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Some limit results on supremum of Shepp statistics for fractional Brownian motion

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Abstract. Define the incremental fractional Brownian field $Z_H(\tau, s) = B_H(s+\tau) - B_H(s)$, where $B_H(s)$ is a standard fractional Brownian motion with Hurst parameter $H \in (0,1)$. In this paper, we first derive an exact asymptotic of distribution of the maximum $M_H(T_u) = \sup_{\tau\in[0,1], s\in[0,xT_u]} Z_H(\tau, s)$, which holds uniformly for $x \in [A, B]$ with A, B two positive constants. We apply the findings to analyse the tail asymptotic and limit theorem of $M_H(\mathcal{T})$ with a random index $\mathcal{T}$. In the end, we also prove an almost sure limit theorem for the maximum $M_{1/2}(T)$ with non-random index T.

§1 Introduction

In this paper, we are interested in the limit properties of the maximum of the Shepp statistics for fractional Brownian motion (fBm). Let $\{B_H(t), t \geq 0\}$ be a standard fBm with Hurst parameter $H \in (0, 1)$. In applied probability (see e.g., [3, 27]), the following random field defined in terms of fBm plays an important role,

$$Z_H(\tau, s) = B_H(s+\tau) - B_H(s), \quad s, \tau \in [0, \infty).$$

The process $\{Z_H(\tau; s), \tau \geq 0\}$ is referd to as Shepp statistics in applied probability by various authors, see for example [5,9,10,12,14,20-23,28].

Let

$$M_H(T) = \sup_{\tau\in[0,1], s\in[0,T]} Z_H(\tau, s)$$

and denote by $\Psi(\cdot)$ the tail distribution function of a standard normal random variable.

Paper [28] presents some asymptotic results on $M_H(T)$ for the case $H = 1/2$.

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Theorem 1.1. *If T_u is a positive constant such that $\lim_{u\to\infty} T_u u^2 = \infty$ and $\lim_{u\to\infty} T_u u^2\Psi(u) = 0$, then*

$$P\left(M_{1/2}(T_u) > u\right) = T_u \widetilde{\mathcal{H}} u^2 \Psi(u)(1+o(1)), \quad u \to \infty, \tag{1}$$

and

$$\lim_{T\to\infty} P\left(\alpha_T(M_{1/2}(T) - \beta_T) \le x\right) = \exp\{-e^{-x}\}, \tag{2}$$

with a positive constant $\widetilde{\mathcal{H}}$ defined by

$$\widetilde{\mathcal{H}} = \lim_{a\to\infty}\lim_{b\to\infty} a^{-1} e^{-\frac{a+b}{2}} \mathrm{E}\Bigg(\exp\Bigg(\max_{\substack{0\le t\le a\\ 0\le s\le b}} B_{1/2}(t+s+a) - B_{1/2}(t)\Bigg)\Bigg)$$

and

$$\alpha_T = \sqrt{2\ln T}, \quad \beta_T = \alpha_T + \alpha_T^{-1}\ln(\widetilde{\mathcal{H}} + \frac{1}{2}(\ln\ln T - \ln \pi)).$$

Result (1) has been extended by [12] to the case $H \in (0, 1/2)$ as follows:

Theorem 1.2. *For any $H \in (0, 1/2)$ and any fixed $T > 0$*

$$P\left(M_H(T) > u\right) = T\frac{1}{H}\left(\frac{1}{2}\right)^{1/H} \mathcal{H}_{2H}^2 u^{\frac{2}{H}-2}\Psi(u)(1+o(1)), \quad u \to \infty. \tag{3}$$

Here $\mathcal{H}_{2H}$ denotes the well-known Pickands constant (see e.g., [16,18,19]), which is defined as $\mathcal{H}_{2H} = \lim_{\lambda\to\infty} \mathcal{H}_{2H}(\lambda)/\lambda$, with

$$\mathcal{H}_{2H}(\lambda) = \mathrm{E}\left(\exp\left(\max_{t\in[0,\lambda]} \sqrt{2}B_H(t) - t^{2H}\right)\right).$$

Recently, [5] completed the above results, where the case of $H \in (1/2, 1)$ was treated as an application of their main result.

