is seen that the few other existing measures satisfy property 6 but the results provided by them are unacceptable. Table 4 highlights the unreasonable results.

Table 4: A comparison of the proposed similarity measure S_{sd} with the existing similarity measures [8](p=1 in S_{DC}, S_M, S_{LS1}, S_{LS2} and S_{LS3}; $w_1 = w_2 = w_3 = 1/3$ in S_{LS3}, S_{ZY}, S_{CC}, S_{CCL} and S_{sd})

Similarity	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7
S_C[3]	**0.5**	**0.5**	0.95	0.95	**0.65**	0.7	0.7
S_{HK}[14]	**0.5**	**0.5**	0.95	0.95	**0.6**	0.7	0.7
S_{LX}[18]	**0.5**	**0.5**	0.95	0.95	**0.625**	0.7	0.7
S_{LO}[17]	0.2929	0.2929	0.9293	0.9293	0.4343	**0.5757**	**0.5757**
S_{DC}[12]	**0.5**	**0.2929**	0.7764	0.7764	0.3675	**0.4523**	**0.4523**
S_M[20]	**0.5**	**0.5**	0.8419	0.8419	0.5528	0.6127	0.6127
S_{LS1}[16]	**0.5**	**0.2929**	0.7764	0.7764	0.3675	**0.4523**	**0.4523**
S_{LS2}[16]	**0.5**	**0.3386**	0.7764	0.7764	0.3482	**0.4084**	**0.4084**
S_{LS3}[16]	**0.5**	**0**	0.6127	0.6127	**N/A**	**0.0513**	**0.0513**
S_{HY1}[15]	**0**	**0**	0.9	0.9	0.2	**0.4**	**0.4**
S_{HY2}[15]	**0**	**0**	0.8495	0.8495	0.1289	**0.2862**	**0.2862**
S_{HY3}[15]	**0**	**0**	**N/A**	**N/A**	**N/A**	**N/A**	**N/A**
S_Y[23]	**N/A**	**N/A**	**1**	**1**	**1**	**0.1806**	**0.1806**
S_{ZY}[25]	**N/A**	**0.5**	**N/A**	**N/A**	**N/A**	0.6225	0.6225
S_{CR}[5]	**0.25**	**0.2343**	**0.3882**	**0.3882**	0.2556	**0.3654**	**0.3654**
S_{BA}[2]	**0.5**	0.2929	0.7764	0.7764	0.3675	**0.4523**	**0.4523**
S_{CC}[4]	**0.25**	**0.75**	**0.9425**	**0.9575**	**0.8**	**0.91**	**0.67**
S_{CCL}[8]	**0.5**	**0.5**	**0.9617**	**0.9383**	**0.6**	**0.74**	**0.66**
S_{sd}	0.25	0.25	0.9075	0.9075	0.4	0.61	0.61

Case 1: A=(x:1,0),B=(x:0,0),Case2:A=(x:0,1),B=(x:0,0),Case3:A=(x:0.1,0),B=(x:0.2,0),
Case 4:A=(x:0,0.1),B=(x:0,0.2) ,Case 5: A=(x:0,0.1), B=(x:0,0.9),
Case 6:A=(x:0.2,0.8) , B=(x:0.2,0.2),Case 7:A=(x:0.8,0.2) , B=(x:0.2,0.2).

It is noted that S_{CC}[4] concentrates only on the absolute difference of the membership values and not on the non-membership values. This drawback of S_{CC}[4] is discussed in table 5 and table 6.

To further validate the argument with the MIFS table 5, lists five different sets. It is noted that case 3 and case 4 are two different sets with different membership and non-membership values but S_{CCL}[8] provides same similarity measure and at the same time gives very high similairty value. Also S_{CC}[4] provides unreasonable results for case 1 and case 2.

