Appl. Math. J. Chinese Univ.
2017, 32(1): 108-116

Traveling wavefronts for a reaction-diffusion-chemotaxis model with volume-filling effect

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Abstract. In this paper, we study the propagation of the pattern for a reaction-diffusion-chemotaxis model. By using a weakly nonlinear analysis with multiple temporal and spatial scales, we establish the amplitude equations for the patterns, which show that a local perturbation at the constant steady state is spread over the whole domain in the form of a traveling wavefront. The simulations demonstrate that the amplitude equations capture the evolution of the exact patterns obtained by numerically solving the considered system.

§1 Introduction and preliminaries

We consider the following reaction-diffusion system with chemotaxis and volume-filling effects

$$\begin{cases} u_t = \nabla \cdot (D(1-u)^{-\alpha}\nabla u - \chi u(1-u)^{\beta}\nabla v) + \mu u(1-u/u_c), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1}$$

where $D(1-u)^{-\alpha}$ stands for the diffusion coefficient depending on cell density, and $u(1-u)^{\beta}$ is the chemotactic sensitivity function with *crowding capacity* 1, i.e., the maximal cell numbers that can be accommodated in a unit volume of space; $(x,t) \in \Omega \times [0,+\infty)$ and Ω is a bounded convex domain in $\mathbb{R}^N$ ($N = 1, 2$ and 3) with smooth boundary $\partial\Omega$; $D > 0$ and α, β are real constants; $\mu > 0$ is the intrinsic growth rate of the cell and u_c denotes the carrying capacity with $0 < u_c < 1$; $\chi > 0$ is called the chemotactic coefficient.

Received: 2015-10-22. Revised: 2016-11-23.
MR Subject Classification: 35K55, 35K45, 35K57.
Keywords: chemotaxis model, traveling wavefront, weakly nonlinear analysis.
Digital Object Identifier(DOI): 10.1007/s11766-017-3409-4.
The work was partially supported by the National Natural Science Foundation of China (11671359), the Provincial Natural Science Foundation of Zhejiang (LY15A010017, LY16A010009), and the Science Foundation of Zhejiang Sci-Tech University 15062173-Y.
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As for the derivation of system (1), we refer to [3, 4, 7, 8] and references therein. The research on (1) has already involved many aspects. For instance, the global existence of classical solutions and non-existence and existence of stationary patterns [4], the local and global bifurcation and their stability of non-constant steady states [3], pattern formation and competition of unstable wave modes [2]. However, there are no references on the wave propagation in a large spatial domain for system (1). There only exist two trivial steady states $(0,0)$ and (u_c, u_c) in (1) when the chemotactic coefficient χ is sufficiently small [2]. Thus, we assert that there is a traveling wave solution for system (1) connecting $(0,0)$ and (u_c, u_c) in the domain $\mathbb{R}$, which can be proved by applying the method in the references [5, 6]. But this problem is left for interested readers. In this paper, we shall focus on how the pattern invades the whole domain when the chemotactic coefficient χ and the spatial domain are large enough. By using the weakly nonlinear analysis [1], we derive the real cubic Ginzburg-Landau equation governing the evolution of pattern amplitude, so that the existence of traveling wavefront in (1) is established. For system (1) we incorporate the initial data

$$(u_0, v_0) \in [W^{1,\infty}(\Omega)]^2 \text{ and } 0 \le u_0(x) < 1,\ v_0(x) \ge 0,\ x \in \overline{\Omega} \tag{2}$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0,\ t > 0,\ x \in \partial\Omega \tag{3}$$

with ν being the outer unit normal vector on $\partial\Omega$. The present work is based on the following results:

Lemma 1.1 (Theorem 2.1 of [4]). *Assume that parameters α and β satisfy*

$$\alpha + \beta > 1, \tag{4}$$

then problem (1)-(2) has a global classical solution $(u(x,t), v(x,t))$. Moreover, there exists a constant $\delta > 0$ such that

$$0 \le u(x,t) \le 1 - \delta, \quad 0 \le v(x,t) \le 1 - \delta, \text{ for all } (x,t) \in \Omega \times (0, \infty). \tag{5}$$