Theorem 1.3. *For any $H \in (1/2, 1)$ and any fixed $T > 0$*

$$P\left(M_H(T) > u\right) = T\mathcal{H}_{2H} u^{\frac{1}{H}}\Psi(u)(1+o(1)), \quad u \to \infty. \tag{4}$$

In this paper, we will make some extensions on the above results. Firstly, we derive an exact asymptotic of $P\left(\sup_{\tau\in[0,1], sw_H(u)\in[0,x]} Z_H(\tau, s) \le u\right)$ as $u \to \infty$, where $w_H(u)$ is some function defined later. Secondly, with motivations from applications in queuing theory, insurance, and hydrodynamics (see [15]), by using the first result, we extend Theorems 1.1-1.3 to the random time case, ie. we will replace T by a positive random variable $\mathcal{T}$, which is independent of $\{B_H(t), t \ge 0\}$ and satisfies one of the following assumptions:

A1. $\mathcal{T}$ is integrable, i.e., $\mathrm{E}\,\mathcal{T} < \infty$;

A2. $\mathcal{T}$ has regularly varying tail distribution with parameter $\lambda \in [0, 1)$, i.e., $P(\mathcal{T} > t) = L(t)t^{-\lambda}$, where $L(\cdot)$ is slowly varying at ∞.

For some related studies, we refer to [1,2,4,6-8,17,29]. Thirdly, inspired by the second problem, we also consider the limit theorem of the maximum of the Shepp statistics with random index. Finally, we extended the second assertion of Theorem 1.1 to the almost sure version. For some related results in this field, we refer to [24] and the references therein.

The paper is organized as follows. Section 2 describes the main results. The proofs of the main results are shown in Section 3.

§2 Main results

Define

$$w_H(u) = m_H^{-1}(u) = \begin{cases} \frac{1}{H}\left(\frac{1}{2}\right)^{1/H}\mathcal{H}_{2H}^2 u^{\frac{2}{H}-2}\Psi(u), & H\in(0,1/2);\\ \widetilde{\mathcal{H}}u^2\Psi(u), & H=1/2;\\ \mathcal{H}_{2H}u^{\frac{1}{H}}\Psi(u), & H\in(1/2,1).\end{cases}$$

We state next our first main result:

Theorem 2.1. *For any $0<A_0<A_\infty<\infty$ and $H\in(0,1)$, we have*

$$P\left(M_H(xm_H(u))\le u\right)\to e^{-x}$$

uniformly for $x\in[A_0,A_\infty]$, as $u\to\infty$.

As an application of Theorem 2.1, we have the following result.

Theorem 2.2. *Let $\mathcal{T}$ be a positive random variable independent of $\{B_H(t),t\ge 0\}$ with $H\in(0,1)$.*

(i) *If $\mathcal{T}$ satisfies Assumption A1, then we have*

$$P\left(M_H(\mathcal{T})>u\right)=\mathrm{E}(\mathcal{T})w_H(u)(1+o(1)),\quad u\to\infty. \tag{5}$$

(ii) *If $\mathcal{T}$ satisfies Assumption A2, then we have*

$$P\left(M_H(\mathcal{T})>u\right)=\Gamma(1-\lambda)P(\mathcal{T}>m_H(u))(1+o(1)),\quad u\to\infty, \tag{6}$$

where Γ is the Gamma function.

Remark 2.1. A similar result for the extreme of standardised Shepp statistics has been obtained by [26], where the maximum is defined as

$$\widetilde{M}_H(\mathcal{T})=\sup_{\tau\in[a,b],s\in[0,\mathcal{T}]}[B_H(s+\tau)-B_H(s)]\tau^{-H}$$

with $0<a<b<\infty$. Note that $[B_H(s+\tau)-B_H(s)]\tau^{-H}$ is a locally stationary Gaussian random filed.

