Appl. Math. J. Chinese Univ.
2019, 34(1): 45-54

Asymptotic behavior for sums of non-identically distributed random variables

YU Chang-jun [1] CHENG Dong-ya[2,3]

Abstract. For any given positive integer m, let $X_i, 1 \leq i \leq m$ be m independent random variables with distributions $F_i, 1 \leq i \leq m$. When all the summands are nonnegative and at least one of them is heavy-tailed, we prove that the lower limit of the ratio $\frac{P(\sum_{i=1}^m X_i > x)}{\sum_{i=1}^m \overline{F_i}(x)}$ equals 1 as $x \to \infty$. When the summands are real-valued, we also obtain some asymptotic results for the tail probability of the sums. Besides, a local version as well as a density version of the above results is also presented.

§1 Introduction

Throughout this paper, let $X_n, n \geq 1$ be independent random variables (r.v.s) with distributions $F_n, n \geq 1$ unless otherwise stated. For any $m \geq 1$, we denote the convolution of the distributions $F_1, \cdots, F_m$ by $F_1 * F_2 * \cdots * F_m$. In the special case that $F_1 = F_2 = \cdots = F_m = F$, the convolution reduces to F^{*m}. We also use F^{*0} to denote a distribution degenerated at 0. We say that a r.v. X (or its distribution F) is heavy-tailed, if $Ee^{\varepsilon X} = \infty$ for all $\varepsilon > 0$, and light-tailed otherwise. Heavy-tailed distributions play a very important role in the distribution theory and have extensive applications in finance and insurance. For systematical research on heavy-tailed distributions, we refer the reader to Resnick (2007), Su et al. (2009) and Foss et al. (2013), among many others.

This paper focuses on the limit of the ratio $\frac{P(\sum_{i=1}^m X_i > x)}{\sum_{i=1}^m \overline{F_i}(x)}$ as $x \to \infty$, with the assumption that at least one of the summands is unbounded on the right, namely $\max_{1 \leq i \leq m} \overline{F_i}(x) > 0$ for all $x > 0$, where $\overline{F_i}(x) = 1 - F_i(x), 1 \leq i \leq m$.

We first introduce some related results. When the summands $X_n, n \geq 1$ are identically distributed with a common distribution F, and are independent of a r.v. τ which is nonnegative

Received: 2016-02-02. Revised 2018-12-08.
MR Subject Classification: Primary 60E05; 60F99.
Keywords: lower limits; upper limits; heavy-tailed distributions; local distributions; densities.
Digital Object Identifier(DOI): https://doi.org/10.1007/s11766-019–3440-8.
Supported by the National Natural Science Foundation of China (no.11401415),Tian Yuan Foundation (nos.11226208 and 11426139), Natural Science Foundation of the Jiangsu Higher Education Institutions of China (no.13KJB110025), Postdoctoral Research Program of Jiangsu Province of China (no.1402111C) and Jiangsu Overseas Research and Training Program for Prominent University Young and Middle-aged Teachers and Presidents.

and integer-valued, Rudin (1973) studied the lower limit for the ratio $\frac{P(\sum_{i=1}^{\tau} X_i > x)}{\overline{F}(x)}$ as $x \to \infty$ for some special kinds of heavy-tailed distributions F; Foss and Korshunov (2007) investigated the case in which F is a general heavy-tailed distribution and $\tau = 2$. Based on these work, Denisov et al. (2008a,b) established more results on the lower limit of the ratio $\frac{P(\sum_{i=1}^{\tau} X_i > x)}{\overline{F}(x)}$. Watanabe and Yamamuro (2010) and Yu et al. (2010) considered the similar problems for real-valued summands and obtained the upper bound of the lower limit and the lower bound of the upper limit for the ratio $\frac{P(\sum_{i=1}^{\tau} X_i > x)}{\overline{F}(x)}$.

However, in the case that the summands are non-identically distributed, the research has developed slowly. Theorem 9 of Foss and Korshunov (2007) obtained the following result.

