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Limit theorems for supremum of Gaussian processes over a random interval

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Abstract. Let $\{X(t), t \geq 0\}$ be a centered stationary Gaussian process with correlation $r(t)$ such that $1 - r(t)$ is asymptotic to a regularly varying function. With $\mathbf{T}$ being a nonnegative random variable and independent of $X(t)$, the exact asymptotics of $P(\sup_{t\in[0,\mathbf{T}]} X(t) > x)$ is considered, as $x \to \infty$.

§1 Introduction

Let $\{X(t), t \geq 0\}$ be a centered stationary Gaussian process whose correlation function $r(t) := \mathrm{Cov}(X(s), X(s+t))$ satisfies the assumption

$$1 - r(t) = |t|^\alpha H(t) + o(|t|^\alpha H(t)) \text{ as } t \to 0, \tag{1}$$

where $0 < \alpha < 2$ and C is a positive constant. A similar assumption can be found in Qualls and Watanabe (1972). Under the assumption, we consider the exact asymptotics of $\lim_{x\to\infty} P(\sup_{t\in[0,\mathbf{T}]} X(t) > x)$ with random variables $\mathbf{T}$ being nonnegative and independent of $X(t)$. The asymptotics of maxima of processes has received a great deal of attention. Pickands (1969) first proved that

$$\lim_{x\to\infty} \frac{P(\sup_{t\in[0,T]} X(t) > x)}{x^{2/\alpha}\psi(x)} = TC^{1/\alpha}Q_\alpha,$$

providing $r(t) < 1$ for all $t > 0$, where Q_α is called the Pickands constant, which was extended to more general cases in Qualls and Watanabe (1972). But neither of two papers considered the random interval case. The momentousness of extremes of random processes or random fields over random intervals comes from theoretic questions in extreme value (see, e.g., Kozubowski et

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al. (2006) for a detailed introduction) and applied problems in other subjects (see, e.g., Dębicki et al. (2004)). For some recent study on extremes of random processes or random fields over random intervals, we refer the readers to Arendarczyk and Dębicki (2012), Tan and Hashorva (2013a,b) and Dębicki et al. (2014). But these results were only based on the assumption of $H(t)$ being a constant in condition (1).

The rest of the paper is organized as follows. In Section 2, notation and the main result are presented. Section 3 contains some lemmas. Proofs of the main result are postponed in Section 4. Throughout, K is a constant which may vary from line to line and $[x]$ denotes the integer part of x.

§2 Notation and main results

Let $\{X(t), t \geq 0\}$ be a centered stationary Gaussian process with a.s. continuous sample paths and correlation function $r(t)$. We first list some definitions of slowly varying functions.

Definition 2.1. A positive function $H(t)$ defined for $t > 0$ varies slowly at zero (at infinity), if for all $x > 0$,

$$\lim_{\substack{t\to 0 \\ (t\to\infty)}} \frac{H(xt)}{H(t)} = 1.$$

The function $H(t)$ varies slowly at zero if and only if

$$H(t) = a(t)\exp(\int_t^1 \varepsilon(x)/x dx),$$

where $\varepsilon(t) \to 0$ and $a(t) \to A$ as $t \to 0$ $(0 < A < \infty)$. More properties of slowly varying functions can be found in Karamata (1930), Adamovic (1966) and Feller (1966).

Definition 2.2. The slowly varying function $H(t)$ is said to be "normalized" if $a(t) \equiv A$ in the formula above.

The following conditions will be used in the main result:

D1. $r(t) = 1 - |t|^\alpha H(|t|) + o(|t|^\alpha H(|t|))$ as $t \to 0$, where $H(t)$ is normalized slowly varying at zero (written $H(t) \in RV_0^0$) and $0 < \alpha < 2$;

D2. $r(t) < 1$ for all $t > 0$;

D3. $r(t)(\log t)^{1+\varepsilon_0} \to 0$ as $t \to \infty$, where $\varepsilon_0 > 0$ can be made arbitrarily small.