Lemma 1.2 (Proposition 2.5 of [3]). *Let $(u(x), v(x))$ be a nonnegative steady state of (1)-(3) with $0 \le u(x) \le 1$. If α and β satisfy (4), then*

$$0 < u(x) < 1, \quad 0 < v(x) < 1, \text{ for all } x \in \overline{\Omega}. \tag{6}$$

In one dimensional case, we take $\Omega = (0, l)$ with $l > 0$. The negative Laplace operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition has a sequence of simple eigenvalues with corresponding eigenfunctions given by

$$\lambda_j = (\pi j/l)^2,\ \varphi_j(x) = \begin{cases} 1, & j = 0, \\ \cos(\pi j x/l), & j > 0, \end{cases} \tag{7}$$

where $j = 0, 1, 2, \cdots$. The result below is from [2] and [3].

Lemma 1.3. *Let parameters D, μ and u_c be fixed. If (4) holds and the chemotactic coefficient χ satisfies*

$$\chi > \frac{(\sqrt{D(1-u_c)^{-\alpha}} + \sqrt{\mu})^2}{u_c(1-u_c)^{\beta}} \stackrel{def}{=} \chi_c, \tag{8}$$

then the constant steady state (u_c, u_c) loses stability and (1)-(3) possesses stationary patterns. Moreover, the first admissible wave number k_{j_m} corresponds to the smallest bifurcation value

$$\chi_m = \frac{(1+k_{j_m}^2)\left[\mu + k_{j_m}^2 D(1-u_c)^{-\alpha}\right]}{k_{j_m}^2 u_c(1-u_c)^{\beta}} \geq \chi_c, \tag{9}$$

where $k_{j_m} = \lambda_j = (\pi j/l)^2$, and we denote k_{j_m} by k_a.

This paper is organized as follows. In Section 2, we shall derive the real Ginzburg-Landau equation governing the evolution of the amplitude of the pattern by using a weakly nonlinear multiple scale analysis. We show that the pattern invades the whole spatial domain as a traveling wavefront. In Section 3, by solving numerically the full system (1)-(3), it is demonstrated that the Ginzburg-Landau equation well approximates the shape and speed of the traveling front. Discussion and problems for further study are also presented.

For ease of statement, in what follows, we will restrict the spatial dimension to $N = 1$ and take $\Omega = [0, l]$, where the positive constant l is large enough. But it will be seen that the same arguments can still apply for $N = 2,\ 3$. In addition, throughout this paper we will always assume (4) and (8) are true.

§2 Traveling front invasion of pattern

In this section, we shall derive the equation describing the evolution of the amplitude of the pattern. By taking the slow and the fast spatial dependence of solution into account, we introduce X as the slow dependence and x as the fast dependence with $X = \varepsilon x$. The pattern stems from the perturbation of the constant steady state (u_c, u_c), and thus we begin with the transformation

$$U = u - u_c,\ V = v - u_c,$$

then system (1) becomes

$$\begin{cases} U_t = d'(u_c+U)(U')^2 + d(u_c+U)U'' \\ \quad - \chi h'(u_c+U)U'V' - \chi h(u_c+U)V'' - \mu U - \dfrac{\mu}{u_c}U^2, \\ V_t = V'' + U - V, \end{cases} \tag{10}$$

where $d(u) = D(1-u)^{-\alpha},\ h(u) = u(1-u)^{\beta}$. For nonlinear analysis, we introduce time and space scales as

$$\begin{cases} t = t(T_1, T_2, T_3, \cdots), & T_i = \varepsilon^i t, i = 1, 2, \cdots, \\ x = x(x, X), & X = \varepsilon x, \end{cases} \tag{11}$$

then the spatial and the temporal derivatives take the form as

$$\begin{cases} \partial_t \to \varepsilon\partial_{T_1} + \varepsilon^2\partial_{T_2} + \varepsilon^3\partial_{T_3} + \cdots, \\ \partial_x \to \partial_x + \varepsilon\partial_X, \\ \partial_{xx} \to \partial_{xx} + 2\varepsilon\partial_{xX} + \varepsilon^2\partial_{XX}, \end{cases} \tag{12}$$