Theorem 1.A *Suppose that X_1 and X_2 are nonnegative r.v.s with distributions F_1 and F_2. If F_1 is heavy-tailed, then*

$$\liminf_{x \to \infty} \frac{P(X_1 + X_2 > x)}{\overline{F_1}(x) + \overline{F_2}(x)} = 1. \tag{1.1}$$

Theorems 3.1 and 3.2 of Yuen and Yin (2012) proved that for some dependent real-valued r.v.s with distributions belonging to some subclasses of the heavy-tailed class, the number of the summands in (1.1) can be arbitrarily finite, namely for any $m \geq 1$,

$$\liminf_{x \to \infty} \frac{P(\sum_{i=1}^{m} X_i > x)}{\sum_{i=1}^{m} \overline{F_i}(x)} = 1. \tag{1.2}$$

This paper mainly attempts to establish (1.2) for all nonnegative heavy-tailed r.v.s. As for real-valued summands with either heavy or light tails, some asymptotic results for the ratio $\frac{P(\sum_{i=1}^{m} X_i > x)}{\sum_{i=1}^{m} \overline{F_i}(x)}$ are also derived. What's more, asymptotic behavior for local distributions and densities of the sums is investigated at the same time.

The main result of the paper is as follows.

Theorem 1.1. *Let $X_i, 1 \leq i \leq m$ be nonnegative r.v.s with distributions $F_i, 1 \leq i \leq m$. If F_1 is heavy-tailed, then (1.2) holds.*

The rest of the paper consists of 3 sections. The proof of Theorem 1.1 is presented in Section 2. Some asymptotic results for the ratio $\frac{P(\sum_{i=1}^{m} X_i > x)}{\sum_{i=1}^{m} \overline{F_i}(x)}$ for real-valued summands with either heavy or light tails are derived in Section 3. In Section 4, some asymptotic results for local distributions and densities of the sums are established.

§2 Proof of Theorem 1.1

Hereafter, all limits are taken as $n \to \infty$ unless otherwise stated. And we write $a_n = o(b_n)$, if $\lim a_n/b_n = 0$; $a_n \sim b_n$, if $\lim a_n/b_n = 1$ and $a_n = O(b_n)$, if $\limsup a_n/b_n < \infty$.

To prove Theorem 1.1, we need some lemmas, which are of independent interest in their own right. It follows from Theorem 1.A that for any heavy-tailed distribution F supported on $[0, \infty)$, there exists a sequence of positive numbers $x_n \uparrow \infty$ such that

$$\overline{F^{*2}}(x_n) \sim 2\overline{F}(x_n). \tag{2.1}$$

Furthermore, the following lemma states that such a distribution is piecewise long-tailed at the left-hand sides of the points $x_n, n \geq 1$.

Lemma 2.1. *Suppose that F is a distribution supported on $[0, \infty)$ and $\{x_n\}_{n\geq 1}$ is a sequence of positive numbers increasing to ∞. If (2.1) holds, then there exists another sequence of positive numbers $t_n \uparrow \infty$ such that $t_n = o(x_n)$ and*

$$\overline{F}(x_n - t_n) \sim \overline{F}(x_n).$$

Since the proof of Lemma 2.1 is similar to that of Lemma 2.2 of Yu et al. (2010), we omit the details. For a heavy-tailed distribution F with a finite mean, the above property was proved by (10) of Foss and Korshunov (2007).

The next lemma is due to Theorem 2.11 of Foss et al. (2013).

Lemma 2.2. *Let $X_i, 1 \leq i \leq m$ be nonnegative r.v.s with distributions $F_i, 1 \leq i \leq m$. If $\sum_{i=1}^m \overline{F_i}(x) > 0$ for all $x > 0$, then*

$$\liminf_{x\to\infty} \frac{P(\sum_{i=1}^m X_i > x)}{\sum_{i=1}^m \overline{F_i}(x)} \geq 1.$$

The following lemma seems to be intuitive, but it is helpful to prove Lemma 2.4. So we deliver a complete proof.