It is intuitive that when $\mathcal{T}$ is a non-negative random variable, then there is a certain connection between the result in (6) and the limit law for the normalised maximum.

Theorem 2.3. *Let $M_H(T)$ be defined as before and let $\mathcal{T}_t$ be a nonnegative random process such that $\mathcal{T}_t/t\to\mathcal{T}$ in probability, as $t\to\infty$. If further, $B_H(t)$ and $\mathcal{T}_t$ are independent, then*

$$\lim_{T\to\infty}\sup_{x\in\mathbb{R}}\left|P\left(\alpha_T\left(M_H(\mathcal{T}_T)-\beta_T\right)\le x\right)-\mathbb{E}\exp\{-\mathcal{T}e^{-x}\}\right|=0, \tag{7}$$

where $\alpha_T=\sqrt{2\ln T}$, and for $H\in(0,1/2)$

$$\beta_T=\alpha_T+\alpha_T^{-1}\Big((\frac{1}{H}-\frac{3}{2})\ln\ln T+\ln(2^{-3/2}\mathcal{H}_{2H}^2H^{-1}(2\pi)^{-1/2})\Big)$$

for $H=1/2$

$$\beta_T=\alpha_T+\alpha_T^{-1}\Big(\frac{1}{2}\ln\ln T+\ln(\widetilde{\mathcal{H}}\pi^{-1/2})\Big)$$

and for $H\in(1/2,1)$

$$\beta_T=\alpha_T+\alpha_T^{-1}\Big((\frac{1}{2H}-\frac{1}{2})\ln\ln T+\ln(2^{(\frac{1}{2H}-\frac{1}{2})}\mathcal{H}_{2H}(2\pi)^{-1/2})\Big).$$

Remark 2.2. (i) Theorem 2.3 still holds if we do not impose the independence between $B_H(t)$ and $\mathcal{T}_T$. We omit that result since it can be shown with similar arguments as in [11,25].
(ii) Let $\mathcal{T}_T = T$ in Theorem 2.3, we get the Gumbel limit theorem for $M_H(T)$ immediately.

Our last result is as follows:

Theorem 2.4. *Under the conditions of Theorem 1.1, we have*

$$\lim_{T\to\infty} \frac{1}{\ln T}\int_1^T \frac{1}{t} I\left(\alpha_t(M_{1/2}(t)-\beta_t)\le x\right)dt = \exp\{-e^{-x}\} \quad a.s.,$$

where $I(\cdot)$ denotes the indicator function and α_t, β_t are defined in Theorem 2.3 for case $H = 1/2$.

§3 Proofs

In this section, we give the proofs of the main results. In the following, C will denote positive constants whose values may vary from place to place.

3.1 Proof of Theorem 2.1

The proof of Theorem 2.1 is split into two cases: case $H \in (0, 1/2]$ and case $H \in (1/2, 1)$ and based on the following five lemmas. In this subsection, define $n_x := xm_H(u)$.

Lemma 3.1. *For any constant $\kappa \in (0, 1-(lnu)^2/u^2]$ and $H \in (0,1)$, we have as $u \to \infty$*

$$P\left(\max_{\tau\in[0,1],s\in[0,n_x]} Z_H(\tau,s)\le u\right) = P\left(\max_{\tau\in[\kappa,1],s\in[0,n_x]} Z_H(\tau,s)\le u\right)(1+o(1)), \tag{8}$$

uniformly for $x \in [A_0, A_\infty]$.