Table 5: A comparison of S_{sd} with the existing similarity measure S_{CC} [4], S_{CCL} [8] for the sets A as FS and B as MIFS

Cases	Set	S_{CC}[4]	S_{CCL}[8]	S_{sd}
1	$A = \{(x, 0.1, 0.9)\}, B = \{(x, 0, 0)\}$	**0.7**	**0.6**	0.3
2	$A = \{(x, 0.2, 0.8)\}, B = \{(x, 0, 0)\}$	**0.65**	**0.7**	0.35
3	$A = \{(x, 0.5, 0.5)\}, B = \{(x, 0, 0)\}$	0.5	**0.833**	0.5
4	$A = \{(x, 0.4, 0.6)\}, B = \{(x, 0, 0)\}$	0.55	**0.833**	0.45
5	$A = \{(x, 0.7, 0.3)\}, B = \{(x, 0, 0)\}$	0.4	**0.8**	0.4

Table 6: A comparison of the IFSs A and B with $\mu_A = \mu_B$

Sets	S_{CC}[4]	S_{CCL}[8]	S_{sd}
$A = \{(x, 0, 0.1)\}, B = \{(x, 0, 0.9)\}$	**0.8**	**0.6**	0.4
$A = \{(x, 0.2, 0.8)\}, B = \{(x, 0.2, 0.2)\}$	**0.91**	**0.74**	0.61

From table 1 to table 6, we see that the proposed similarity measure S_{sd} can overcome the drawbacks of the existing similarity measures S_C[3], S_{HK}[14], S_{LX}[18], S_{LO}[17], S_{DC}[12], S_M[20], S_{LS1}[16], S_{LS2}[16], S_{LS3}[16], S_{HY1}[15], S_{HY2}[15], S_{HY3}[15], S_Y[23], S_{ZY}[25], S_{CR}[5],S_{BA}[2], S_{CC}[4] and S_{CCL}[8].

§7 Applications

In this section, we apply the proposed similarity measure between intuitionistic fuzzy sets to deal with pattern recognition problems and medical diagnosis problems adopted from [23].

Example 7.1. *Let us consider three known patterns P_1, P_2 and P_3 represented by the IFSs in the universe of discourse X, respectively, where $X = \{x_1, x_2, x_3\}$, shown as follows:*
$P_1 = \{(x_1, 1, 0), (x_2, 0.8, 0), (x_3, 0.7, 0.1)\}$,
$P_2 = \{(x_1, 0.8, 0.1), (x_2, 1, 0), (x_3, 0.9, 0)\}$,
$P_3 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0), (x_3, 1, 0)\}$.
We have to classify an unknown pattern represented by an IFS Q, into one of the pattern P_1, P_2, P_3 shown as follows: $Q = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.8, 0.1)\}$.
From table 7, we can see that the similarity measures [2-5,8,12,14-18,20,23] in literature and the proposed similarity measure S_{sd} classified the unknown pattern represented by an intuitionistic fuzzy set Q in to the pattern P_3. Therefore, the unknown pattern represented by an IFS Q, is classified in to the pattern P_3. Except the measure S_{ZY} [25] all the existing measures in the literature classify the result in to the pattern P_3. The values highlighted denotes unreasonable results, "N/A" denotes it cannot calculate the degree of similarity due to "the division by zero problem"

Table 7: A comparison of S_{sd} with the ones of the existing similarity measures for Example 7.1

Similarity measures	$S(C_1, Q)$	$S(C_2, Q)$	$S(C_3, Q)$	Classification result
S_C[3]	**0.7833**	**0.7833**	0.85	P_3
S_{HK}[14]	**0.7833**	**0.7833**	0.85	P_3
S_{LX}[18]	**0.7833**	**0.7833**	0.85	P_3
S_{LO}[17]	0.7323	0.7585	0.8419	P_3
S_{DC}[12]	**0.7833**	**0.7833**	0.85	P_3
S_M[20]	**0.7833**	**0.7833**	0.85	P_3
S_{LS1}[16]	**0.7833**	**0.7833**	0.85	P_3
S_{LS2}[16]	**0.7833**	**0.7833**	0.85	P_3
S_{LS3}[16]	**0.8389**	**0.8389**	0.8944	P_3
S_{HY1}[15]	**0.7333**	**0.7333**	0.8333	P_3
S_{HY2}[15]	**0.6297**	**0.6297**	0.7571	P_3
S_{HY3}[15]	**0.5789**	**0.5789**	0.7143	P_3
S_Y[23]	0.9353	0.9519	0.9724	P_3
S_{ZY}[25]	**N/A**	**N/A**	**N/A**	cannot be determined
S_{CR}[5]	0.7591	0.7699	0.8471	P_3
S_{BA}[2]	**0.7833**	**0.7833**	0.85	P_3
S_{CC}[4]	0.7467	0.7508	0.8325	P_3
S_{CCL}[8]	0.7706	0.7710	0.845	P_3
S_{sd}	0.8008	0.7975	0.8708	P_3