Lemma 2.3. *Suppose that $\{a_{in}\}_{n\geq 1}$ and $\{b_{in}\}_{n\geq 1}, 1 \leq i \leq m$ are sequences of positive numbers. If for all $1 \leq i \leq m$,*

$$\liminf \frac{a_{in}}{b_{in}} \geq 1 \tag{2.2}$$

and

$$\sum_{i=1}^m a_{in} \sim \sum_{i=1}^m b_{in}, \tag{2.3}$$

then

$$\max_{1\leq i\leq m} \left|a_{in} - b_{in}\right| = o\left(\sum_{i=1}^m b_{in}\right). \tag{2.4}$$

Proof. We assume that (2.4) does not hold, then there exist some $\varepsilon_0 > 0$ and a sequence of positive integers $n_k \uparrow \infty$ as $k \to \infty$ such that for all $n_k, k \geq 1$,

$$\max_{1\leq i\leq m} \left|a_{i,n_k} - b_{i,n_k}\right| \geq \varepsilon_0 \sum_{i=1}^m b_{i,n_k}. \tag{2.5}$$

Since the numbers $b_{in},\ n \geq 1,\ 1 \leq i \leq m$ are positive, it follows from (2.2) that for $\frac{\varepsilon_0}{2m} > 0$, there exists $N > 0$ such that for all $n \geq N$,

$$\min_{1\leq i\leq m} (a_{in} - b_{in}) \geq -\frac{\varepsilon_0}{2m} \sum_{i=1}^m b_{in}. \tag{2.6}$$

Without loss of generality, we may assume that $n_1 > N$. Thus by (2.5) and (2.6), for all $n_k, k \geq 1$,

$$\max_{1\leq i\leq m} (a_{i,n_k} - b_{i,n_k}) = \max_{1\leq i\leq m} \left|a_{i,n_k} - b_{i,n_k}\right| \geq \varepsilon_0 \sum_{i=1}^m b_{i,n_k}. \tag{2.7}$$

Therefore, it follows from (2.6) and (2.7) that

$$\begin{aligned}\sum_{i=1}^{m} a_{i,n_k} - \sum_{i=1}^{m} b_{i,n_k} &\geq \max_{1\leq i\leq m}(a_{i,n_k} - b_{i,n_k}) + (m-1)\min_{1\leq i\leq m}(a_{i,n_k} - b_{i,n_k})\\ &\geq \varepsilon_0 \sum_{i=1}^{m} b_{i,n_k} - (m-1)\frac{\varepsilon_0}{2m}\sum_{i=1}^{m} b_{i,n_k}\\ &> \frac{\varepsilon_0}{2}\sum_{i=1}^{m} b_{i,n_k},\end{aligned}$$

which contradicts (2.3). Thus (2.4) is proved. □

The following lemma plays an important role in the proof of Theorems 1.1 and 3.1.

Lemma 2.4. *Suppose that the distributions $F_i, 1 \leq i \leq m$ are supported on $[0,\infty)$. If F_1 is heavy-tailed, then there exists a sequence of positive numbers $x_n \uparrow \infty$ such that for two subsets $\{i_1, i_2\cdots, i_k\}$ and $\{j_1, j_2\cdots, j_l\}$ of the set $\{1, 2\cdots, m\}$, where $1 \leq k,\ l \leq m$, we have*

$$\begin{aligned}&\overline{(F_{i_1} * F_{i_2} * \cdots * F_{i_k}) * (F_{j_1} * F_{j_2} * \cdots * F_{j_l})}(x_n)\\ =\ &\overline{F_{i_1} * F_{i_2} * \cdots * F_{i_k}}(x_n) + \overline{F_{j_1} * F_{j_2} * \cdots * F_{j_l}}(x_n) + o(\overline{F_1 * F_2 * \cdots * F_m}(x_n)).\end{aligned}$$

Proof. Let

$$\begin{aligned}H &= (2^m-1)^{-1}\sum_{j=1}^{m}\sum_{1\leq a_1<a_2<\cdots,a_j\leq m} F_{a_1} * F_{a_2} * \cdots * F_{a_j}\\ &\equiv (2^m-1)^{-1}\sum_{i=1}^{2^m-1} G_i.\end{aligned}$$