Proof. The result follows from Lemma 3.1 of [12]. □

Fix positive constants L and $\delta \in (0, L)$. For $k \ge 1$ and some constant $\kappa > 0$ we define

$$\mathbf{I}_k = [\kappa,1]\times[(k-1)L, kL-\delta), \quad \mathbf{I}_k^* = [\kappa,1]\times[kL-\delta, kL).$$

Setting

$$K_x := \left[\frac{n_x}{L}\right] \in \mathbb{N}, \tag{9}$$

where $[x]$ denotes the integral part of x, we have

$$[0,1]\times[\kappa,n_x] = \bigcup_{k=1}^{K_x}(\mathbf{I}_k\cup\mathbf{I}_k^*)\cup\mathbf{I}_{K_x+1}, \text{ where } \mathbf{I}_{K_x+1} = [\kappa,1]\times[K_xL, n_x]$$

implying the length of the last interval $|[K_xL, n_x]| \le L$. Since δ and L are independent of u, we can apply Theorems 1.1-1.3 for all rectangles $\mathbf{I}_k$ and $\mathbf{I}_k^*$, respectively. In the sequel, K_x always refers to the definition given in (9).

Lemma 3.2. *For $H \in (0,1)$, with the definitions of $\mathbf{I}_k$, $k \ge 1$ it follows that for $u \to \infty$ and $\delta \downarrow 0$*

$$P\left(\max_{\tau\in[0,1],s\in[0,n_x]} Z_H(\tau,s)\le u\right) = P\left(\max_{(\tau,s)\in\cup_{k=1}^{K_x}\mathbf{I}_k} Z_H(\tau,s)\le u\right)(1+o(1)), \tag{10}$$

uniformly for $x \in [A_0, A_\infty]$.

Proof. By the construction of the rectangles we may write

$$P\left(\max_{\tau\in[0,1],s\in[0,n_x]} Z_H(\tau,s)\le u\right) = P\left(\max_{(\tau,s)\in\cup_{k=1}^{K_x}(\mathbf{I}_k\cup\mathbf{I}_k^*)\cup\mathbf{I}_{K_x+1}} Z_H(\tau,s)\le u\right).$$

Thus, in order to prove (10), it suffices to show that

$$\Pi_u := \left|P\left(\max_{(\tau,s)\in\cup_{k=1}^{K_x}(\mathbf{I}_k\cup\mathbf{I}_k^*)\cup\mathbf{I}_{K_x+1}} Z_H(\tau,s)\le u\right) - P\left(\max_{(\tau,s)\in\cup_{k=1}^{K_x}\mathbf{I}_k} Z_H(\tau,s)\le u\right)\right| \to 0. \quad (11)$$

uniformly for $x\in[A_0,A_\infty]$ holds as $u\to\infty$ and $\delta\downarrow 0$. Since for all u large

$$\begin{aligned}\Pi_u &\le P\left(\max_{(\tau,s)\in\cup_{k=1}^{K_x}\mathbf{I}_k^*\cup\mathbf{I}_{K_x+1}} Z_H(\tau,s)>u\right)\\ &\le \sum_{k=1}^{K_x} P\left(\max_{(\tau,s)\in\mathbf{I}_k^*} Z_H(\tau,s)>u\right) + P\left(\max_{(\tau,s)\in\mathbf{I}_{K_x+1}} Z_H(\tau,s)>u\right). \quad (12)\end{aligned}$$

By Theorems 1.1-1.3 the right-hand side of (12) is bounded by

$$\begin{aligned}C\delta K_x w_H(u) + CLw_H(u) &= C\delta\frac{K_x}{n_x}n_x w_H(u) + C\frac{L}{n_x}n_x w_H(u)\\ &= C\frac{\delta}{L} + C\frac{L}{n_x}(1+o(1))\\ &\le C\frac{\delta}{L} + C\frac{L}{A_\infty m_H(u)}(1+o(1)). \quad (13)\end{aligned}$$

Since L is a positive constant, we conclude that the right-hand side of (13) tends to 0 uniformly for $x\in[A_0,A_\infty]$ as $u\to\infty$ and $\delta\downarrow 0$. Thus (11) follows, and hence the proof is complete. □