Example 7.2. *Let $X = \{x_1(Temperature), x_2(Headache), x_3(StomachPain), x_4(Cough), x_5(ChestPain)\}$ be a set of symptoms.*
Consider a set $Q = \{Q_1(ViralFever), Q_2(Malaria), Q_3(Typhoid), Q_4(StomachProblem), Q_5(ChestProblem)\}$ of diagnosis,

where the elements $Q_1(ViralFever)$, $Q_2(Malaria)$, $Q_3(Typhoid)$, $Q_4(StomachProblem)$ *and* $Q_5(ChestProblem)$ *in* Q *are represented by the intuitionistic fuzzy sets in the universe of discourse* X*, shown as follows:*

$Q_1(ViralFever) = \{(x_1, 0.4, 0), (x_2, 0.3, 0.5), (x_3, 0.1, 0.7), (x_4, 0.4, 0.3), (x_5, 0.1, 0.7)\}$

$Q_2(Malaria) = \{(x_1, 0.7, 0), (x_2, 0.2, 0.6), (x_3, 0, 0.9), (x_4, 0.7, 0), (x_5, 0.1, 0.8)\}$

$Q_3(Typhoid) = \{(x_1, 0.3, 0.3), (x_2, 0.6, 0.1), (x_3, 0.2, 0.7), (x_4, 0.2, 0.6), (x_5, 0.1, 0.9)\}$

$Q_4(StomachProblem) = \{(x_1, 0.1, 0.7), (x_2, 0.2, 0.4), (x_3, 0.8, 0), (x_4, 0.2, 0.7), (x_5, 0.2, 0.7)\}$,

$Q_5(ChestProblem) = \{(x_1, 0.1, 0.8), (x_2, 0, 0.8), (x_3, 0.2, 0.8), (x_4, 0.2, 0.8), (x_5, 0.8, 0.1)\}$

Assume that a patient with respect to all the symptoms can be represented by the following intuitionistic fuzzy set:

$P(Patient) = \{(x_1, 0.8, 0.1), (x_2, 0.6, 0.1), (x_3, 0.2, 0.8), (x_4, 0.6, 0.1), (x_5, 0.1, 0.6)\}$.

We want to classify the patient P(Patient) with respect to all the symptoms, represented by an intuitionistic fuzzy set, into one of the diagnosis $Q_1(ViralFever), Q_2(Malaria), Q_3(Typhoid)$, $Q_4(StomachProblem) and Q_5(ChestProblem)$ *. Then based on equation (4), we can get*

$S_{sd}(P(Patient), Q_1(ViralFever)) = 0.8435$,

$S_{sd}(P(Patient), Q_2(Malaria)) = 0.8315$,

$S_{sd}(P(Patient), Q_3(Typhoid)) = 0.8075$,

$S_{sd}(P(Patient), Q_4(StomachProblem)) = 0.6260$,

$S_{sd}(P(Patient), Q_5(ChestProblem)) = 0.5650$.

Among all the values $S_{sd}(P(Patient), Q_1(ViralFever))$ *is largest. Hence the patient P(Patient) with respect to all the symptoms is classified into the diagnosis* $Q_1(ViralFever)$*. This result coincides with the existing measures.*

§8 Conclusion

In this paper, a new similarity measure between intuitionistic fuzzy sets based on the mid points of the transformed triangular fuzzy numbers is defined for identifying the similarity between intuitionistic fuzzy sets. Several novel measures are available in literature to access the similarity of intuitionistic fuzzy sets but the proposed measure correlates better than the other measures. The proposed similarity measure deals effectively with some demanding situations. The experimental results discussed in tables 1-7 clearly indicates that the proposed similarity measure satisfies the basic properties, as well overcome the drawbacks of the existing similarity measures. In the illumination of this study, the proposed similarity measure can be effectively used in real applications of decision making, medical diagnosis and pattern recognition.

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