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On vector variational-like inequalities and vector optimization problems with (G, α)-invexity

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Abstract. The aim of this paper is to study the relationship among Minty vector variational-like inequality problem, Stampacchia vector variational-like inequality problem and vector optimization problem involving (G, α)-invex functions. Furthermore, we establish equivalence among the solutions of weak formulations of Minty vector variational-like inequality problem, Stampacchia vector variational-like inequality problem and weak efficient solution of vector optimization problem under the assumption of (G, α)-invex functions. Examples are provided to elucidate our results.

§1 Introduction

Variational inequalities introduced by Stampacchia [19] represents a natural generalization to the variational theory of boundary value problems for partial differential equations. Since then, the study of variational inequalities and its generalizations has increased tremendously due to its wide applications in many areas such as mechanics, physics, mathematical programming, theory of control, complementarity problems and economics.

Another form of variational inequalities was presented by Minty [15], which have proved to characterize a stronger notion of equilibrium than Stampacchia variational inequalities. Gianessi [8,9] introduced vector variational inequalities in finite dimensional Euclidean space as an extension of Stampacchia [19] and Minty [15] variational inequalities. Vector variational-like inequalities have been a powerful tool to solve vector optimization problems. Hence, vector variational inequality problems have been considerably generalized in different ways by many researchers [5,6,10,11,13,14,17,18,20,22].

Yang *et al.* [21] established relations between a Minty vector variational inequality and a vector optimization problem under pseudoconvexity or pseudomonotonicity assumptions. Gang

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and Liu [7] proved the relationship among Minty vector variational-like inequality problem, Stampacchia vector variational-like inequality problem and vector optimization problem using the concept of pseudoinvexity or η-pseudomonotonicity. Thereafter, Al-Homidan and Ansari [1] studied the relationship among generalized Minty vector variational-like inequality problem, generalized Stampacchia vector variational-like inequality problem and vector optimization problem for nondifferentiable and nonconvex functions.

Convexity has been generalized in many ways for the purpose of weakening its limitations in mathematical programming, see for instance [2,16]. Antczak [3] introduced a new class of generalized convexity called G-invexity and further extended it to the vectorial case by defining vector G-invexity [4]. Motivated by Antczak, Liu *et al.* [12] introduced the concept of (G, α)-invexity. Inspired from the ongoing research work, we study in the paper the relationship between solutions of Minty vector variational-like inequalities, Stampacchia vector variational-like inequalities and the considered vector optimization problems under the assumption of (G, α)-invexity.

This paper is organized as follows: In Section 2, we recall some definitions and establish results which will be useful in the sequel of the paper. Section 3 is devoted to the study of relationship among Minty vector variational-like inequality problem, Stampacchia vector variational-like inequality problem and vector optimization problem involving (G, α)-invex functions. An example is constructed to elucidate our result. In Section 4, we consider weak Minty vector variational-like inequality problem, weak Stampacchia vector variational-like problem and establish the relationship of their solutions with the weak efficient solution of vector optimization problem under the assumption of (G, α)-invexity. We present another example to illustrate the established result. Finally, in Section 5, we conclude our paper.

§2 Notations and preliminaries

Definition 2.1. [2] Let X be a nonempty subset of R^n and u be any arbitrary point of X. The set X is said to be invex at $u \in X$ with respect to $\eta : X \times X \mapsto R^n$, if for each $x \in X$ and $\lambda \in [0, 1]$, we have

$$u + \lambda\eta(x, u) \in X.$$

The set X is said to be invex with respect to η if X is invex at every $u \in X$ with respect to η.

Throughout this paper, unless specifically stated otherwise, let X be a nonempty subset of R^n, $f = (f_1, \ldots, f_m) : X \mapsto R^m$ be a vector-valued differentiable function defined on X, $I_{f_i}(X)$, $i = 1, \ldots, m$, be the range of f_i, that is, the image of X under f_i. Let $G_f = (G_{f_1}, \ldots, G_{f_m}) : R \mapsto R^m$ be a differentiable vector-valued function such that any of its component $G_{f_i} : I_{f_i}(X) \mapsto R$ is a strictly increasing function on its domain. Let $M = \{1, \ldots, m\}$ and $\langle ., . \rangle$ denotes the scalar product in R^n. Let $\eta : X \times X \mapsto R^n$ and $\alpha_i : X \times X \mapsto R^+ \setminus \{0\},\ i \in M$.

Definition 2.2. [12] Let $f = (f_1, \ldots, f_m) : X \mapsto R^m$ be a vector-valued differentiable function defined on a nonempty open set $X \subset R^n$, $I_{f_i}(X), i \in M$, be the range of f_i. If there exists a differentiable vector-valued function $G_f = (G_{f_1}, \ldots, G_{f_m}) : R \mapsto R^m$ such that any of its component $G_{f_i} : I_{f_i}(X) \mapsto R$ is a strictly increasing function on its domain, a vector-valued function $\eta : X \times X \mapsto R^n$ and real function $\alpha_i : X \times X \mapsto R_+ (i \in M)$ such that, for all $x \in X (x \neq u)$,

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(u))(>) \geq \alpha_i(x,u) \left\langle G'_{f_i}(f_i(u)) \nabla f_i(u), \eta(x,u) \right\rangle \ ; \ i = 1, \dots, m, \quad (1)$$

then f is said to be a (strictly) vector (G_f, α)-invex function at u on X with respect to η. If (1) is satisfied for each $u \in X$, then f is (strictly) vector (G_f, α)-invex on X with respect to η.

For more details on vector (G, α)-invex function, we refer to [12].

Definition 2.3. [1] A mapping $\eta : X \times X \mapsto R^n$ is said to be skew if for all $x, u \in X$,

$$\eta(x,u) + \eta(u,x) = 0.$$

Definition 2.4. [11] A mapping $\alpha : X \times X \mapsto R$ is symmetric if for all $x, u \in X$, we have

$$\alpha(x,u) = \alpha(u,x).$$

Condition C [1] Let X be an invex set of R^n with respect to $\eta : X \times X \mapsto R^n$. Then, for all, $x, u \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0,1]$, we have
(a) $\eta(u, u + \lambda\eta(x,u)) = -\lambda\eta(x,u)$,
(b) $\eta(x, u + \lambda\eta(x,u)) = (1-\lambda)\eta(x,u)$,
(c) $\eta(u + \lambda_1\eta(x,u), u + \lambda_2\eta(x,u)) = (\lambda_1 - \lambda_2)\eta(x,u)$.

Condition D [11] Let X be an invex subset of R^n with respect to η and let $\alpha_i, i \in M$ be scalar valued mappings. Then, for all $i \in M$, $x, y \in X$ and $\lambda \in [0,1]$, we have
(a) $\alpha_i(u, u + \lambda\eta(x,u)) \geq \alpha_i(x,u)$,
(b) $\alpha_i(x, u + \lambda\eta(x,u)) \geq \alpha_i(x,u)$,
(c) $\dfrac{\alpha_i(u, u + \lambda\eta(x,u))}{\alpha_i(u + \lambda\eta(x,u), u)} \geq \alpha_i(x,u)$.