Remark 2.1. D1 is a weak assumption on the correlation function when considering extremes of Gaussian processes. An example that is easily verified to satisfy such an assumption is $r(t) \equiv \exp(-e^{|t|}|t|^\alpha)$ with $\alpha = 0.5$.

Let $\mathbf{T}$ be a nonnegative random variable independent from $X(t)$, and we consider two common types of probability distributions:

C1. $\mathbf{T}$ is integrable, i.e., $E\mathbf{T} < \infty$;

C2. $\mathbf{T}$ has regularly varying tail distribution with parameter $\lambda \in (0, 1)$, i.e., $P(\mathbf{T} > t) = L(t)t^{-\lambda}$, where $L(\cdot)$ is slowly varying at ∞ (written $(L(t)t^{-\lambda}) \in RV_{-\lambda}^\infty$).

In the sequel, we use the constants $Q_\alpha(a)$ and Q_α, for given $\alpha \in (0, 2]$, which defined by the following limits

$$Q_\alpha = \lim_{T\to\infty} \frac{1}{T} \int_0^\infty e^s P(\sup_{0<t<T} Y(t) > s) ds$$

and

$$Q_\alpha(a) = \lim_{n\to\infty} \frac{1}{n} \int_0^\infty e^s P(\max_{1\le k\le n} Y(ka) > s) ds$$

with $a > 0$ being constant, where $\{Y(t), t \ge 0\}$ is non-stationary Gaussian process with $Y(0) = 0$ a.s., $E(Y(t)) = -|t|^\alpha/2$, $\mathrm{Cov}(Y(t), Y(s)) = (|t|^\alpha + |s|^\alpha - |t-s|^\alpha)/2$. Moreover, let $\psi(x) = (2\pi)^{-1/2} x^{-1} \exp(-x^2/2)$ and $\widetilde{\sigma}^{\leftarrow}(\cdot)$ be the inverse of $\widetilde{\sigma}(s) = 2^{1/2} s^{\alpha/2} (H(s))^{1/2}$, $s > 0$.

Under the above-mentioned conditions, we derive the following result.

Theorem 2.1. *Let $\{X(t), t \ge 0\}$ be a separable and centered stationary Gaussian process with a.s. continuous sample paths and correlation function $r(t)$ satisfying D1 and D2.*

(i) If the random variable $\boldsymbol{T}$ satisfies C1, then

$$\lim_{x\to\infty} \frac{P(\sup_{t\in[0,\boldsymbol{T}]} X(t) > x)}{\psi(x)/\widetilde{\sigma}^{\leftarrow}(1/x)} = E(\boldsymbol{T}) Q_\alpha.$$

(ii) If D3 further holds and the random variable $\boldsymbol{T}$ satisfies C2, then

$$\lim_{x\to\infty} \frac{P(\sup_{t\in[0,\boldsymbol{T}]} X(t) > x)}{(\psi(x)/\widetilde{\sigma}^{\leftarrow}(1/x))^\lambda L(x\widetilde{\sigma}^{\leftarrow}(1/x) \exp(\frac{x^2}{2}))} = \Gamma(1-\lambda) Q_\alpha^\lambda.$$

§3 Some lemmas

In this section we introduce and prove some auxiliary results as needed in the course of the next section.

For convenience, we rewrite Theorem 2.1 in Qualls and Watanabe (1972) as Lemma 3.1 below.

Lemma 3.1. *If conditions D1, D2 and $\lim_{t\to 0} r(t) \log(t) \to 0$ hold, then*

$$\lim_{x\to\infty} \frac{P(\sup_{t\in[0,T]} X(t) > x)}{\psi(x)/\widetilde{\sigma}^{\leftarrow}(1/x)} = T Q_\alpha. \tag{2}$$

Lemma 3.2. *Let $t > 0$ be fixed and $\Delta(x) = \widetilde{\sigma}^{\leftarrow}(1/x)$ for all $x \ge 1/\widetilde{\sigma}(\widetilde{\delta})$ (let $\widetilde{\delta}$ be a threshold value such that $\widetilde{\sigma}(s)$ is monotone on some small interval $(0, \delta)$). Let $q(x) = a\Delta(x)$ with $a > 0$ being constant. If conditions D1 and D2 hold, then for each interval I of length t,*

$$0 \le P(X(kq(x)) \le x, kq(x) \in I) - P(\sup_{s\in I} X(s) \le x) \le t\rho(a)\frac{1}{m(x)} + o\Big(\frac{1}{m(x)}\Big),$$

where $\rho(a) \to 0$ as $a \to 0$ and $m(x) = (Q_\alpha \psi(x)/\widetilde{\sigma}^{\leftarrow}(\frac{1}{x}))^{-1}$.