Since F_1 is heavy-tailed, the distribution H is heavy-tailed, too. Just as discussed at the beginning of the section, there exists a sequence of positive numbers $x_n \uparrow \infty$ such that

$$\overline{H^{*2}}(x_n) \sim 2\overline{H}(x_n). \tag{2.8}$$

By the definition of H, we have

$$\frac{\overline{H^{*2}}(x_n)}{2\overline{H}(x_n)} = \frac{\sum_{i=1}^{2^m-1}\sum_{j=1}^{2^m-1}\overline{G_i * G_j}(x_n)}{2(2^m-1)\sum_{i=1}^{2^m-1}\overline{G_i}(x_n)} = \frac{\sum_{i=1}^{2^m-1}\sum_{j=1}^{2^m-1}\overline{G_i * G_j}(x_n)}{\sum_{i=1}^{2^m-1}\sum_{j=1}^{2^m-1}\big(\overline{G_i}(x_n) + \overline{G_j}(x_n)\big)}. \tag{2.9}$$

Obviously, for two distributions V_1 and V_2, $\overline{V_1 * V_2}(x) > 0$ for all $x > 0$ if and only if $\overline{V}_1(x) + \overline{V}_2(x) > 0$ for all $x > 0$. So in (2.9), the numbers of the non-zero terms in the numerator and denominator are equal. Moreover, by Lemma 2.2, for any $1 \leq i, j \leq 2^m - 1$, if $\overline{G_i}(x) + \overline{G_j}(x) > 0$ for all $x > 0$, then

$$\liminf \frac{\overline{G_i * G_j}(x_n)}{\overline{G_i}(x_n) + \overline{G_j}(x_n)} \geq 1. \tag{2.10}$$

Hence, by (2.8)-(2.10), Lemma 2.3 and the definition of H, for any $1 \leq i,\ j \leq 2^m - 1$,

$$\begin{aligned}\overline{G_i * G_j}(x_n) &= \overline{G_i}(x_n) + \overline{G_j}(x_n) + o\big(2\overline{H}(x_n)\big)\\ &= \overline{G_i}(x_n) + \overline{G_j}(x_n) + o(\overline{F_1 * F_2 * \cdots * F_m}(x_n)).\end{aligned} \tag{2.11}$$

One might easily find that for two subsets $\{i_1, i_2 \cdots, i_k\}$ and $\{j_1, j_2 \cdots, j_l\}$ of the set $\{1, 2 \cdots, m\}$, there exist two numbers $1 \le k_0, l_0 \le 2^m - 1$ such that the distributions $F_{i_1} * F_{i_2} * \cdots * F_{i_k}$ and $F_{j_1} * F_{j_2} * \cdots * F_{j_l}$ are identical with the distributions G_{k_0} and G_{l_0}, respectively. Thus we finish the proof by (2.11). $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.4, there exists a sequence of positive numbers $x_n \uparrow \infty$ such that for all $2 \le k \le m$,

$$\overline{F_1 * F_2 * \cdots * F_k}(x_n) = \overline{F_1 * F_2 * \cdots * F_{k-1}}(x_n) + \overline{F_k}(x_n) + o(\overline{F_1 * F_2 * \cdots * F_m}(x_n)).$$

Thus we immediately get

$$\overline{F_1 * F_2 * \cdots * F_m}(x_n) = \sum_{i=1}^{m} \overline{F_i}(x_n) + o(\overline{F_1 * F_2 * \cdots * F_m}(x_n)),$$

which, together with Lemma 2.2, implies (1.2). Thus we finish the proof. $\square$

§3 Asymptotic results for tail distributions of sums of real-valued r.v.s

In Section 1, we get the exact lower limit for the ratio $\frac{P(\sum_{i=1}^m X_i > x)}{\sum_{i=1}^m \overline{F_i}(x)}$ as $x \to \infty$, where $X_i, 1 \le i \le m$ are nonnegative and F_1 is heavy-tailed. In this section, we attempt to deal with real-valued r.v.s. $X_i, 1 \le i \le m$. Instead of the exact lower limit, we get asymptotic upper bound for the lower limit and asymptotic lower bound for the upper limit of the ratio. Related discussion for randomly stopped sums with identically distributed summands may be found in Lemma 4 of Denisov (2008a), Proposition 4.1 (ii) and Lemma 5.1 of Watanabe and Yamamuro (2010), and Theorems 1.1 and 1.2 of Yu et al. (2010).