Definition 2.5. [11] Let S be a nonempty subset of X and let $T_i : X \mapsto 2^X$ be a set-valued mapping for every $i \in M$. The mapping $T : (T_1, \dots, T_m)$ is said to be V-invariant monotone on S with respect to η and $\alpha_i, i \in M$, iff for all $i \in M$, $x, y \in S$, $x_i^* \in T_i(x)$ and $y_i^* \in T_i(y)$, we have

$$\alpha_i(x,y)\langle y_i^*, \eta(x,y)\rangle + \alpha_i(y,x)\langle x_i^*, \eta(y,x)\rangle \leq 0.$$

Definition 2.6. [2] Let $S \subset R^n$ be a nonempty invex set with respect to η, and x and u two arbitrary points of S. A set P_{ux} is said to be a closed η-path joining the points u and $v = u + \eta(x,u)$ (contained in S) if

$$P_{ux} = \{y = u + \lambda\eta(x,u) : \lambda \in [0,1]\}.$$

Analogously, an open η-path joining the points u and $v = u + \eta(x,u)$ (contained in S) we call a set of the form

$$P_{ux}^0 = \{y = u + \lambda\eta(x,u) : \lambda \in (0,1)\}.$$

We prove the following mean value theorem which will be used to prove one of the important results in the sequel.

Theorem 2.1. *Let $X \subset R^n$ be a nonempty invex set with respect to $\eta : X \times X \mapsto R^n$ and P_{xy} be an arbitrary η-path contained in* int$X$*. Moreover, we assume that $f : X \mapsto R^m$ is defined on X and differentiable on* int$X$*. Then, for any $x, y \in X$, there exists $z_i \in P^0_{xy},\ i \in M$ such that the following relation*

$$G_{f_i}(f_i(x + \eta(y, x))) - G_{f_i}(f_i(x)) = \langle G'_{f_i}(f_i(z_i))\nabla f_i(z_i), \eta(y, x)\rangle$$

holds for all $i \in M$. In other words, for any $w = x + \eta(y, x) \in X$, there exists $\lambda^0_i \in (0, 1),\ z_i = x + \lambda^0_i\eta(y, x),\ i \in M$ such that

$$G_{f_i}(f_i(w)) - G_{f_i}(f_i(x)) = \langle G'_{f_i}(f_i(z_i))\nabla f_i(z_i), \eta(y, x)\rangle,$$
$$\text{i.e.,}\quad G_{f_i}(f_i(x + \eta(y, x))) - G_{f_i}(f_i(x)) = \langle G'_{f_i}(f_i(z_i))\nabla f_i(z_i), \eta(y, x)\rangle.$$

Proof. We define functions $g_i : [0, 1] \mapsto R,\ i \in M$ as follows

$$g_i(\lambda) = G_{f_i}(f_i(x + \lambda\eta(y, x))) - G_{f_i}(f_i(x)) - \lambda[G_{f_i}(f_i(x + \eta(y, x))) - G_{f_i}(f_i(x))],\ i \in M. \quad (2)$$

Clearly $g_i(0) = 0,\ g_i(1) = 0$. Hence, using Rolle's Theorem, it follows that there exists $\lambda^0_i \in (0, 1),\ i \in M$ such that

$$g_i(1) - g_i(0) = g'_i(\lambda^0_i)(1 - 0),\ i \in M$$
$$\Rightarrow\ g'_i(\lambda^0_i) = 0,\ i \in M. \quad (3)$$

From relations (2) and (3), we obtain

$$\eta(y, x)^T G'_{f_i}(f_i(x + \lambda^0_i\eta(y, x)))\nabla f_i(x + \lambda^0_i\eta(y, x)) - G_{f_i}(f_i(x + \eta(y, x))) + G_{f_i}(f_i(x)) = 0$$
$$\Rightarrow G_{f_i}(f_i(x + \eta(y, x))) - G_{f_i}(f_i(x)) = \langle G'_{f_i}(f_i(x + \lambda^0_i\eta(y, x)))\nabla f_i(x + \lambda^0_i\eta(y, x)), \eta(y, x)\rangle.$$

We set $z_i = x + \lambda^0_i\eta(y, x),\ i \in M$. Since $\lambda^0_i \in (0, 1)$, $z_i \in P^0_{xy}$ for all $i \in M$ and

$$G_{f_i}(f_i(x + \eta(y, x))) - G_{f_i}(f_i(x)) = \langle G'_{f_i}(f_i(z_i))\nabla f_i(z_i), \eta(y, x)\rangle,\quad i \in M.$$

This completes the proof. □

Lemma 2.1. *Let $f : X \mapsto R^m$ be a differentiable function defined on a nonempty invex set $X \subset R^n$ with respect to η. If f is a vector (G_f, α)-invex function with respect to η, then the differential of G_{f_i} is V-invariant monotone on X with respect to η and $\alpha_i, i \in M$, i.e., for all $x, y \in X,\ i \in M$,*

$$\alpha_i(x, y)\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(x, y)\rangle + \alpha_i(y, x)\langle G'_{f_i}(f_i(x))\nabla f_i(x), \eta(y, x)\rangle \leq 0.$$

Proof. Since f is a vector (G_f, α)-invex function on the invex set X, we have

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(y)) \geq \alpha_i(x, y)\left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(x, y)\right\rangle,\ i \in M. \quad (4)$$

Changing the role of x and y in the above inequality, we get

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(x)) \geq \alpha_i(y, x)\left\langle G'_{f_i}(f_i(x))\nabla f_i(x), \eta(y, x)\right\rangle,\ i \in M. \quad (5)$$

Adding (4) and (5), it follows that

$$\alpha_i(x, y)\left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(x, y)\right\rangle + \alpha_i(y, x)\left\langle G'_{f_i}(f_i(x))\nabla f_i(x), \eta(y, x)\right\rangle \leq 0,\ i \in M.$$

This completes the proof. □

Lemma 2.2. *Let $f : X \mapsto R^m$ be a differentiable function defined on a nonempty invex set $X \subset R^n$ with respect to η. Assume that η satisfies Condition C and $\alpha_i,\ i \in M$ satisfies*

Condition D. If f is a vector (G_f,α)-invex function with respect to η, then the following relation

$$G_{f_i}(f_i(y+\lambda\eta(x,y))) \leq \lambda G_{f_i}(f_i(x)) + (1-\lambda)G_{f_i}(f_i(y)),\ i \in M,$$

holds for all $x,y \in X$ and $\lambda \in [0,1]$.