Proof. By the stationarity of $\{X(t), t \ge 0\}$, we have

$$\begin{aligned} 0 &\le P(X(kq(x)) \le x, kq(x) \in I) - P(\sup_{s\in I} X(s) \le x) \\ &\le P(X(0) > x) + P(X(kq(x)) \le x, kq(x) \in [0, t]) - P(\sup_{s\in[0,t]} X(s) \le x), \end{aligned} \tag{3}$$

where $P(X(0) > x) = o(1/m(x))$. Furthermore, Lemma 2.3 in Qualls and Watanabe (1972) yields

$$m(x)P\Big(\max_{0\le kq(x)\le t} X(kq(x)) > x\Big) = tQ_\alpha^{-1}\frac{Q_\alpha(a)}{a} + o(1) \tag{4}$$

and according to (2), we have

$$m(x)P\Big(\sup_{s\in[0,t]} X(s) > x\Big) = t + o(1). \tag{5}$$

Noting

$$\begin{aligned} & P(X(kq(x)) \le x, kq(x) \in [0,t]) - P(\sup_{s\in[0,t]} X(s) \le x) \\ = \ & P\Big(\sup_{s\in[0,t]} X(s) > x\Big) - P\Big(\max_{0\le kq(x)\le t} X(kq(x)) > x\Big), \end{aligned}$$

then, using (4) and (5) yields that (3) does not exceed

$$\frac{1}{m(x)}\Big(t\Big(1 - Q_\alpha^{-1}\frac{Q_\alpha(a)}{a}\Big) + o(1)\Big).$$

According to Qualls and Watanabe (1972), we have $\lim_{a\to 0} Q_\alpha(a)/a = Q_\alpha$ and hence $\rho(a) := 1 - Q_\alpha^{-1}(Q_\alpha(a)/a) \to 0$ as $a \to 0$. So, the proof is completed. □

Without loss of generality, in the sequel we suppose $x \ge 1/\widetilde{\sigma}(\widetilde{\delta})$.

Lemma 3.3. *If $0 < \alpha < 2$, then for any $\varepsilon > 0$,*

$$\lim_{x\to\infty}\Big(\frac{1}{x}\Big)^{-\varepsilon}\frac{\widetilde{\sigma}^{\leftarrow}(1/x)}{(1/x)^{2/\alpha}} = \infty \ \textit{and} \ \lim_{x\to\infty}\Big(\frac{1}{x}\Big)^{\varepsilon}\frac{\widetilde{\sigma}^{\leftarrow}(1/x)}{(1/x)^{2/\alpha}} = 0.$$

Proof. Obviously, $\widetilde{\sigma}(s)$ satisfies the conditions of Theorem 1.5.12 of Bingham et al. (1987) since $\widetilde{\sigma}(s) \in RV^0_{\alpha/2}$ and is monotone increasing on $(0, \widetilde{\delta})$ with $\widetilde{\sigma}(0) = 0$. So, we have $\widetilde{\sigma}^{\leftarrow}(y) \in RV^0_{2/\alpha}$ and thus $\widetilde{\sigma}^{\leftarrow}(y)/y^{2/\alpha} \in RV^0_0$. Using (1.4) on page 581 in Qualls and Watanabe (1972), we complete the proof. □

Lemma 3.4. *Let $\varepsilon > 0$ be given, and suppose that D1, D2 and D3 hold. Let $T \sim \tau/\mu$ for $\tau > 0$ fixed, as $T \to \infty$ and with $\mu = Q_\alpha\psi(x)/\widetilde{\sigma}^{\leftarrow}(1/x)$ and let $q = a\widetilde{\sigma}^{\leftarrow}(1/x)$. Then*

$$\frac{T}{q}\sum_{\varepsilon\le kq\le T} |r(kq)| \exp\Big(-\frac{x^2}{1+|r(kq)|}\Big) \to 0, \ \textit{as } T \to \infty. \tag{6}$$