where $\varepsilon \ll 1$; moreover, expand χ and $W = (U \;\; V)^T$ in the small parameter ε as

$$\begin{cases} \chi = \chi_c + \varepsilon\chi_1 + \varepsilon^2\chi_2 + \varepsilon^2\chi_2 + \cdots, \\ W = \varepsilon W_1 + \varepsilon^2 W_2 + \varepsilon^3 W_3 + \cdots, \end{cases} \tag{13}$$

where $W_i = (W_{1i} \; W_{2i})^T$, T denotes the transpose of a vector. Substituting (12) and (13) into (1) and collecting the terms with the same order in ε, we have

$$\mathcal{K}(\chi_m)W_1 = \mathbf{0}, \ldots\ldots\ldots O(\varepsilon), \tag{14}$$

$$\mathcal{K}(\chi_m)W_2 = F(W_1), \ldots\ldots\ldots O(\varepsilon^2), \tag{15}$$

$$\mathcal{K}(\chi_m)W_3 = G(W_1, W_2), \ldots\ldots\ldots O(\varepsilon^3), \tag{16}$$

where χ_m is defined as (9),

$$\mathcal{K}(\chi_m) = \begin{pmatrix} d(u_c)\partial_{xx} - \mu & -\chi_m h(u_c)\partial_{xx} \\ 1 & \partial_{xx} - 1 \end{pmatrix}, \qquad F = (F_1 \;\; F_2)^T, \qquad G = (G_1 \;\; G_2)^T,$$

$$\begin{cases} F_1 = \frac{\partial W_{11}}{\partial T_1} + \frac{\mu}{u_c}W_{11}^2 - 2d(u_c)\partial_{xX}W_{11} - d'(u_c)\partial_x(W_{11}\partial_x W_{11}) \\ \quad +\chi_m\left[2h(u_c)\partial_{xX}W_{21} + h'(u_c)\partial_x(W_{11}\partial_x W_{21})\right] + \chi_1 h(u_c)\partial_{xx}W_{21}, \\ F_2 = \frac{\partial W_{21}}{\partial T_1} - 2\partial_{xX}W_{21}, \end{cases}$$

and

$$\begin{cases} G_1 = \frac{\partial W_{11}}{\partial T_2} + \frac{\partial W_{12}}{\partial T_1} + \frac{2\mu}{u_c}W_{11}W_{12} - d(u_c)(2\partial_{xX}W_{12} + \partial_{XX}W_{11}) \\ \quad -d'(u_c)\left[2\partial_x(W_{11}\partial_X W_{11}) + \partial_{xx}(W_{11}W_{12})\right] - \frac{1}{2}d''(u_c)\partial_x(W_{11}^2\partial_x W_{11}) \\ \quad +\chi_m h'(u_c)\left[\partial_x(W_{11}\partial_x W_{22}) + \partial_x(W_{11}\partial_X W_{21}) + \partial_X(W_{11}\partial_x W_{21}) + \partial_x(W_{12}\partial_x W_{21})\right] \\ \quad +\chi_m\left[h(u_c)(2\partial_{xX}W_{22} + \partial_{XX}W_{21}) + \frac{1}{2}h''(u_c)\partial_x(W_{11}^2\partial_x W_{21})\right] \\ \quad +\chi_1\left[h(u_c)(\partial_{xx}W_{22} + 2\partial_{xX}W_{21}) + h'(u_c)\partial_x(W_{11}\partial_x W_{21})\right] + \chi_2 h(u_c)\partial_{xx}W_{21}, \\ G_2 = \frac{\partial W_{21}}{\partial T_2} + \frac{\partial W_{22}}{\partial T_1} - 2\partial_{xX}W_{22} - \partial_{XX}W_{21}. \end{cases}$$

Again substituting (12) and (13) into (3), at $x = 0, \; l$ we have

$$\begin{cases} \frac{\partial W_1}{\partial x} = 0, \\ \frac{\partial W_2}{\partial x} = -\frac{\partial W_1}{\partial X}, \\ \frac{\partial W_3}{\partial x} = -\frac{\partial W_2}{\partial X}, \\ \cdots \end{cases} \tag{17}$$