In Lemmas 3.3-3.5, we suppose $H\in(0,1/2]$. As in [13] we shall apply Berman's inequality to prove that the maximum on the rectangles $\mathbf{I}_k$ are asymptotically independent. Since the Berman's inequality only holds for sequences of Gaussian random variables, for some small $d>0$ and any u, we define a family of grid points as follows. Let

$$q = q(u) = du^{-\frac{1}{H}} \quad\text{and}\quad d = d(u) = (\ln u)^{-1}$$

and define the grid of points

$$s_{k,l} = (k-1)L + lq \quad\text{and}\quad \tau_j = 1 - jq, \quad (14)$$

with $(\tau_j, s_{k,l})\in\mathbf{I}_k$ for all integers $j,l\ge 0$, $k\ge 1$. These grid points are denoted simply by $(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}$ for fixed k, without mentioning the dependence on u.

Lemma 3.3. *Suppose $H\in(0,1/2]$. If φ denotes the probability density function of a standard Gaussian random variable, for the process $Z_H(\tau,s)$ we get for any $k\le K_x$*

$$P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s)\le u - d^H/u, \max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s)>u\right) = O(d^H K_x^{-1}), \quad (15)$$

as $u\to\infty$.

Proof. The proof is similar to that of Lemmas A1 and A2 in [26], so we omit it. □

Remark 3.1. When $H\in(1/2,1)$, the field $Z_H(\tau,s)$ degenerates into one dimension case in some sense, so Lemma 3.3 doesn't hold for case $H\in(1/2,1)$. We will see this point from the proof of case $H\in(1/2,1)$ of Theorem 2.1 below.

Lemma 3.4. *For any $H \in (0, 1/2]$ and any $k \geq 1$ we have as $u \to \infty$*

$$0 \leq P\left(\max_{(\tau,s)\in\cup\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - P\left(\max_{(\tau,s)\in\cup\mathbf{I}_k} Z_H(\tau,s) \leq u\right) \to 0 \tag{16}$$

and also

$$0 \leq \prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - \prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) \leq u\right) \to 0, \tag{17}$$

uniformly for $x \in [A_0, A_\infty]$.

Proof. By Theorems 1.1-1.3, we have

$$P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) > u\right) = (1+o(1))(L-\delta)w_H(u)$$

for any k, as $u \to \infty$. We show now that for any k also

$$P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) > u\right) = (1+o(1))P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) > u\right) \tag{18}$$

holds as $u \to \infty$. This is true since by Lemma 3.3 (set $u_d := u - d^H/u$)

$$\begin{aligned}
P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) > u\right) &\leq P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) > u_d\right)\\
&\quad + P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u_d, \max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) > u\right)\\
&\leq (1+O(d^H))P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) > u_d\right)\\
&= (1+O(d^H))P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) > u\right)
\end{aligned}$$

using $(u_d)^2 = u^2 - 2d^H + o(u^{-2})$ for large u. From this result, it is easy to see that for large u

$$P\left(u_d < \max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) = O(1)d^H(L-\delta)w_H(u).$$

Consequently,

$$\begin{aligned}
0 &\leq P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) \leq u\right)\\
&= P\left(u_d < \max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) + P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u_d\right)\\
&\quad - P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) \leq u\right)\\
&\leq O(d^H)Lw_H(u),
\end{aligned} \tag{19}$$

where $d^H = (d(u))^H \to 0$ as $u \to \infty$, which completes the proof of (18).