Proof. Since f is a vector (G_f,α)-invex function on the invex set X, we have

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(y)) \geq \alpha_i(x,y)\left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(x,y)\right\rangle,\ i \in M.$$

We set $\bar{y} = y + \lambda\eta(x,y)$. Since X is an invex set, $\bar{y} \in X$ for all $x,y \in X$. Thus replacing y by $\bar{y}$ in the above inequality, it follows that

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(\bar{y})) \geq \alpha_i(x,\bar{y})\langle G'_{f_i}(f_i(\bar{y}))\nabla f_i(\bar{y}), \eta(x, y+\lambda\eta(x,y))\rangle,\ i \in M.$$

In view of Conditions C(b) and D(b), the above inequality yields

$$G_{f_i}(f_i(x)) - G_{f_i}(f_i(\bar{y})) \geq (1-\lambda)\alpha_i(x,y)\langle G'_{f_i}(f_i(\bar{y}))\nabla f_i(\bar{y}), \eta(x,y)\rangle,\ i \in M. \tag{6}$$

Again by (G_f,α)-invexity of f, we have

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{y})) \geq \alpha_i(y,\bar{y})\left\langle G'_{f_i}(f_i(\bar{y}))\nabla f_i(\bar{y}), \eta(y, y+\lambda\eta(x,y))\right\rangle,\ i \in M.$$

In view of Conditions C(a) and D(a), the above inequality yields

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{y})) \geq -\lambda\alpha_i(x,y)\langle G'_{f_i}(f_i(\bar{y}))\nabla f_i(\bar{y}), \eta(x,y)\rangle,\ i \in M. \tag{7}$$

On multiplying (6) by λ, (7) by $(1-\lambda)$ and then adding, we obtain

$$\lambda G_{f_i}(f_i(x)) + (1-\lambda)G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{y})) \geq 0,\ i \in M$$

$$\Rightarrow \quad G_{f_i}(f_i(y+\lambda\eta(x,y))) \leq \lambda G_{f_i}(f_i(x)) + (1-\lambda)G_{f_i}(f_i(y)),\ i \in M.$$

This completes the proof. □

Lemma 2.3. *Let ϕ be a real strictly increasing and differentiable function defined on interval $(a,b) \subset R$. Then,*

$$\phi'(x) \geq 0, \quad \forall\ x \in (a,b).$$

Let $f : R^n \mapsto R^m$ be a vector-valued function. We consider the following vector optimization problem:

$$\textbf{(VOP)} \qquad \text{Minimize } f(x) = (f_1(x), \ldots, f_m(x)) \quad \text{subject to } x \in X.$$

Definition 2.7. A point $\bar{x} \in X$ is said to be an efficient solution of (VOP), if

$$f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \ldots, f_m(y) - f_m(\bar{x})) \notin -R^m_+ \setminus \{0\},\quad \forall\ y \in X.$$

Definition 2.8. A point $\bar{x} \in X$ is said to be a weakly efficient solution of (VOP), if

$$f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \ldots, f_m(y) - f_m(\bar{x})) \notin -\text{int}R^m_+,\quad \forall\ y \in X,$$

where int denotes interior of a set.

Clearly, every efficient solution is a weakly efficient solution to (VOP) but the converse is not true in general.

Definition 2.9. A feasible point $\bar{x} \in X$ is said to be a vector critical point for (VOP), if there exists a vector $\nu \in R^m$ with $\nu \geq 0$ such that $\nu^T\nabla f(\bar{x}) = 0$.

Gordan's Theorem. *For each given $m \times n$ matrix A, either $Ax > 0$ has a solution $x \in R^n$ or $A^T y = 0, \quad y \geq 0$ has a solution $y \in R^m$ but never both.*

§3 Relationships between vector optimization problems and vector variational-like inequality problems

In this section, we consider both Minty vector variational-like inequalities and Stampacchia vector variational-like inequalities and establish the relationship of their solutions with the efficient solutions of vector optimization problems involving vector (G, α)-invex functions.

(MVVLIP) A Minty vector variational-like inequality problem is to find a point $y \in X$ such that

$$(\langle \nabla f_1(\bar{x}), \eta(\bar{x}, y)\rangle, \ldots, \langle \nabla f_m(\bar{x}), \eta(\bar{x}, y)\rangle) \notin -R^m_+ \setminus \{0\}, \ \forall\ \bar{x} \in X.$$

(SVVLIP) A Stampacchia vector variational-like inequality problem is to find a point $\bar{x} \in X$ such that

$$(\langle \nabla f_1(\bar{x}), \eta(y, \bar{x})\rangle, \ldots, \langle \nabla f_m(\bar{x}), \eta(y, \bar{x})\rangle) \notin -R^m_+ \setminus \{0\}, \ \forall\ y \in X.$$

Theorem 3.1. *Let $f : X \mapsto R^m$ be a differentiable function defined on the nonempty invex set X such that any η-path is contained in int$X$. Assume that η is skew and satisfies Condition C and $\alpha_i, i \in M$ is symmetric and satisfies Condition D. If f is (G_f, α)-invex with respect to η on X and $G'_{f_i}(f_i(y)) > 0, \ \forall\ i \in M, \ \forall\ y \in X$, then $\bar{x}$ is an efficient solution of (VOP) if and only if $\bar{x}$ solves (MVVLIP).*

Proof. Suppose $\bar{x} \in X$ solves (MVVLIP) but it is not an efficient solution of (VOP). Then, there exists $z \in X$ such that

$$(f_1(z) - f_1(\bar{x}), \ldots, f_m(z) - f_m(\bar{x})) \in -R^m_+ \setminus \{0\}$$
$$\Rightarrow \qquad f_i(z) - f_i(\bar{x}) \leq 0, \ \forall\ i \in M,$$

with strict inequality for at least one $i \in M$.

Since $G_{f_i} : I_{f_i}(X) \mapsto R, \ i \in M$ is a strictly increasing function on $I_{f_i}(X)$, therefore from the above inequalities it follows that

$$G_{f_i}(f_i(z)) - G_{f_i}(f_i(\bar{x})) \leq 0, \ \forall\ i \in M, \tag{8}$$

with strict inequality for at least one $i \in M$.

We set $z(\lambda) = \bar{x} + \lambda\eta(z, \bar{x})$ for all $\lambda \in [0, 1]$. Since X is an invex set with respect to η, $z(\lambda) \in X$ for all $\lambda \in [0, 1]$. Using Lemma 2.2, we have

$$G_{f_i}(f_i(z(\lambda))) = G_{f_i}(f_i(\bar{x} + \lambda\eta(z, \bar{x}))) \leq \lambda G_{f_i}(f_i(z)) + (1 - \lambda)G_{f_i}(f_i(\bar{x})), \ \forall\ i \in M$$
$$\Rightarrow \qquad G_{f_i}(f_i(\bar{x} + \lambda\eta(z, \bar{x}))) - G_{f_i}(f_i(\bar{x})) \leq \lambda[G_{f_i}(f_i(z)) - G_{f_i}(f_i(\bar{x}))], \ \forall\ i \in M.$$

In particular for $\lambda = 1$, we have

$$G_{f_i}(f_i(\bar{x} + \eta(z, \bar{x}))) - G_{f_i}(f_i(\bar{x})) \leq [G_{f_i}(f_i(z)) - G_{f_i}(f_i(\bar{x}))], \ \forall\ i \in M. \tag{9}$$

By the mean value theorem (Theorem 2.1), there exists $\lambda_i \in (0, 1)$ such that

$$G_{f_i}(f_i(\bar{x} + \eta(z, \bar{x}))) - G_{f_i}(f_i(\bar{x})) = \langle G'_{f_i}(f_i(z(\lambda_i)))\nabla f_i(z(\lambda_i)), \eta(z, \bar{x})\rangle, \ \forall\ i \in M. \tag{10}$$