Equation (14) with (17) has a solution

$$W_1 = \rho A(X, T_1, T_2)\cos(k_a x), \quad \rho = (1 + k_a^2 \;\; 1)^T. \tag{18}$$

Here k_a is defined in Lemma 1.3 and $k_a = j_m\pi/l$. It is obvious that the vector ρ is defined up to a constant one. By (18), we have

$$\begin{cases} F_1 &= \frac{\mu}{2u_c}A^2(1+k_a^2)^2 + \left[\frac{\partial A}{\partial T_1}(1+k_a^2) - \chi_1 A h(u_c)k_a^2\right]\cos(k_a x) \\ &\quad + \left[d'(u_c)A^2(1+k_a^2)^2k_a^2 - \chi_m h'(u_c)A^2(1+k_a^2)k_a^2 + \frac{\mu}{2u_c}A^2(1+k_a^2)^2\right]\cos(2k_a x) \\ &\quad + \left[2d(u_c)(1+k_a^2)\frac{\partial A}{\partial X}k_a - \chi_m 2h(u_c)\frac{\partial A}{\partial X}k_a\right]\sin(k_a x), \\ F_2 &= \frac{\partial A}{\partial T_1}\cos(k_a x) + 2k_a\frac{\partial A}{\partial X}\sin(k_a x). \end{cases}$$

By a simple computation, the adjoint equation of (14) has a solution

$$\overline{W}_1 = (\overline{W}_{11}\ \overline{W}_{21})^T = \overline{\rho}A(X,T_1,T_2)\cos(k_a x),\quad \overline{\rho} = \begin{pmatrix} \dfrac{1+k_a^2}{\chi_m h(u_c)k_a^2} & 1 \end{pmatrix}^T, \tag{19}$$

where $\overline{\rho}$ is the kernel of the adjoint of the operator below

$$\mathcal{K}(\chi_m)|_{w_1} = \begin{pmatrix} -d(u_c)k_a^2 - \mu & \chi_m h(u_c)k_a^2 \\ 1 & -k_a^2 - 1 \end{pmatrix}.$$

Then the solvability condition of (15), i.e.,

$$\langle F, \overline{W}_1\rangle = 0, \tag{20}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product on $L^2[0, \frac{2\pi}{k_m}]$. We should point out that, notice of the Neumann boundary condition (3), the Fredholm alternative need to be used on the interval $[0, \frac{2\pi}{k_m}]$. Then we can obtain a solution of (1)-(3) on the whole domain l by using the refection and periodic extension. It is easy to check that equation (20) leads to $\chi_1 = 0,\ T_1 = 0$. According to the expression of F, (15) has a solution in the form of

$$\begin{cases} W_{12} = A^2(b_{11} + b_{12}\cos(2k_a x)) + \frac{\partial A}{\partial X}b_{13}\sin(k_a x), \\ W_{22} = A^2(b_{21} + b_{22}\cos(2k_a x)) + \frac{\partial A}{\partial X}b_{23}\sin(k_a x), \end{cases} \tag{21}$$

where b_{ij} $(i = 1,2,\ j = 1,2,3)$ satisfy the following equations, respectively,

$$\begin{aligned} \kappa_0(\chi_m)(b_{11}, b_{21})^T &= \begin{pmatrix} \frac{\mu}{2u_c}(1+k_a^2)^2 \\ 0 \end{pmatrix}, \\ \kappa_2(\chi_m)(b_{12}, b_{22})^T &= \begin{pmatrix} (1+k_a^2)^2k_a^2d'(u_c) - \chi_m h'(u_c)(1+k_a^2)k_a^2 + \frac{\mu}{2u_c}(1+k_a^2)^2 \\ 0 \end{pmatrix}, \\ \kappa_1(\chi_m)(b_{13}, b_{23})^T &= \begin{pmatrix} 2d(u_c)(1+k_a^2)k_a - 2\chi_m h(u_c)k_a \\ 2k_a \end{pmatrix} \end{aligned}$$

and

$$\kappa_p(\chi_m) = \begin{pmatrix} -p^2k_a^2d(u_c) - \mu & p^2k_a^2\chi_m h(u_c) \\ 1 & -p^2k_a^2 - 1 \end{pmatrix},\ p = 0,1,2.$$