Proof. We first need to show $x^2 \sim 2\log T$ and $\exp(-x^2/2) \le KT^{-1}$. To accomplish this, we begin at the condition $T \sim \tau/\mu$, i.e., $T\mu = TQ_\alpha\psi(x)/\widetilde{\sigma}^{\leftarrow}(1/x) \to \tau > 0$. Taking logarithms yields that

$$\log T + \log(Q_\alpha(2\pi)^{-1}) - \frac{x^2}{2} - \log x - \log(\widetilde{\sigma}^{\leftarrow}(1/x)) \to \log\tau, \tag{7}$$

i.e.,

$$x^2 = 2\log T + 2\log((2\pi)^{-1}Q_\alpha) - 2\log(x\widetilde{\sigma}^{\leftarrow}(1/x)) - 2\log\tau + o(1). \tag{8}$$

Let $0 < \varepsilon_1 \le \max\{\varepsilon_0/2, (2-\alpha)/\alpha\}$ $(0 < \alpha < 2)$. According to Lemma 3.3, there exist positive constants M_1 and M_2 such that, for sufficiently large x,

$$M_1 x^{-\varepsilon_1 + (\alpha-2)/\alpha} < x\widetilde{\sigma}^{\leftarrow}(1/x) < M_2 x^{\varepsilon_1 + (\alpha-2)/\alpha}, \tag{9}$$

which furthermore yields

$$(M_1 x^{-\varepsilon_1 - 2/\alpha})^{1/x^2} < (\widetilde{\sigma}^{\leftarrow}(1/x))^{1/x^2} < (M_2 x^{\varepsilon_1 - 2/\alpha})^{1/x^2}. \tag{10}$$

(10) implies that $(\widetilde{\sigma}^{\leftarrow}(1/x))^{1/x^2} \to 1$ and hence $\log(\widetilde{\sigma}^{\leftarrow}(1/x))/x^2 \to 0$, as $x \to \infty$. Combining this with (7), we have the result $x^2 \sim 2\log T$, which induces

$$\log x = \frac{1}{2}\log 2 + \frac{1}{2}\log\log T + o(1). \tag{11}$$

Being taken logarithms of, (9) yields

$$\log M_1 + (-\varepsilon_1 + (\alpha-2)/\alpha)\log x < \log(x\widetilde{\sigma}^{\leftarrow}(1/x)) < \log M_2 + (\varepsilon_1 + (\alpha-2)/\alpha)\log x. \tag{12}$$

Substituting (12) and (11) into (8), we have

$$2\log T + (-\varepsilon_1 + \frac{2-\alpha}{\alpha})\log\log T + O(1) < x^2 < 2\log T - (\frac{\alpha-2}{\alpha} - \varepsilon_1)\log\log T + O(1). \tag{13}$$

Since $\varepsilon_1 + \frac{\alpha-2}{\alpha} \leq 0$, the first inequality in (13) further implies that

$$\exp(-x^2) \leq KT^{-2}(\log T)^{\varepsilon_1 + \frac{\alpha-2}{\alpha}} \leq KT^{-2}. \tag{14}$$

So, we have $\exp(-x^2/2) \leq KT^{-1}$.