Combining (9) and (10), we get that

$$\langle G'_{f_i}(f_i(z(\lambda_i)))\nabla f_i(z(\lambda_i)), \eta(z,\bar{x})\rangle \leq [G_{f_i}(f_i(z)) - G_{f_i}(f_i(\bar{x}))],\ \forall\ i\in M. \tag{11}$$

Firstly, we suppose that $\lambda_1 = \ldots = \lambda_m = \lambda$. On multiplying both sides of the above inequality by $-\lambda$, using skewness of η and Condition C, we obtain

$$\langle G'_{f_i}(f_i(z(\lambda)))\nabla f_i(z(\lambda)), \eta(z(\lambda),\bar{x})\rangle \leq \lambda[G_{f_i}(f_i(z)) - G_{f_i}(f_i(\bar{x}))],\ \forall\ i\in M.$$

By (8), the above inequality yields

$$\langle G'_{f_i}(f_i(z(\lambda)))\nabla f_i(z(\lambda)), \eta(z(\lambda),\bar{x})\rangle \leq 0,\ \forall\ i\in M,$$

with strict inequality for at least one $i\in M$. Since each component G_{f_i} $(i\in M)$ is strictly increasing, it follows that

$$\langle \nabla f_i(z(\lambda)), \eta(z(\lambda),\bar{x})\rangle \leq 0,\ \forall\ i\in M,$$

with strict inequality for at least one $i\in M$. This contradicts the fact that $\bar{x}\in X$ solves (MVVLIP). Hence we get the result.

Now, we consider the case when $\lambda_1,\ldots,\lambda_m$ are not all equal. Without loss of generality, we assume $\lambda_1\neq\lambda_2$. From (11), we have

$$\langle G'_{f_1}(f_1(z(\lambda_1)))\nabla f_1(z(\lambda_1)), \eta(z,\bar{x})\rangle \leq [G_{f_1}(f_1(z)) - G_{f_1}(f_1(\bar{x}))] \tag{12}$$

and

$$\langle G'_{f_2}(f_2(z(\lambda_2)))\nabla f_2(z(\lambda_2)), \eta(z,\bar{x})\rangle \leq [G_{f_2}(f_2(z)) - G_{f_2}(f_2(\bar{x}))]. \tag{13}$$

Since f is vector (G_f,α)-invex function with respect to η, then, by Lemma 2.1, the differential of G_{f_1} and G_{f_2} are V-invariant monotone on X with respect to η and $\alpha_i, i=1,2$ respectively, i.e., for all $x,y\in X$, we have

$$\begin{aligned}&\alpha_1(z(\lambda_2),z(\lambda_1))\langle G'_{f_1}(f_1(z(\lambda_1)))\nabla f_1(z(\lambda_1)), \eta(z(\lambda_2),z(\lambda_1))\rangle\\ &\quad + \alpha_1(z(\lambda_1),z(\lambda_2))\langle G'_{f_1}(f_1(z(\lambda_2)))\nabla f_1(z(\lambda_2)), \eta(z(\lambda_1),z(\lambda_2))\rangle \leq 0\\ \text{and}\quad &\alpha_2(z(\lambda_2),z(\lambda_1))\langle G'_{f_2}(f_2(z(\lambda_1)))\nabla f_2(z(\lambda_1)), \eta(z(\lambda_2),z(\lambda_1))\rangle\\ &\quad + \alpha_2(z(\lambda_1),z(\lambda_2))\langle G'_{f_2}(f_2(z(\lambda_2)))\nabla f_2(z(\lambda_2)), \eta(z(\lambda_1),z(\lambda_2))\rangle \leq 0.\end{aligned}$$

Using Condition C for η and symmetry of $\alpha_i, i\in M$ in the above two inequalities, we obtain

$$\begin{aligned}&(\lambda_2-\lambda_1)\alpha_1(z(\lambda_2),z(\lambda_1))\langle G'_{f_1}(f_1(z(\lambda_1)))\nabla f_1(z(\lambda_1)), \eta(z,\bar{x})\rangle\\ &\quad + (\lambda_1-\lambda_2)\alpha_1(z(\lambda_2),z(\lambda_1))\langle G'_{f_1}(f_1(z(\lambda_2)))\nabla f_1(z(\lambda_2)), \eta(z,\bar{x})\rangle \leq 0 \qquad (14)\\ \text{and }&(\lambda_2-\lambda_1)\alpha_2(z(\lambda_1),z(\lambda_2))\langle G'_{f_2}(f_2(z(\lambda_1)))\nabla f_2(z(\lambda_1)), \eta(z,\bar{x})\rangle\\ &\quad + (\lambda_1-\lambda_2)\alpha_2(z(\lambda_1),z(\lambda_2))\langle G'_{f_2}(f_2(z(\lambda_2)))\nabla f_2(z(\lambda_2)), \eta(z,\bar{x})\rangle \leq 0. \qquad (15)\end{aligned}$$

If $(\lambda_1-\lambda_2)>0$, dividing (14) by $(\lambda_1-\lambda_2)\alpha_1(z(\lambda_2),z(\lambda_1))$, it follows that

$$\langle G'_{f_1}(f_1(z(\lambda_1)))\nabla f_1(z(\lambda_1)), \eta(z,\bar{x})\rangle \geq \langle G'_{f_1}(f_1(z(\lambda_2)))\nabla f_1(z(\lambda_2)), \eta(z,\bar{x})\rangle.$$

By (12), the above inequality yields

$$\langle G'_{f_1}(f_1(z(\lambda_2)))\nabla f_1(z(\lambda_2)), \eta(z,\bar{x})\rangle \leq [G_{f_1}(f_1(z)) - G_{f_1}(f_1(\bar{x}))]. \tag{16}$$

If $(\lambda_2-\lambda_1)>0$, dividing (15) by $(\lambda_2-\lambda_1)\alpha_2(z(\lambda_1),z(\lambda_2))$, it follows that

$$\langle G'_{f_2}(f_2(z(\lambda_1)))\nabla f_2(z(\lambda_1)), \eta(z,\bar{x})\rangle \leq \langle G'_{f_2}(f_2(z(\lambda_2)))\nabla f_2(z(\lambda_2)), \eta(z,\bar{x})\rangle.$$

By (13), the above inequality yields

$$\langle G'_{f_2}(f_2(z(\lambda_1)))\nabla f_2(z(\lambda_1)), \eta(z,\bar{x})\rangle \leq [G_{f_2}(f_2(z)) - G_{f_2}(f_2(\bar{x}))]. \tag{17}$$

Thus for the case $\lambda_1 \neq \lambda_2$, we set $\bar{\lambda} = \min\{\lambda_1, \lambda_2\}$. Hence, from (16) and (17), we conclude that

$$\langle G'_{f_i}(f_i(z(\bar{\lambda})))\nabla f_i(z(\bar{\lambda})), \eta(z,\bar{x})\rangle \leq [G_{f_i}(f_i(z)) - G_{f_i}(f_i(\bar{x}))],\ i = 1, 2.$$

On continuing this process, we obtain $\lambda^* \in (0,1)$ such that $\lambda^* = \min\{\lambda_1, \ldots, \lambda_m\}$ and

$$\langle G'_{f_i}(f_i(z(\lambda^*)))\nabla f_i(z(\lambda^*)), \eta(z,\bar{x})\rangle \leq [G_{f_i}(f_i(z)) - G_{f_i}(f_i(\bar{x}))],\ \forall\, i \in M.$$

Multiplying the above inequalities by $-\lambda^*$ and by using Condition C, skewness of η and (8), we get

$$\langle G'_{f_i}(f_i(z(\lambda^*)))\nabla f_i(z(\lambda^*)), \eta(z(\lambda^*),\bar{x})\rangle \leq 0,\ \forall\, i \in M,$$

with strict inequality for at least one $i \in M$. Since $G'_{f_i}(f_i(z(\lambda^*))) > 0$, it follows that

$$\langle \nabla f_i(z(\lambda^*)), \eta(z(\lambda^*),\bar{x})\rangle \leq 0,\ \forall\, i \in M,$$

with strict inequality for at least one $i \in M$. This contradicts the fact that $\bar{x}$ solves (MVVLIP). Hence we get the result.