We first study the heavy-tailed case.

3.1 The heavy-tailed case

Theorem 3.1. *Let $X_i, 1 \le i \le m$ be real-valued r.v.s with distributions $F_i, 1 \le i \le m$. If F_1 is heavy-tailed, then*

$$\liminf_{x\to\infty} \frac{P\left(\sum_{i=1}^m X_i > x\right)}{\sum_{i=1}^m \overline{F_i}(x)} \le 1 \tag{3.1}$$

and

$$\limsup_{x\to\infty} \frac{P(\sum_{i=1}^m X_i > x)}{\sum_{i=1}^m \overline{F_i}(x)} \ge 1. \tag{3.2}$$

Proof. Let F_i^+ denote the distribution of $X_i^+ = X_i I_{\{X_i \ge 0\}}, 1 \le i \le m$, where the notation I_A represents the indictor function of the set A. Obviously,

$$P\left(\sum_{i=1}^m X_i > x\right) \le P\left(\sum_{i=1}^m X_i^+ > x\right). \tag{3.3}$$

Dividing both sides of (3.3) by $\sum_{i=1}^{m}\overline{F_i}(x)$ and taking lower limits as $x\to\infty$, we immediately get (3.1) by Theorem 1.1.

Next we prove (3.2). By Lemma 2.4, there exists a sequence of positive numbers $x_n\uparrow\infty$ such that

$$\overline{(F_1^+ * F_2^+ * \cdots * F_m^+)^{*2}}(x_n)\sim 2\overline{F_1^+ * F_2^+ * \cdots * F_m^+}(x_n), \tag{3.4}$$

and for all $2\le k\le m$,

$$\begin{aligned}\overline{F_1^+ * F_2^+ * \cdots * F_k^+}(x_n) &= \overline{F_1^+ * F_2^+ * \cdots * F_{k-1}^+}(x_n)+\overline{F_k^+}(x_n)\\ &+ o(\overline{F_1^+ * F_2^+ * \cdots * F_m^+}(x_n)).\end{aligned} \tag{3.5}$$

By (3.5) and the proof of Theorem 1.1, we have

$$\overline{F_1^+ * F_2^+ * \cdots * F_m^+}(x_n) \sim \sum_{i=1}^{m}\overline{F_i^+}(x_n). \tag{3.6}$$

By (3.4) and Lemma 2.1, there exists a sequence of positive numbers $\{t_n\}_{n\ge1}$ such that $t_n\uparrow\infty$, $t_n=o(x_n)$ and

$$\overline{F_1^+ * F_2^+ * \cdots * F_m^+}(x_n-t_n)\sim\overline{F_1^+ * F_2^+ * \cdots * F_m^+}(x_n). \tag{3.7}$$

Without loss of generality, we assume that $x_n>t_n$ for all n, then by (3.6) and (3.7),

$$\begin{aligned}P\left(\sum_{i=1}^{m}X_i>x_n-t_n\right) &\ge \sum_{i=1}^{m}\overline{F_i}(x_n)\cdot\prod_{j\ne i}\left(F_j\left(\frac{t_n}{m}\right)-F_j\left(-\frac{t_n}{m}\right)\right)\\ &\sim \sum_{i=1}^{m}\overline{F_i^+}(x_n)\\ &\sim \overline{F_1^+ * F_2^+ * \cdots * F_m^+}(x_n-t_n).\end{aligned} \tag{3.8}$$

Meanwhile, Lemma 2.2 implies that

$$\liminf\frac{\overline{F_1^+ * F_2^+ * \cdots * F_m^+}(x_n-t_n)}{\sum_{i=1}^{m}\overline{F_i}(x_n-t_n)}\ge 1. \tag{3.9}$$

Combining (3.8) with (3.9), we immediately get (3.2). □

3.2 The light-tailed case

Motivated by Lemma 9 of Foss et al. (2007) and Lemma 4 of Denisov et al. (2008a), we establish some light-tailed results corresponding to Theorem 3.1 in this subsection.