Next we turn our attention to the left-hand side of (6). As in Leadbetter et al. (1983), we partition the sum in (6) at T^β, where β is a constant such that $\beta < (1-\delta)/(1+\delta)$, $\delta = \sup\{|r(t)| : t \geq \varepsilon\} < 1$. Since $\exp(-x^2/2) \leq K/T$ and $x^2 \sim 2\log T$, the first sum has the following bound

$$\begin{aligned}
&\frac{T}{q}\sum_{\varepsilon \leq kq \leq T^\beta} |r(kq)| \exp\Big(-\frac{x^2}{1+|r(kq)|}\Big) \leq \frac{T^{\beta+1}}{q^2}\exp\Big(-\frac{x^2}{1+\delta}\Big) \\
\leq\ & \frac{K}{q^2}T^{\beta+1-2/(1+\delta)} = \frac{K}{(q(\widetilde{\sigma}^{\leftarrow}(1/x))^{-1})^2}\cdot\frac{1}{(\widetilde{\sigma}^{\leftarrow}(1/x))^2}T^{\beta+1-2/(1+\delta)} \\
=\ & \frac{K}{(q(\widetilde{\sigma}^{\leftarrow}(1/x))^{-1})^2}\cdot\frac{1}{(\widetilde{\sigma}^{\leftarrow}(1/x))^2 x^{2\varepsilon_1+4/\alpha}}\cdot\frac{x^{2\varepsilon_1+4/\alpha}}{(\log T)^{\varepsilon_1+2/\alpha}}(\log T)^{\varepsilon_1+2/\alpha}T^{\beta+1-2/(1+\delta)} \\
\to\ & 0, \text{ as } T \to \infty.
\end{aligned} \tag{15}$$

This holds because, in the last product of (15), the first term is bounded under the condition of Lemma 3.4; by using (9), we get $M_1^2 \leq (x^{\varepsilon_1+2/\alpha}\widetilde{\sigma}^{\leftarrow}(1/x))^2 \leq M_2^2 x^{4\varepsilon_1}$ and hence $1/(x^{\varepsilon_1+2/\alpha}\widetilde{\sigma}^{\leftarrow}(1/x))^2 = O(1)$; the third term is bounded since $x^2 \sim 2\log T$; and the fourth term is $o(1)$.

Consider the case of $kq \geq T^\beta$. Define $\delta(t) := \sup\{|r(s)\log s|; s \geq t\}$. For sufficiently large T, we have $|r(t)| \leq \delta(T^\beta)/\log T^\beta$ and thus

$$\exp\Big(-\frac{x^2}{1+|r(kq)|}\Big) \leq \exp\Big(-x^2\Big(1-\frac{\delta(T^\beta)}{\log T^\beta}\Big)\Big).$$

Therefore, the second sum does not exceed

$$\begin{aligned}
&\frac{T}{q}\sum_{T^\beta < kq \leq T} |r(kq)| \exp\Big(-x^2\Big(1-\frac{\delta(T^\beta)}{\log T^\beta}\Big)\Big) \\
\leq\ & \Big(\frac{T}{q}\Big)^2 \exp\Big(-x^2\Big(1-\frac{\delta(T^\beta)}{\log T^\beta}\Big)\Big)\cdot\frac{1}{(\log T^\beta)^{1+\varepsilon_0}}\cdot\frac{q}{T}\sum_{T^\beta<kq\leq T}|r(kq)|(\log(kq))^{1+\varepsilon_0}.
\end{aligned} \tag{16}$$

According to (13) and (14), we have, for $\beta < 1$ and $\delta(t) \to 0$ as $t \to \infty$,

$$\exp\Big(-x^2\Big(1-\frac{\delta(T^\beta)}{\log T^\beta}\Big)\Big) \leq K\exp(-x^2) \leq KT^{-2}(\log T)^{\varepsilon_1-(2-\alpha)/\alpha}.$$

Noting $r(t)(\log t)^{1+\varepsilon_0} \to 0$, we also have

$$\frac{q}{T} \sum_{T^\beta < kq \leq T} |r(kq)|(\log(kq))^{1+\varepsilon_0} \to 0.$$