Conversely, let $\bar{x}$ be an efficient solution of (VOP). Then, for all $y \in X$, we have

$$f(y) - f(\bar{x}) = (f_1(y) - f_1(\bar{x}), \ldots, f_m(y) - f_m(\bar{x})) \notin -R^m_+ \setminus \{0\}. \tag{18}$$

Suppose, contrary to the result, that $\bar{x}$ does not solve (MVVLIP) with respect to η. Then, there exists $y \in X$ such that

$$\langle \nabla f_i(y), \eta(y,\bar{x})\rangle \leq 0,\ \forall\, i \in M,$$

with strict inequality for at least one $i \in M$. Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$, each component G_{f_i} is strictly increasing and by assumption $G'_{f_i}(f_i(y)) > 0$, the above inequality can be rewritten as

$$\alpha_i(\bar{x},y)\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(y,\bar{x})\rangle \leq 0,\ \forall\, i \in M,$$

with strict inequality for at least one $i \in M$. By the skewness of η, it follows that

$$\alpha_i(\bar{x},y)\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x},y)\rangle \geq 0,\ \forall\, i \in M, \tag{19}$$

with strict inequality for at least one $i \in M$. Since f is vector (G_f, α)-invex with respect to η, we have

$$G_{f_i}(f_i(\bar{x})) - G_{f_i}(f_i(y)) \geq \alpha_i(\bar{x},y)\left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x},y)\right\rangle,\ \forall\, i \in M. \tag{20}$$

From (19) and (20), we conclude that

$$G_{f_i}(f_i(\bar{x})) - G_{f_i}(f_i(y)) \geq 0,\ \forall\, i \in M,$$

$$\Rightarrow \qquad f_i(y) - f_i(\bar{x}) \leq 0,\ \forall\, i \in M,$$

with strict inequality for at least one $i \in M$. This contradicts the fact that $\bar{x}$ is an efficient solution of (VOP). □

Theorem 3.2. *Let $f : X \mapsto R^m$ be a differentiable function defined on the nonempty invex set X. If f is vector (G_f, α)-invex with respect to η on X, $G'_{f_i}(f_i(y)) > 0,\ \forall\, i \in M,\ \forall\, y \in X$ and $\bar{x}$ solves (SVVLIP), then $\bar{x}$ is an efficient solution of (VOP). Furthermore if η is skew, then $\bar{x}$ solves (MVVLIP).*

Proof. Suppose, contrary to the result, that $\bar{x}$ is not an efficient solution of (VOP). Then, there exists $y \in X$ such that

$$(f_1(y) - f_1(\bar{x}), \ldots, f_m(y) - f_m(\bar{x})) \in -R^m_+ \setminus \{0\}$$
$$\Rightarrow \qquad f_i(y) - f_i(\bar{x}) \leq 0, \ \forall \ i \in M,$$

with strict inequality for at least one $i \in M$. Since $G_{f_i} : I_{f_i}(X) \mapsto R, \ i \in M$ is a strictly increasing function on $I_{f_i}(X)$, therefore from the above inequalities it follows that

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{x})) \leq 0, \ \forall \ i \in M, \tag{21}$$

with strict inequality for at least one $i \in M$. Since f is vector (G_f, α)-invex with respect to η, we have

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{x})) \geq \alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y, \bar{x}) \right\rangle, \ \forall \ i \in M. \tag{22}$$

On combining (21) and (22), we obtain

$$\alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y, \bar{x}) \right\rangle \leq 0, \ \forall \ i \in M,$$

with strict inequality for at least one $i \in M$. Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$ and $G'_{f_i}(f_i(\bar{x})) > 0$, the above inequality can be rewritten as

$$\langle \nabla f_i(\bar{x}), \eta(y, \bar{x}) \rangle \leq 0, \ \forall \ i \in M,$$

with strict inequality for at least one $i \in M$. This implies there exists $y \in X$ such that

$$(\langle \nabla f_1(\bar{x}), \eta(y, \bar{x}) \rangle, \ldots, \langle \nabla f_m(\bar{x}), \eta(y, \bar{x}) \rangle) \in -R^m_+ \setminus \{0\},$$

which contradicts the fact that $\bar{x}$ solves (SVVLIP). Hence we get the result.

Suppose, contrary to the result, that $\bar{x}$ is not a solution of (MVVLIP). Then, there exists $y \in X$ such that

$$\langle \nabla f_i(y), \eta(y, \bar{x}) \rangle \leq 0, \ \forall \ i \in M,$$

with strict inequality for at least one $i \in M$. Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$, each component G_{f_i} is strictly increasing and by assumption $G'_{f_i}(f_i(y)) > 0$, the above inequality can be rewritten as

$$\alpha_i(\bar{x}, y) \langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(y, \bar{x}) \rangle \leq 0, \ \forall \ i \in M,$$

with strict inequality for at least one $i \in M$. By the skewness of η, it follows that

$$\alpha_i(\bar{x}, y) \langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y) \rangle \geq 0, \ \forall \ i \in M, \tag{23}$$

with strict inequality for at least one $i \in M$. Since f is vector (G_f, α)-invex with respect to η, by Lemma 2.1, the differential of G_{f_i} is V-invariant monotone on X with respect to η and $\alpha_i, \ i \in M$, respectively. Therefore for all $y \in X$, we have

$$\alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y, \bar{x}) \right\rangle + \alpha_i(\bar{x}, y) \left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y) \right\rangle \leq 0, \ \forall \ i \in M.$$

By the skewness of η, the above inequalities imply

$$\alpha_i(\bar{x}, y) \left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y) \right\rangle \leq \alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(\bar{x}, y) \right\rangle, \ \forall \ i \in M. \tag{24}$$

From (23) and (24), we conclude that

$$\alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(\bar{x}, y) \right\rangle \geq 0, \ \forall \ i \in M,$$

with strict inequality for at least one $i \in M$. Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$, $G'_{f_i}(f_i(\bar{x})) > 0$ and

η is skew, the above inequality can be rewritten as

$$\langle \nabla f_i(\bar{x}), \eta(y, \bar{x}) \rangle \leq 0, \ \forall \ i \in M,$$

with strict inequality for at least one $i \in M$. This contradicts the fact that $\bar{x}$ solves (SVVLIP). This completes the proof. $\square$

Now, we present an example to illustrate the above theorem.