For any distribution F, denote its Laplace transform at the point $\alpha\ge0$ by

$$\widehat{F}(\alpha)=\int_{-\infty}^{\infty}e^{\alpha y}F(dy).$$

Let

$$\gamma_F=\sup\left\{\alpha\ge0:\widehat{F}(\alpha)<\infty\right\}.$$

Our main result is as follows.

Theorem 3.2. *Let $X_i, 1 \le i \le m$ be real-valued r.v.s with distributions $F_i, 1 \le i \le m$. If $0 < \gamma = \min_{1\le i\le m} \gamma_{F_i} < \infty$ and $\max_{1\le i\le m} \widehat{F}_i(\gamma) < \infty$, then*

$$\liminf_{x\to\infty} \frac{P(\sum_{i=1}^m X_i > x)}{\sum_{i=1}^m \prod_{j\neq i} \widehat{F_j}(\gamma)\overline{F_i}(x)} \le 1 \tag{3.10}$$

and

$$\limsup_{x\to\infty} \frac{P(\sum_{i=1}^m X_i > x)}{\sum_{i=1}^m \prod_{j\neq i} \widehat{F_j}(\gamma)\overline{F_i}(x)} \ge 1. \tag{3.11}$$

Proof. The proofs of (3.10) and (3.11) are similar, so we only prove (3.10).

Write the left-hand side of (3.10) as $\underline{c}$, then for any $\varepsilon > 0$, there exists $x_1 > 0$ such that when $x > x_1$,

$$P\left(\sum_{i=1}^m X_i > x\right) > (\underline{c} - \varepsilon)\sum_{i=1}^m \prod_{j\neq i} \widehat{F_j}(\gamma)\overline{F_i}(x). \tag{3.12}$$

For all $1 \le i \le m$, define

$$H_i(dx) = (\widehat{F}_i(\gamma))^{-1} e^{\gamma x} F_i(dx), x \in (-\infty, \infty),$$

then

$$\overline{H_i}(x) = (\widehat{F}_i(\gamma))^{-1}\left(\overline{F_i}(x)e^{\gamma x} + \gamma\int_x^\infty \overline{F_i}(t)e^{\gamma t}dt\right). \tag{3.13}$$

It is obvious that

$$H_1 * H_2 * \cdots * H_m(dx) = \prod_{i=1}^m (\widehat{F}_i(\gamma))^{-1} e^{\gamma x} F_1 * F_2 * \cdots * F_m(dx), x \in (-\infty, \infty),$$

thus we have

$$\begin{aligned} & \overline{H_1 * H_2 * \cdots * H_m}(x) \\ = & \prod_{j=1}^m (\widehat{F_j}(\gamma))^{-1} \int_x^\infty e^{\gamma t} F_1 * F_2 * \cdots * F_m(dt) \\ = & \prod_{j=1}^m (\widehat{F_j}(\gamma))^{-1} \left(e^{\gamma x}\overline{F_1 * F_2 * \cdots * F_m}(x) + \gamma\int_x^\infty e^{\gamma t}\overline{F_1 * F_2 * \cdots * F_m}(t)dt\right). \end{aligned} \tag{3.14}$$

Substituting (3.12) into (3.14), we have

$$\begin{aligned} & \overline{H_1 * H_2 * \cdots * H_m}(x) \\ \ge & (\underline{c} - \varepsilon)\left(\sum_{i=1}^m (\widehat{F}_i(\gamma))^{-1}\overline{F_i}(x)e^{\gamma x} + \gamma\int_x^\infty \sum_{i=1}^m (\widehat{F}_i(\gamma))^{-1}\overline{F_i}(t)e^{\gamma t}dt\right) \\ = & (\underline{c} - \varepsilon)\sum_{i=1}^m (\widehat{F}_i(\gamma))^{-1}\left(\overline{F_i}(x)e^{\gamma x} + \gamma\int_x^\infty \overline{F_i}(t)e^{\gamma t}dt\right) \\ = & (\underline{c} - \varepsilon)\sum_{i=1}^m \overline{H_i}(x), \end{aligned}$$

where in the last step (3.13) was used. Thus we have

$$\underline{c}-\varepsilon \le \liminf_{x\to\infty} \frac{\overline{H_1 * H_2 * \cdots * H_m}(x)}{\sum_{i=1}^m \overline{H_i}(x)}. \tag{3.15}$$