So, combining these with the result: $x^2 \sim 2\log T$ and $q(\widetilde{\sigma}^{\leftarrow}(1/x))^{-1} \to a$, it follows that the right-hand side of (16) does not exceed

$$\begin{aligned} & \frac{T^2}{q^2} T^{-2} (\log T)^{\varepsilon_1 - (2-\alpha)/\alpha} \frac{1}{(\log T^\beta)^{1+\varepsilon_0}} \cdot o(1) \\ = \ & \frac{1}{(q(\widetilde{\sigma}^{\leftarrow}(1/x))^{-1})^2} \cdot \frac{1}{(\widetilde{\sigma}^{\leftarrow}(1/x))^2 x^{2\varepsilon_1 + 4/\alpha}} \cdot \frac{x^{2\varepsilon_1+4/\alpha}}{(\log T)^{\varepsilon_1 + 2/\alpha}} \frac{(\log T)^{2\varepsilon_1}}{(\log T)^{\varepsilon_0}} \cdot o(1) \\ \to \ & 0, \text{ as } T \to \infty, \end{aligned}$$

where, noting $2\varepsilon_1 \leq \varepsilon_0$, the last '$\to$' can be proved in the same manner as in the proof of (15). The proof is completed. □

Lemma 3.5. *If conditions D1, D2 and D3 hold, then for* $0 < C_0 < C_\infty < \infty$,

$$P(\sup_{s \in [0, \tau m(x)]} X(s) \leq x) \to e^{-\tau}, \tag{17}$$

as $x \to \infty$*, uniformly for* $\tau \in [C_0, C_\infty]$.

Proof. The proof is a little similar to that of Lemma 4.3 in Arendarczyk and Dębicki (2012). We mainly give the key steps. Let $n_\tau = [\tau m(x)]$. The left-hand side of (17) has the following upper and lower bounds:

$$P(\sup_{s \in [0, n_\tau + 1]} X(s) \leq x) \leq P(\sup_{s \in [0, \tau m(x)]} X(s) \leq x) \leq P(\sup_{s \in [0, n_\tau]} X(s) \leq x). \tag{18}$$

We only prove $P(\sup_{s \in [0, n_\tau]} X(s) \leq x) \to e^{-\tau}$ as $x \to \infty$ since similarly, we can also prove $P(\sup_{s \in [0, n_\tau + 1]} X(s) \leq x) \to e^{-\tau}$ as $x \to \infty$. Divide interval $[0, n_\tau]$ into intervals of length 1, and split each of them into subintervals I_k^* and I_k of length ε and $1 - \varepsilon$, respectively. First of all, we have

$$\limsup_{x \to \infty} |P(\sup_{s \in [0, n_\tau]} X(s) \leq x) - P(\sup_{s \in \bigcup_{j=1}^{n_\tau} I_j} X(s) \leq x)| = 0, \tag{19}$$

uniformly for $\tau \in [C_0, C_\infty]$. This follows since, by the stationarity of $X(t)$, we have

$$\begin{aligned} 0 \ & \leq \ P(\sup_{s \in \bigcup_{j=1}^{n_\tau} I_j} X(s) \leq x) - P(\sup_{s \in [0, n_\tau]} X(s) \leq x) \\ & \leq \ n_\tau P(\sup_{s \in I_1^*} X(s) > x) \leq C_\infty m(x) P(\sup_{s \in I_1^*} X(s) > x) = \varepsilon C_\infty (1 + o(1)), \end{aligned} \tag{20}$$

$x \to \infty$, where the last equality in (20) is due to (2).

Let $a > 0$ and $q = a\widetilde{\sigma}^{\leftarrow}(1/x)$. Secondly, we prove that

$$\limsup_{x \to \infty} |P(\sup_{s \in \bigcup_{j=1}^{n_\tau} I_j} X(s) \leq x) - P(X(kq) \leq x, kq \in \bigcup_{j=1}^{n_\tau} I_j)| = 0, \tag{21}$$

uniformly for $\tau \in [C_0, C_\infty]$. To show (21), using the proof similar to that of Lemma 4.3 in Arendarczyk and Dębicki (2012), we have that for sufficiently small a

$$0 \leq P(X(kq) \leq x, kq \in \bigcup_{j=1}^{n_\tau} I_j) - P(\sup_{s \in \bigcup_{j=1}^{n_\tau} I_j} X(s) \leq x) \leq C_\infty \rho(a),$$

as $x \to \infty$, where the second inequality follows from Lemma 3.2 with $\rho(a) \to 0$, as $a \to 0$.