Example 3.1 Let $X = \{(x_1, x_2) \in R^2 : x_1 \geq 1, x_2 \geq 1 \wedge x_1 \geq x_2\}$. Consider the following vector optimization problem:

$$\textbf{(VOP)} \qquad \text{Minimize } f(x) = \left(\ln\left(\frac{x_1^2}{x_2^2}\right), \exp\left(\frac{x_2}{x_1}\right)\right)$$
$$\text{subject to } x \in X,$$

where $f : X \mapsto R^2$ is a differentiable function. We define

$$G_{f_1}(t) = \sqrt{e^t}, \ G_{f_2}(t) = \ln(t), \alpha_1(x, \bar{x}) = \frac{\bar{x}_2}{x_2}, \ \alpha_2(x, \bar{x}) = \frac{\bar{x}_1}{x_1} \ \text{ and } \eta(x, \bar{x}) = x - \bar{x}.$$

Then, f is (G_f, α)-invex with respect to η on X as shown below. Here,

$$G'_{f_1}(t) = \frac{\sqrt{e^t}}{2}, \ G'_{f_2}(t) = \frac{1}{t}$$

$$\Rightarrow \qquad G'_{f_1}(f_1(x)) = \left(\frac{x_1}{2x_2}\right), \ G'_{f_2}(f_2(x)) = \frac{1}{\exp\left(\dfrac{x_2}{x_1}\right)}$$

$$\text{and} \quad \nabla f_1(x) = 2\left(\frac{1}{x_1}, -\frac{1}{x_2}\right), \nabla f_2(x) = \exp\left(\frac{x_2}{x_1}\right)\left(-\frac{x_2}{x_1^2}, \frac{1}{x_1}\right).$$

Now,

$$\begin{aligned}
&G_{f_1}(f_1(x)) - G_{f_1}(f_1(\bar{x})) - \alpha_1(x, \bar{x}) \left\langle G'_{f_1}(f_1(\bar{x})) \nabla f_1(\bar{x}), \eta(x, \bar{x}) \right\rangle \\
&= \left(\frac{x_1}{x_2}\right) - \left(\frac{\bar{x}_1}{\bar{x}_2}\right) - \frac{\bar{x}_2}{x_2}\left\langle \left(\frac{\bar{x}_1}{2\bar{x}_2}\right) 2\left(\frac{1}{\bar{x}_1}, -\frac{1}{\bar{x}_2}\right), (x_1 - \bar{x}_1, x_2 - \bar{x}_2) \right\rangle \\
&= \frac{x_1\bar{x}_2 - \bar{x}_1 x_2}{x_2\bar{x}_2} - \frac{\bar{x}_2}{x_2}\left(\frac{x_1\bar{x}_2 - \bar{x}_1 x_2}{\bar{x}_2^2}\right) \\
&= \frac{x_1\bar{x}_2 - \bar{x}_1 x_2 - x_1\bar{x}_2 + \bar{x}_1 x_2}{x_2\bar{x}_2} \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&G_{f_2}(f_2(x)) - G_{f_2}(f_2(\bar{x})) - \alpha_2(x, \bar{x}) \left\langle G'_{f_2}(f_2(\bar{x})) \nabla f_2(\bar{x}), \eta(x, \bar{x}) \right\rangle \\
&= \left(\frac{x_2}{x_1}\right) - \left(\frac{\bar{x}_2}{\bar{x}_1}\right) - \frac{\bar{x}_1}{x_1}\left\langle \frac{1}{\exp\left(\dfrac{\bar{x}_2}{\bar{x}_1}\right)} \exp\left(\frac{\bar{x}_2}{\bar{x}_1}\right)\left(-\frac{\bar{x}_2}{\bar{x}_1^2}, \frac{1}{\bar{x}_1}\right), (x_1 - \bar{x}_1, x_2 - \bar{x}_2) \right\rangle \\
&= \frac{\bar{x}_1 x_2 - x_1\bar{x}_2}{x_1\bar{x}_1} - \frac{\bar{x}_1}{x_1}\left(\frac{-x_1\bar{x}_2 + \bar{x}_1 x_2}{\bar{x}_1^2}\right) \\
&= \frac{\bar{x}_1 x_2 - x_1\bar{x}_2 + x_1\bar{x}_2 - \bar{x}_1 x_2}{x_1\bar{x}_1} \\
&= 0.
\end{aligned}$$

We observe that $\bar{x} = (1,1)$ solves the (SVVLIP), since

$$
\begin{aligned}
&(\langle \nabla f_1(\bar{x}), \eta(y,\bar{x})\rangle, \langle \nabla f_2(\bar{x}), \eta(y,\bar{x})\rangle)\\
&= \left(\left\langle 2\left(\frac{1}{\bar{x}_1}, -\frac{1}{\bar{x}_2}\right), (y_1-\bar{x}_1, y_2-\bar{x}_2) \right\rangle, \left\langle \exp\left(\frac{\bar{x}_2}{\bar{x}_1}\right)\left(-\frac{\bar{x}_2}{\bar{x}_1^2}, \frac{1}{\bar{x}_1}\right), (y_1-\bar{x}_1, y_2-\bar{x}_2) \right\rangle \right)\\
&= (2(y_1-y_2), e(y_2-y_1)) \notin -R^2_+ \setminus \{0\}, \quad \forall\, y \in X.
\end{aligned}
$$

Hence, by Theorem 3.2, $\bar{x} = (1,1)$ is an efficient solution of (VOP). Also, η is skew, therefore $\bar{x}$ solves (MVVLIP) which can be easily verified.

Remark 3.1 Every vector G-invex function is (G,α)-invex function with $\alpha_i = 1$ but the converse may not hold in general. The function f considered in the above example clearly verifies that it is not vector G_f-invex function at $\bar{x} = (1,1)$ with respect to the same η on X. Also f defined in Example 3.1 is not V-invex at $\bar{x} = (1,1)$, since

$$
\begin{aligned}
&f_1(x) - f_1(\bar{x}) - \alpha_1(x,\bar{x})\langle \nabla f_1(\bar{x}), \eta(x,\bar{x})\rangle\\
&= \ln\left(\frac{x_1^2}{x_2^2}\right) - \ln\left(\frac{\bar{x}_1^2}{\bar{x}_2^2}\right) - \frac{\bar{x}_2}{x_2}\left\langle 2\left(\frac{1}{\bar{x}_1}, -\frac{1}{\bar{x}_2}\right), (x_1-\bar{x}_1, x_2-\bar{x}_2)\right\rangle\\
&= \ln\left(\frac{x_1^2}{x_2^2}\right) - \ln\left(\frac{\bar{x}_1^2}{\bar{x}_2^2}\right) - \frac{2\bar{x}_2}{x_2}\left(\frac{x_1\bar{x}_2 - \bar{x}_1 x_2}{\bar{x}_1\bar{x}_2}\right).
\end{aligned}
$$

At $\bar{x} = (1,1)$ and $x = (2,1)$,

$$f_1(x) - f_1(\bar{x}) - \alpha_1(x,\bar{x})\langle \nabla f_1(\bar{x}), \eta(x,\bar{x})\rangle = \ln(4) - 2 = -0.613706.$$