Now G can be simplified as

$$\begin{cases} G_1 &= \left[(1+k_a^2)\frac{\partial A}{\partial T_2} - \chi_2 h(u_c)k_a^2 A + G_1^{(1)}A^3 - G_1^{(2)}\frac{\partial^2 A}{\partial X^2}\right]\cos(k_a x) \\ & \quad +G_1^{(3)}A\frac{\partial A}{\partial X}\sin(2k_a x) + G^*, \\ G_2 &= \left(\frac{\partial A}{\partial T_2} - \frac{\partial^2 A}{\partial X^2} - 2b_{23}k_a\frac{\partial^2 A}{\partial X^2}\right)\cos(k_a x) + 8k_a b_{22}A\frac{\partial A}{\partial X}\sin(2k_a x), \end{cases}$$

where G^* satisfies $\langle G^*, \overline{W}_{11}\rangle = 0$,

$$\begin{aligned} G_1^{(1)} &= \frac{2\mu}{u_c}(1+k_a^2)(b_{11}+\frac{1}{2}b_{12}) + \left[d'(u_c)(b_{11}+\frac{1}{2}b_{12}) + d''(u_c)\frac{1}{8}(1+k_a^2)^2\right](1+k_a^2)k_a^2 \\ &- \chi_m h'(u_c)k_a^2\left[b_{11} - \frac{1}{2}b_{12} + b_{22}(1+k_a^2)\right] - \chi_m\frac{1}{8}h''(u_c)(1+k_a^2)^2k_a^2, \\ G_1^{(2)} &= d(u_c)(1+k_a^2+2b_{13}k_a) - \chi_m h(u_c)(1+2b_{23}k_a), \\ G_1^{(3)} &= \frac{\mu}{u_c}(1+k_a^2)b_{13} + 8d(u_c)k_a b_{12} + 2d'(u_c)(k_a b_{13}+1+k_a^2)(1+k_a^2)k_a \\ &- \chi_m 8h(u_c)k_a b_{22} - \chi_m h'(u_c)\left[k_a b_{13} + (2+k_a b_{23})(1+k_a^2)\right]k_a. \end{aligned}$$

Applying the solvability condition of (16), i.e., $\langle G, \overline{W}_1\rangle = 0$, we have the amplitude equation of the pattern

$$\frac{\partial A}{\partial T_2} = \delta\frac{\partial^2 A}{\partial X^2} + \tau A - \gamma A^3, \tag{22}$$

where

$$\begin{aligned} \tau &= \frac{\chi_2 h(u_c)(1+k_a^2)k_a^2}{(1+k_a^2)^2 + \chi_m h(u_c)k_a^2}, \quad \gamma = \frac{(1+k_a^2)G_1^{(1)}}{(1+k_a^2)^2 + \chi_m h(u_c)k_a^2}, \\ \delta &= \frac{\chi_m h(u_c)(1+2b_{23}k_a)k_a^2 + (1+k_a^2)G_1^{(2)}}{(1+k_a^2)^2 + \chi_m h(u_c)k_a^2}, \\ G_1^{(4)} &= \frac{4}{3k_a l}\left(8k_a b_{22} + \frac{1+k_a^2}{\chi_m h(u_c)k_a^2}G_1^{(3)}\right)[(-1)^{j_m} - 1]. \end{aligned}$$

We know that (22) is the typical Ginzburg-Landau equation. By the "tanh" method, we can find the following exact solution of (22) in $\mathbb{R}$

$$A(X,T_2) = \frac{1}{2}\sqrt{\frac{\tau}{\gamma}}\left(1 - \tanh\left(\sqrt{\frac{\tau}{\delta}}\frac{Y-Y_0}{2\sqrt{2}}\right)\right), \tag{23}$$

where $Y = X - cT_2$, $c = 3\sqrt{\tau\delta/2}$. It is a traveling wave solution of (22) connecting 0 and $\sqrt{\tau/\gamma}$.