Once again, we prove that

$$|P(X(kq)\le x, kq\in\bigcup_{j=1}^{n_\tau} I_j) - \prod_{j=1}^{n_\tau} P(X(kq)\le x, kq\in I_j)| \to 0, \tag{22}$$

as $x\to\infty$ uniformly for $x\in[C_0, C_\infty]$. Let $\Lambda=(\lambda_{ij})$ be the covariance matrix of $X(kq)$, $kq\in\bigcup_{j=1}^{n_\tau} I_j$ and let $\Sigma=(\sigma_{ij})$ be the covariance matrix of $Z(kq)$, $kq\in\bigcup_{j=1}^{n_\tau} I_j$ of independent standard normal random variables. Applying Berman's inequality, we have

$$\begin{aligned}
& |P(X(kq)\le x, kq\in\bigcup_{j=1}^{n_\tau} I_j) - \prod_{j=1}^{n_\tau} P(X(kq)\le x, kq\in I_j)| \\
= & |P(X(kq)\le x, kq\in\bigcup_{j=1}^{n_\tau} I_j) - P(Z(kq)\le x, kq\in\bigcup_{j=1}^{n_\tau} I_j)| \\
\le & \frac{1}{2\pi}\sum_{1\le i<j\le L} |\lambda_{ij}-\sigma_{ij}|(1-\rho_{ij}^2)^{-1/2}\exp\Big(-\frac{x^2}{1+\rho_{ij}}\Big),
\end{aligned} \tag{23}$$

where L is the total number of kq-points in $\bigcup_{j=1}^{n_\tau} I_j$ and $\rho_{ij}=\max(|\lambda_{ij}|,|\sigma_{ij}|)$. Now noting the definition of the sequence $Z(kq)$ and matrix Σ, we have $|\lambda_{ii}-\sigma_{ii}|=0$ and $|\lambda_{ij}-\sigma_{ij}|\le|r(kq)|$ for $k=i-j$. Moreover, from the construction of the intervals I_j, the minimum value of kq is at least ε. Combining these with the observation that $\sup\{|r(t)|:|t|\ge\varepsilon\}:=\rho<1$, we get an upper bound for (23):

$$\begin{aligned}
& \frac{1}{2\pi(1-\rho^2)^{\frac12}}\frac{n_\tau}{q}\sum_{\varepsilon\le kq\le\tau m(x)} |r(kq)|\exp\Big(-\frac{x^2}{1+|r(kq)|}\Big) \\
\le & \frac{1}{2\pi(1-\rho^2)^{\frac12}}\frac{C_\infty m(x)}{q}\sum_{\varepsilon\le kq\le C_\infty m(x)} |r(kq)|\exp\Big(-\frac{x^2}{1+|r(kq)|}\Big) \\
\to & 0,
\end{aligned}$$

as $x\to\infty$, where the last limit is due to Lemma 3.4 since $C_\infty m(x)\mu=C_\infty$. So (22) holds.

Finally, we prove that

$$\limsup_{x\to\infty}|\prod_{j=1}^{n_\tau} P(X(kq)\le x, kq\in I_j) - (P(\sup_{s\in[0,1]} X(s)\le x))^{n_\tau}| \to 0, \tag{24}$$

as $x\to\infty$, uniformly for $\tau\in[C_0, C_\infty]$. In order to prove (24), using the proof similar to that of Lemma 4.3 in Arendarczyk and Dębicki (2012) yields

$$0\le\prod_{j=1}^{n_\tau} P(X(kq)\le x, kq\in I_j) - \prod_{j=1}^{n_\tau} P(\sup_{s\in I_j} X(s)\le x)\le C_\infty\rho(a),$$

as $x\to\infty$. Besides, the stationarity of $\{X(t), t\ge0\}$ yields

$$\prod_{j=1}^{n_\tau} P(\sup_{s\in I_j} X(s)\le x) = (P(\sup_{s\in I_1} X(s)\le x))^{n_\tau}$$

and (see the proof of (19))

$$\begin{aligned}
0 \le & (P(\sup_{s\in I_1} X(s)\le x))^{n_\tau} - (P(\sup_{s\in[0,1]} X(s)\le x))^{n_\tau} \\
\le & n_\tau P(\sup_{s\in I^*} X(s)>x))\le\varepsilon C_\infty(1+o(1)),
\end{aligned}$$

as $x \to \infty$. The rest of the proof is the same as that of Lemma 4.3 in Arendarczyk and Dębicki (2012), so we omit it. □