Theorem 3.3. *Let $f : X \mapsto R^m$ be a differentiable function defined on the nonempty invex set X. If f is (G_f,α)-invex at $\bar{x}$ with respect to η on X and $\bar{x}$ is a vector critical point of (VOP) then $\bar{x}$ is a weak efficient solution of (VOP).*

Proof. Let $\bar{x}$ be a vector critical point of (VOP). Suppose, contrary to the result, that $\bar{x}$ is not a weak efficient solution of (VOP). Then, there exists $y \in X$ such that

$$
\begin{aligned}
&(f_1(y) - f_1(\bar{x}), \ldots, f_m(y) - f_m(\bar{x})) \in -\mathrm{int}R^m_+\\
\Rightarrow\quad & f_i(y) - f_i(\bar{x}) < 0, \ \forall\, i \in M.
\end{aligned}
$$

Since $G_{f_i} : I_{f_i}(X) \mapsto R,\ i \in M$ is a strictly increasing function on $I_{f_i}(X)$, therefore from the above inequalities it follows that

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{x})) < 0, \ \forall\, i \in M. \tag{25}$$

By vector (G_f,α)-invexity of f with respect to η, we have

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{x})) \geq \alpha_i(y,\bar{x})\left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y,\bar{x})\right\rangle, \ \forall\, i \in M. \tag{26}$$

On combining (25) and (26), we obtain

$$\alpha_i(y,\bar{x})\left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y,\bar{x})\right\rangle < 0, \ \forall\, i \in M.$$

Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$ and each component G_{f_i} is strictly increasing, the above inequality can be rewritten as

$$\langle \nabla f_i(\bar{x}), \eta(y,\bar{x})\rangle < 0, \ \forall\, i \in M.$$

This implies that $\langle \nabla f(\bar{x}), \eta(y,\bar{x})\rangle_m < 0$ has a solution in R^n. Therefore, by Gordan's Theorem, the above inequality implies that the system

$$\lambda^T \nabla f(\bar{x}) = 0, \lambda \in R^m, \lambda \geq 0,$$

has no solution for λ,which contradicts the fact that $\bar{x}$ is a vector critical point of (VOP). This completes the proof. □

§4 Relationships between vector optimization problems and weak vector variational-like inequality problems

In this section, we consider weak Minty vector variational-like inequalities and weak Stampacchia vector variational-like inequalities and establish the relationship of their solutions with the weak efficient solutions of vector optimization problems involving vector (G, α)-invex functions.

(WMVVLIP) A weak Minty vector variational-like inequality problem is to find a point $y \in X$ such that

$$(\langle \nabla f_1(\bar{x}), \eta(\bar{x}, y)\rangle, \ldots, \langle \nabla f_m(\bar{x}), \eta(\bar{x}, y)\rangle) \notin -\text{int}R^m_+, \quad \forall\, \bar{x} \in X.$$

(WSVVLIP) A weak Stampacchia vector variational-like inequality problem is to find a point $\bar{x} \in X$ such that

$$(\langle \nabla f_1(\bar{x}), \eta(y, \bar{x})\rangle, \ldots, \langle \nabla f_m(\bar{x}), \eta(y, \bar{x})\rangle) \notin -\text{int}R^m_+, \quad \forall\, y \in X.$$

Theorem 4.1. *Let $f : X \mapsto R^m$ be a differentiable function defined on the nonempty invex set X. If f is vector (G_f, α)-invex at $\bar{x}$ with respect to η on X such that η is skew and $\bar{x}$ solves (WSVVLIP), then $\bar{x}$ is a weak efficient solution of (VOP).*

Proof. Suppose, contrary to the result, that $\bar{x}$ is not a weak efficient solution of (VOP). Then, there exists $y \in X$ such that

$$(f_1(y) - f_1(\bar{x}), \ldots, f_m(y) - f_m(\bar{x})) \in -\text{int}R^m_+$$
$$\Rightarrow \qquad f_i(y) - f_i(\bar{x}) < 0,\ \forall\, i \in M.$$

Since $G_{f_i} : I_{f_i}(X) \mapsto R,\ i \in M$ is a strictly increasing function on $I_{f_i}(X)$, therefore from the above inequalities it follows that

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{x})) < 0,\ \forall\, i \in M. \tag{27}$$

By vector (G_f, α)-invexity of f with respect to η, we have

$$G_{f_i}(f_i(y)) - G_{f_i}(f_i(\bar{x})) \geq \alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y, \bar{x})\right\rangle,\ \forall\, i \in M. \tag{28}$$

On combining (27) and (28), we obtain

$$\alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y, \bar{x})\right\rangle < 0,\ \forall\, i \in M.$$

Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$ and each component G_{f_i} is strictly increasing, the above inequality can be rewritten as

$$\langle \nabla f_i(\bar{x}), \eta(y, \bar{x})\rangle < 0,\ \forall\, i \in M.$$

This implies there exists $y \in X$ such that

$$(\langle \nabla f_1(\bar{x}), \eta(y,\bar{x})\rangle, \ldots, \langle \nabla f_m(\bar{x}), \eta(y,\bar{x})\rangle) \in -\text{int}R^m_+,$$

which contradicts the fact that $\bar{x}$ solves (WSVVLIP). This completes the proof. □

The following example is constructed to illustrate the above theorem.

Example 4.1 Let $X = [-1, 0]$. Consider the following vector optimization problem:

$$\textbf{(VOP)} \qquad \text{Minimize } f(x) = (\ln(x^2+1), \tan x)$$
$$\text{subject to } x \in X,$$

where $f : X \mapsto R^2$ is a differentiable function. We define

$$G_{f_1}(t) = e^t,\ G_{f_2}(t) = \arctan(t),$$
$$\alpha_1(x,\bar{x}) = \alpha_2(x,\bar{x}) = 1 \text{ and } \eta(x,\bar{x}) = x - \bar{x}.$$

Then, f is (G_f,α)-invex with respect to η at $\bar{x} = 0$ as shown below. Here,

$$G'_{f_1}(t) = e^t,\ G'_{f_2}(t) = \frac{1}{1+t^2}$$
$$\Rightarrow \quad G'_{f_1}(f_1(x)) = x^2+1,\ G'_{f_2}(f_2(x)) = \cos^2 x$$
$$\text{and} \qquad \nabla f_1(x) = \frac{2x}{x^2+1},\ \nabla f_2(x) = \sec^2 x.$$

Now,

$$\begin{aligned} &G_{f_1}(f_1(x)) - G_{f_1}(f_1(\bar{x})) - \alpha_1(x,\bar{x}) \left\langle G'_{f_1}(f_1(\bar{x}))\nabla f_1(\bar{x}), \eta(x,\bar{x})\right\rangle \\ &= (x^2+1) - (\bar{x}^2+1) - \left\langle (\bar{x}^2+1)\left(\frac{2\bar{x}}{\bar{x}^2+1}\right), (x-\bar{x})\right\rangle \\ &= x^2 \geq 0. \end{aligned}$$