§3 Simulation and discussion

In this section, simulation will show that the local perturbation of the steady state (u_c, u_c) propagates through the whole domain in the form of a traveling wave and equation (22) can excellently capture the envelope evolution and the progressing of the perturbation as a wave.

We take the system parameters as $D = 0.1$, $\mu = 0.9$, $u_c = 0.3$, $\alpha = 1.3$ and $\beta = -0.1$. The size of the domain is chosen as $l = 4\pi$ and the small parameter $\varepsilon = 0.12$. Then by a numerical computation, we have that the first admissible mode $j_m = 6$ as well as $\tau = 0.9068$, $\gamma = 70.5696$, $\delta = 0.3388$, $\rho = (3.25\ 1)^T$, $\chi_m = 5.8435$, $k_a = 1.5$. Thus the amplitude of the stationary pattern is approximately equal to $\varepsilon\rho\sqrt{\tau/\gamma} + \varepsilon^2(b_{11} + b_{12}\quad b_{21} + b_{22})^T\tau/\gamma = (0.0472\quad 0.0110)^T$. The chemotactic coefficient χ is given by $\chi = (1+\varepsilon^2)\chi_m = 5.9276$. We have perturbed the steady state $(u_c, u_c) = (0.3, 0.3)$ at the left end of the spatial interval, see Fig 1.