§4 The proofs of main result

Proof of (i). Replacing

$$\Big|\frac{P(\sup_{s\in[0,t]} X(s) > u)}{C^{1/\alpha} Q_\alpha u^{2/\alpha}\psi(u)}\Big| \le D \text{ with } \Big|\frac{P(\sup_{s\in[0,t]} X(s) > u)}{Q_\alpha \psi(u)/\widetilde{\sigma}^{\leftarrow}(1/u)}\Big| < D,$$

the remainder of the proof is the same as that of Theorem 3.1 in Arendarczyk and Dębicki (2012). So, we omit it.

Proof of (ii). Let $F_T(t)$ be the cumulative distribution function of $\mathbf{T}$ and let $0 < C_0 < C_\infty$. We have the following decomposition which is from Arendarczyk and Dębicki (2012):

$$\begin{aligned} P(\sup_{t\in[0,\mathbf{T}]} X(t) > x) &= \int_0^{C_0 m(x)} P(\sup_{t\in[0,s]} X(t) > x) dF_T(s) + \int_{C_0 m(x)}^{C_\infty m(x)} P(\sup_{t\in[0,s]} X(t) > x) dF_T(s) \\ &\quad + \int_{C_\infty m(x)}^{\infty} P(\sup_{t\in[0,s]} X(t) > x) dF_T(s) \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

First, by the stationarity of the process $\{X(t), t \ge 0\}$, using the proof similar to that of (14) in Arendarczyk and Dębicki (2012) yields that

$$J_1 \le P(\sup_{t\in[0,1]} X(t) > x)\Big(\int_0^{C_0 m(x)} P(\mathbf{T} > s) ds - C_0 m(x) P(\mathbf{T} > C_0 m(x)) + 1\Big). \tag{25}$$

Since $P(\mathbf{T} > s) \in RV_{-\lambda}^{\infty}$, (2) and (25) yield

$$J_1 \le \frac{\lambda}{1+\lambda} C_0 P(\mathbf{T} > C_0 m(x))(1 + o(1)) = \frac{\lambda}{1+\lambda} C_0^{1-\lambda} P(\mathbf{T} > m(x))(1 + o(1)),$$

$x \to \infty$.

Secondly, we have

$$J_3 \le P(\mathbf{T} > C_\infty m(x)) = C_\infty^{-\lambda} P(\mathbf{T} > m(x))(1 + o(1)),$$

as $x \to \infty$.

Finally, let $\varepsilon > 0$. Using Lemma 3.5, for sufficiently large x, we have

$$\begin{aligned} &(1-\varepsilon)\Big(\int_{C_0}^{C_\infty} e^{-s} s^{-\lambda} ds - (1 - e^{-C_\infty}) C_\infty^{-\lambda} + (1 - e^{-C_0}) C_0^{-\lambda}\Big) \\ \le\ &\liminf_{x\to\infty} \frac{J_2}{P(\mathbf{T} > m(x))} \le \limsup_{x\to\infty} \frac{J_2}{P(\mathbf{T} > m(x))} \\ \le\ &(1+\varepsilon)\Big(\int_{C_0}^{C_\infty} e^{-s} s^{-\lambda} ds - (1 - e^{-C_\infty}) C_\infty^{-\lambda} + (1 - e^{-C_0}) C_0^{-\lambda}\Big). \end{aligned}$$

Hence, passing with $\varepsilon \to 0$, $C_0 \to 0$ and $C_\infty \to \infty$, it follows that $J_1/P(\mathbf{T} > m(x)) \to 0$, $J_3/P(\mathbf{T} > m(x)) \to 0$ and

$$J_2/P(\mathbf{T} > m(x)) \to \Gamma(1 - \lambda),$$

as $x \to \infty$. According to the definition of $m(x)$ and the condition C2, a simple calculation can complete the proof. □

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