Similarly,

$$\begin{aligned} &G_{f_2}(f_2(x)) - G_{f_2}(f_2(\bar{x})) - \alpha_2(x,\bar{x}) \left\langle G'_{f_2}(f_2(\bar{x}))\nabla f_2(\bar{x}), \eta(x,\bar{x})\right\rangle \\ &= x - \bar{x} - \langle \cos^2\bar{x}.\sec^2\bar{x}, (x-\bar{x})\rangle \\ &= 0. \end{aligned}$$

We observe that $\bar{x} = 0$ solves the (WSVVLIP), since

$$\begin{aligned} &(\langle \nabla f_1(\bar{x}), \eta(y,\bar{x})\rangle, \langle \nabla f_2(\bar{x}), \eta(y,\bar{x})\rangle) \\ &= \left(\left\langle \frac{2\bar{x}}{\bar{x}^2+1}, y-\bar{x}\right\rangle, \langle \sec^2\bar{x}, y-\bar{x}\rangle\right) \\ &= (0, y) \notin -\text{int}R^2_+,\ \forall\, y \in X. \end{aligned}$$

Clearly, $\bar{x} = 0$ does not solve (SVVLIP). Since all the assumptions of Theorem 4.1 are satisfied, $\bar{x} = 0$ is a weak efficient solution of (VOP).

Theorem 4.2. *Let $f : X \mapsto R^m$ be a differentiable function defined on the nonempty invex set X. If f is vector (G_f,α)-invex with respect to skew function η on X such that $G'_{f_i}(f_i(y)) > 0$ and $\bar{x}$ solves (WSVVLIP), then $\bar{x}$ solves (WMVVLIP).*

Proof. Suppose, contrary to the result, that $\bar{x}$ is not a solution of (WMVVLIP). Then, there exists $y \in X$ such that

$$\langle \nabla f_i(y), \eta(y, \bar{x}) \rangle < 0, \ \forall \ i \in M.$$

Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$, each component G_{f_i} is strictly increasing and by assumption $G'_{f_i}(f_i(y)) > 0$, the above inequality can be rewritten as

$$\alpha_i(\bar{x}, y)\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(y, \bar{x}) \rangle < 0, \ \forall \ i \in M.$$

In view of the skewness of η, it follows that

$$\alpha_i(\bar{x}, y)\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y) \rangle > 0, \ \forall \ i \in M. \tag{29}$$

As f is vector (G_f, α)-invex with respect to η, by Lemma 2.1, the differential of G_{f_i} is V-invariant monotone on X with respect to η and α_i, $i \in M$, respectively. Therefore for all $y \in X$, we have

$$\alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(y, \bar{x}) \right\rangle + \alpha_i(\bar{x}, y) \left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y) \right\rangle \leq 0, \ \forall \ i \in M.$$

Again η is skew, the above inequalities imply

$$\alpha_i(\bar{x}, y) \left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y) \right\rangle \leq \alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(\bar{x}, y) \right\rangle, \ \forall \ i \in M. \tag{30}$$

From (29) and (30), we conclude that

$$\alpha_i(y, \bar{x}) \left\langle G'_{f_i}(f_i(\bar{x}))\nabla f_i(\bar{x}), \eta(\bar{x}, y) \right\rangle > 0, \ \forall \ i \in M.$$

Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$, each component G_{f_i} is strictly increasing and η is skew, the above inequality reduces to

$$\langle \nabla f_i(\bar{x}), \eta(y, \bar{x}) \rangle < 0, \ \forall \ i \in M,$$

which contradicts the fact that $\bar{x}$ solves (WSVVLIP). □

Theorem 4.3. *Let X be an invex set with respect to η, where η satisfies Condition C, and let ∇f be continuous. If $y \in X$ is a solution of (WMVVLIP), then y is a solution of (WSVVLIP).*

Proof. The proof follows from Theorem 4.2 [20]. □

Theorem 4.4. *Let $f : X \mapsto R^m$ be a differentiable function defined on the nonempty invex set X. Assume that ∇f is a continuous function, η is skew and satisfies Condition C. If f is vector (G_f, α)-invex with respect to η on X such that $G'_{f_i}(f_i(y)) > 0$, then $\bar{x}$ solves (WSVVLIP), if and only if $\bar{x}$ solves (WMVVLIP).*

Proof. The proof follows from Theorem 4.2 and Theorem 4.3. □

Theorem 4.5. *Let $f : X \mapsto R^m$ be a differentiable function defined on the nonempty invex set X. If f is vector (G_f, α)-invex with respect to skew function η on X such that $G'_{f_i}(f_i(y)) > 0$ and $\bar{x}$ is a weak efficient solution of (VOP), then $\bar{x}$ solves (WMVVLIP).*

Proof. Suppose, contrary to the result, that $\bar{x}$ is not a solution of (WMVVLIP). Then, there exists $y \in X$ such that

$$\begin{aligned} & (\langle \nabla f_1(y), \eta(y, \bar{x}) \rangle, \ldots, \langle \nabla f_m(y), \eta(y, \bar{x}) \rangle) \in -\text{int} R^m_+ \\ \Rightarrow \quad & \langle \nabla f_i(y), \eta(y, \bar{x}) \rangle < 0, \ \forall \ i \in M. \end{aligned}$$

Since $\alpha_i : X \times X \mapsto R_+ \setminus \{0\}$, each component G_{f_i} is strictly increasing and by assumption $G'_{f_i}(f_i(y)) > 0$, the above inequality can be rewritten as

$$\alpha_i(\bar{x}, y)\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(y, \bar{x})\rangle < 0, \ \forall\ i \in M.$$

By the skewness of η, it follows that

$$\alpha_i(\bar{x}, y)\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y)\rangle > 0, \ \forall\ i \in M. \tag{31}$$

By (G_f, α)-invexity of f with respect to η, we have

$$G_{f_i}(f_i(\bar{x})) - G_{f_i}(f_i(y)) \geq \alpha_i(\bar{x}, y)\left\langle G'_{f_i}(f_i(y))\nabla f_i(y), \eta(\bar{x}, y)\right\rangle, \ \forall\ i \in M. \tag{32}$$

From (31) and (32), we conclude that

$$G_{f_i}(f_i(\bar{x})) - G_{f_i}(f_i(y)) > 0, \ \forall\ i \in M.$$

Again, since each component G_{f_i} is strictly increasing, the above inequality implies

$$f_i(y) - f_i(\bar{x}) < 0, \ \forall\ i \in M$$

$$\Rightarrow \quad (f_1(y) - f_1(\bar{x}), \ldots, f_m(y) - f_m(\bar{x})) \in -\text{int}R^m_+$$

which contradicts the fact that $\bar{x}$ is a weak efficient solution of (VOP). □

§5 Conclusion

In this paper, we studied the relationship among Minty vector variational-like inequality problem, Stampacchia vector variational-like inequality problem and efficient solutions of vector optimization problem involving (G, α)-invex functions. We constructed an example to illustrate our derived result. Furthermore, we managed to establish equivalence among the vector critical points, weak efficient points of the vector optimization problem and solutions of the weak Stampacchia and weak Minty vector variational-like inequality problems under the assumption of (G, α)-invexity. Our results generalize some well-known results.

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