Next, utilizing (19) we obtain

$$\begin{aligned}
0 &\leq P\left(\max_{(\tau,s)\in\cup\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - P\left(\max_{(\tau,s)\in\cup\mathbf{I}_k} Z_H(\tau,s) \leq u\right)\\
&\leq \sum_{k=1}^{K_x}\left(P\left(\max_{(\tau,s)\in\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - P\left(\max_{(\tau,s)\in\mathbf{I}_k} Z_H(\tau,s) \leq u\right)\right)\\
&\leq O(d^H)LK_x w_H(u)
\end{aligned}$$

$$
\begin{aligned}
&= O(d^H) n_x w_H(u)\\
&\leq O(d^H) A_\infty \to 0
\end{aligned}
$$

uniformly for $x \in [A_0, A_\infty]$, as $u \to \infty$, which completes the proof of (16). Since

$$
\begin{aligned}
&\prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in \mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - \prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in \mathbf{I}_k} Z_H(\tau,s) \leq u\right)\\
&\leq \sum_{i=1}^{K_x}\left(P\left(\max_{(\tau,s)\in \mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - P\left(\max_{(\tau,s)\in \mathbf{I}_k} Z_H(\tau,s) \leq u\right)\right)
\end{aligned}
$$

the proof of (17) follows easily with similar arguments. □

Lemma 3.5. *Suppose $H \in (0, 1/2]$. With the above definitions and properties of $Z_H(\tau,s)$ we have*

$$
P\left(\max_{(\tau,s)\in \cup\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - \prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in \mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) \to 0, \quad u \to \infty, \tag{20}
$$

uniformly for $x \in [A_0, A_\infty]$.

Proof. The correlation function $r(\tau, s; \tau', s')$ of $Z_H(\tau, s)$ equals

$$
r(\tau, s; \tau', s') = \frac{1}{2\tau^H \tau'^H}[|s+\tau-s'|^{2H} + |s-s'-\tau'|^{2H} - |s+\tau-s'-\tau'|^{2H} - |s-s'|^{2H}] \tag{21}
$$

and thus by Taylor expansion

$$
r(\tau, s; \tau', s') = 1 - \frac{1}{2}(1+o(1))(|s+\tau-s'-\tau'|^{2H} + |s-s'|^{2H})
$$

as $\tau', \tau \uparrow 1$, $s - s' \to 0$ and $\tau - \tau' \to 0$. Applying Berman's inequality (see [16,19]) we have

$$
\begin{aligned}
&P\left(\max_{(\tau,s)\in \cup\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right) - \prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in \mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \leq u\right)\\
&= P\left(\max_{(\tau,s)\in \cup\mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s)\tau^{-H} \leq u\tau^{-H}\right) - \prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in \mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s)\tau^{-H} \leq u\tau^{-H}\right)\\
&\leq \sum_{k\neq k'} \sum_{\substack{(\tau_j, s_{k,l})\in \mathbf{I}_k\cap\mathcal{R}\\ (\tau_j', s_{k',l'})\in \mathbf{I}_k'\cap\mathcal{R}}} |r(\tau_j, s_{k,l}, \tau_j', s_{k',l'})| \exp\left(-\frac{(\tau_j^{-2H} + \tau_j'^{-2H})u^2}{2(1 + r(\tau_j, s_{k,l}, \tau_j', s_{k',l'}))}\right)\\
&\leq \sum_{k\neq k'} \sum_{\substack{(\tau_j, s_{k,l})\in \mathbf{I}_k\cap\mathcal{R}\\ (\tau_j', s_{k',l'})\in \mathbf{I}_k'\cap\mathcal{R}}} |r(\tau_j, s_{k,l}, \tau_j', s_{k',l'})| \exp\left(-\frac{u^2}{1 + r(\tau_j, s_{k,l}, \tau_j', s_{k',l'})}\right).
\end{aligned}
$$