Note that at least one of the distributions $H_1, \cdots, H_m$ is heavy-tailed, so by Theorem 3.1, we have

$$\liminf_{x\to\infty} \frac{\overline{H_1 * H_2 * \cdots * H_m}(x)}{\sum_{i=1}^m \overline{H_i}(x)} \le 1. \tag{3.16}$$

Since ε is arbitrary, by (3.15) and (3.16), we have $\underline{c} \le 1$. This completes the proof. $\square$

Remark 3.1. We point out that if we allow $\gamma = 0$ in Theorem 3.2, then $\widehat{F}_i(\gamma) = 1, 1 \le i \le m$ and at least one of the summands is heavy-tailed, thus the conclusion coincides with Theorem 3.1.

§4 Asymptotic results for local distributions and densities of sums of r.v.s

In this section, we present the local and density versions of Theorems 3.1 and 3.2. To this end, we give some notations.

For any $0 < T < \infty$, let $\Delta_T = (0, T]$ and $x + \Delta_T = (x, x+T]$. For a distribution F, let $F(x+\Delta_T) = F(x+T) - F(x), x \in (-\infty, \infty)$. For two measurable functions f and g, let

$$f \otimes g(x) \equiv \int_{-\infty}^{\infty} f(x-y)g(y)dy.$$

So if $F_i, 1 \le i \le m$ are absolutely continuous with densities $f_i, 1 \le i \le m$, then $f_1 \otimes f_2 \cdots \otimes f_m$ is the density of $\sum_{i=1}^m X_i$. Similar to the notations in Section 3, for any $\alpha \ge 0$, we write

$$\widehat{f}(\alpha) = \int_{-\infty}^{\infty} e^{\alpha y} f(y)dy$$

and let

$$\gamma_f = \sup\left\{\alpha \ge 0 : \widehat{f}(\alpha) < \infty\right\}.$$

One might easily find that, if f is the density of F, then for any $\alpha \ge 0$, $\widehat{f}(\alpha) = \widehat{F}(\alpha)$ and $\gamma_f = \gamma_F$.

We first study the asyptotics of the local distribution $P\left(\sum_{i=1}^m X_i \in x + \Delta_T\right)$ as $x \to \infty$.

Theorem 4.1. *Let $X_i, 1 \le i \le m$ be real-valued r.v.s. with distributions $F_i, 1 \le i \le m$. Suppose that $\gamma = \min_{1\le i\le m} \gamma_{F_i} < \infty$ and $\max_{1\le i\le m} \widehat{F}_i(\gamma) < \infty$. If $\sum_{i=1}^m F_i(x+\Delta_T)$ is eventually positive, then*

$$\liminf_{x\to\infty} \frac{P(\sum_{i=1}^m X_i \in x+\Delta_T)}{\sum_{i=1}^m \prod_{j\ne i} \widehat{F_j}(\gamma) F_i(x+\Delta_T)} \le 1 \tag{4.1}$$

and

$$\limsup_{x\to\infty} \frac{P(\sum_{i=1}^m X_i \in x+\Delta_T)}{\sum_{i=1}^m \prod_{j\ne i} \widehat{F_j}(\gamma) F_i(x+\Delta_T)} \ge 1. \tag{4.2}$$

Proof. We only prove (4.1), and (4.2) can be proved similarly.