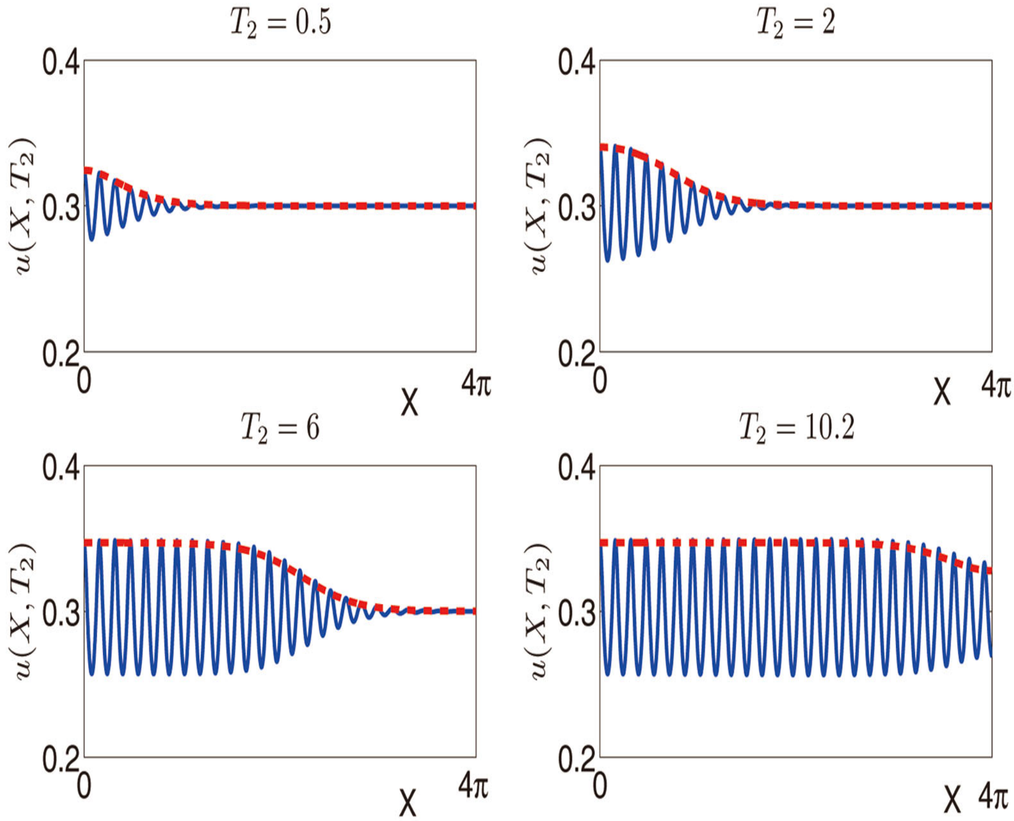

Figure 1: (Color online) A modulated progressing wave by which the pattern invades the whole domain. The red dashed line is $u = u_c + \varepsilon(1 + k_a^2)A(X, T_2) + \varepsilon^2(b_{11} + b_{12})A^2(X, T_2)$, where $A(X, T_2)$ is a numerical solution of (22) with the initial data $\frac{1}{2}\sqrt{\tau/\gamma}(1 - \tanh(\sqrt{\tau/(8\delta)}X))$. The blue solid line is a numerical solution of system (1) with the initial data $(u_0\ v_0)^T = (u_c\ u_c)^T + \varepsilon\rho A(X, 0)\cos(k_a x)$.

In our numerical simulations, we find a traveling wave solution (u, v) connecting $(0, 0)$ and $(u_c, u_c) = (0.3, 0.3)$ for $\chi < \chi_m$, see Fig 2, and a traveling wave solution (u, v) connecting the constant steady state $(u_c, u_c) = (0.3, 0.3)$ and the non-constant steady state with the amplitude $\varepsilon(1 + k_a^2, 1)\sqrt{\tau/\gamma} + \varepsilon^2(b_{11} + b_{12}, b_{21} + b_{22})\tau/\gamma = (0.0472, 0.0110)$ for $\chi > \chi_m$, see Fig 3. The former can be strictly proved by the method in [5,6] and is left for interested readers. The theoretical study for the latter is in progress and will be presented in future papers.

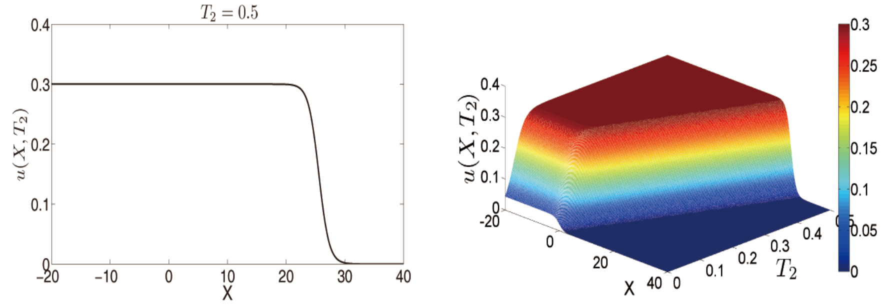

Figure 2: A traveling wavefront u connecting 0 to $u_c = 0.3$. The initial data is $(u_0\ v_0)^T = \varepsilon\rho A(X, 0)$. All the parameters are the same as in Fig.1 except $\chi = 4 < \chi_m, l = 40$ and the running time is $T_2 = 0.5$

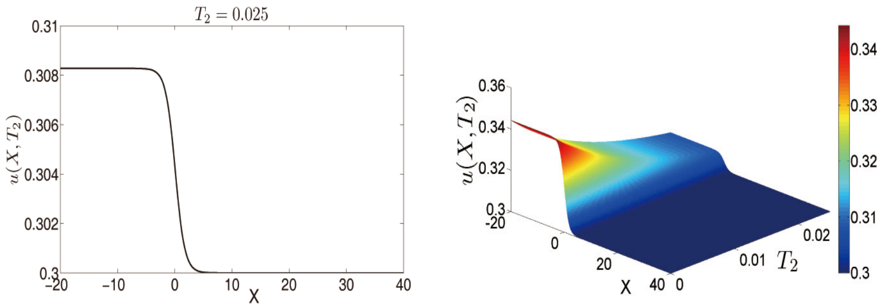

Figure 3: A traveling wave solution u connecting $u_c = 0.3$ to a stationary pattern with the amplitude 0.0472. The initial data is $(u_0\ v_0)^T = (u_c\ u_c)^T + \varepsilon\rho A(X, 0)$. All the parameters are the same as in Fig.1 except $\chi = (1 + 0.12^2)\chi_m = 5.9276 > \chi_m, l = 40$ and the running time is $T_2 = 0.025$

Acknowledgments. The authors would like to thank the anonymous referees for their valuable comments, which greatly improved the exposition of the paper.

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