Suppose that (4.1) does not hold, then there exist two numbers $\varepsilon_0 > 0$ and $N_0 > 0$ such that, for all $x > N_0$,

$$P\left(\sum_{i=1}^{m} X_i \in x + \Delta_T\right) > (1+\varepsilon_0)\sum_{i=1}^{m}\prod_{j\neq i}\widehat{F_j}(\gamma)F_i(x+\Delta_T).$$

Thus

$$\begin{aligned} P\left(\sum_{i=1}^{m} X_i > x\right) &= \sum_{n=0}^{\infty} P\left(\sum_{i=1}^{m} X_i \in x + nT + \Delta_T\right) \\ &> (1+\varepsilon_0)\sum_{n=0}^{\infty}\sum_{i=1}^{m}\prod_{j\neq i}\widehat{F_j}(\gamma)F_i(x+nT+\Delta_T) \\ &= (1+\varepsilon_0)\sum_{i=1}^{m}\prod_{j\neq i}\widehat{F_j}(\gamma)\overline{F_i}(x), \end{aligned}$$

which, in the case $\gamma = 0$, contradicts Theorem 3.1 and in the case $\gamma > 0$, contradicts Theorem 3.2. This completes the proof. □

Next, we present the density version of Theorems 3.1 and 3.2, which is inspired by Lemma 2 of Foss and Korshunov (2007) and Theorems 3.1 and 3.2 of Yu et al. (2010). Since the proof is quite similar to that of the local version, we omit the details.

Theorem 4.2. *Let $X_i, 1 \le i \le m$ be real-valued r.v.s. with distribution densities $f_i, 1 \le i \le m$. Suppose that $\gamma = \min_{1\le i\le m} \gamma_{f_i} < \infty$ and $\max_{1\le i\le m} \widehat{f_i}(\gamma) < \infty$. If $\sum_{i=1}^{m} f_i(x)$ is eventually positive, then*

$$\liminf_{x\to\infty} \frac{f_1 \otimes f_2 \cdots \otimes f_m(x)}{\sum_{i=1}^{m}\prod_{j\neq i}\widehat{f_j}(\gamma)f_i(x)} \le 1$$

and

$$\limsup_{x\to\infty} \frac{f_1 \otimes f_2 \cdots \otimes f_m(x)}{\sum_{i=1}^{m}\prod_{j\neq i}\widehat{f_j}(\gamma)f_i(x)} \ge 1.$$

Acknowledgements The authors would like to express their deep gratitude to the referees for their valuable comments which help a lot in the improvement of the paper.

References

[1] D Denisov, S Foss, D Korshunov. *On lower limits and equivalence for distribution tails of random stopped sums*, Bernoulli, 2008a, 14: 391-404.

[2] D Denisov, S Foss, D Korshunov. *Lower limits for distribution tails of randomly stopped sums*, Theor Probab Appl, 2008b, 52: 690-699.

[3] S Foss, D Korshunov. *Lower limits and equivalences for convolution tails*, Ann Probab, 2007, 35: 366-383.

[4] S Foss, D Korshunov, S Zachary. *An introduction to heavy-tailed and subexponential distributions*, 2013, 2nd ed, New York, Springer.

[5] S I Resnick. *Heavy-tail phenomena: probabilistic and statistical modeling*, 2007, Springer.

[6] W Rudin. *Limits of ratios of tails of measures*, Ann Probab, 1973, 1: 982-994.

[7] Z Su, C Su, Z Hu, J Liu. *On domination problem of non-negative distributions*, Front Math China, 2009, 4: 681-696.

[8] T Watanabe, K Yamamuro. *Local subexponentiality and self-decomposability*, J Theor Probab, 2010, 23: 1039-1067.

[9] C Yu, Y Wang, Z Cui. *Lower limits and upper limits for tails of random sums supported on* $\mathbf{R}$, Statist Probab Lett, 2010, 80: 1111-1120.

[10] K C Yuen, C Yin. *Asymptotic results for tail probabilities of sums of dependent and heavy-tailed random variables*, Chinese Annals of Mathematics, Series B, 2012, 33: 557-568.

[1] School of Sciences, Nantong University, Nantong 226019, China

[2] School of Mathematical Sciences, Soochow University, Suzhou 215006, China

[3] The Statistics and Operations Research Department, the University of North Carolina at Chapel Hill, NC 27514, USA

Email: dycheng@suda.edu.cn