Since $|s_{k,l} - s_{k',l'}| \geq \delta$ by definition, $r(\tau_j, s_{k,l}, \tau_j', s_{k',l'}) \leq \rho < 1$. In view of (21) for any τ, τ' with $0 < \kappa < \tau, \tau' < 1$

$$
r(\tau, s; \tau', s') \leq C|s - s'|^{2H-2}
$$

for all s, s' with $|s - s'|$ sufficiently large. Consequently,

$$
\sup_{|s_{k,l} - s_{k',l'}| \geq s} r(\tau_j, s_{k,l}, \tau_j', s_{k',l'}) \leq Cs^{\lambda}
$$

for $\lambda = 2H - 2 < 0$, if s is large. Here we only deal with the case $H \in (0, 1/2)$, since the case $H = 1/2$ is obvious. Set $\beta < (1-\rho)/(1+\rho)$ and split the last sum into two parts S_1

and S_2 with $|s_{k,l}-s_{k',l'}|<n_x^{\beta}$ and $|s_{k,l}-s_{k',l'}|\geq n_x^{\beta}$, respectively. For the first sum there are $n_x^{1+\beta}/q^2$ combinations of two points $s_{k,l},s_{k',l'}\in\cup_k\mathbf{I}_k$. Together with the τ_j combinations there are $(n_x^{1+\beta}/q^2)(1/q^2)$ terms in the sum S_1. Note that

$$n_x w_H(u)=O(1),\quad u\to\infty,$$

which implies for $H\in(0,1/2)$

$$u^2=2\ln n_x+(\frac{2}{H}-3)\ln\ln n_x+O(1).$$

Thus, S_1 is bounded by

$$\begin{aligned}
&\rho\frac{n_x^{1+\beta}}{q^4}\exp\left(-\frac{u^2}{1+\rho}\right)\\
\leq\;&\rho\exp\left((1+\beta)\ln n_x+4\ln[1/2\ln(2\ln n_x)(2\ln n_x)^{1/2H}]-\frac{2(1+o(1))}{1+\rho}\ln n_x\right)\\
=\;&\rho\exp\left((\ln n_x)\left[(1+\beta)-\frac{2(1+o(1))}{1+\rho}+4\frac{\ln[1/2\ln(2\ln n_x)(2\ln n_x)^{1/2H}]}{\ln n_x}\right]\right)\\
\leq\;&\rho\exp\left((\ln(A_\infty m_H(u)))\left[(1+\beta)-\frac{2(1+o(1))}{1+\rho}+4\frac{\ln[(2\ln(A_\infty m_H(u)))^{1/2H+1}]}{\ln(A_0 m_H(u))}\right]\right)\to 0
\end{aligned}$$

uniformly for $x\in[A_0,A_\infty]$, as $u\to\infty$ since $1+\beta<2/(1+\rho)$ by the choice of β, using $d=(\ln u)^{-1}$ and $q=du^{-1/H}$.

For the second sum S_2 with $|s_{k,l}-s_{k',l'}|\geq n_x^{\beta}$, we use that

$$\sup_{|s_{k,l}-s_{k',l'}|\geq n_x^{\beta}} r(\tau_j,s_{k,l},\tau_j',s_{k',l'})\leq Cn_x^{\beta\lambda},$$

with $\lambda=2H-2<0$. In this case there are $(n_x/q)^2$ many combinations of two points $s_{k,l},s_{k',l'}\in\cup_k\mathbf{I}_k$. Hence S_2 is bounded by

$$\begin{aligned}
&n_x^{\beta\lambda}\frac{n_x^2}{q^2}\frac{1}{q^2}\exp\left(-\frac{u^2}{1+Cn_x^{\beta\lambda}}\right)\\
\leq\;&C\exp\left(\beta\lambda\ln n_x+2\ln n_x+4\ln[1/2\ln(2\ln n_x)(2\ln n_x)^{1/2H}]-\frac{2(1+o(1))}{1+Cn_x^{\beta\lambda}}\ln n_x\right)\\
=\;&C\exp\left((\ln n_x)\left[\beta\lambda+2-\frac{2(1+o(1))}{1+Cn_x^{\beta\lambda}}+4\frac{\ln[1/2\ln(2\ln n_x)(2\ln n_x)^{1/2H}]}{\ln n_x})\right]\right)\\
\leq\;&C\exp\left((\ln(A_\infty m_H(u)))\left[\beta\lambda+2-\frac{2(1+o(1))}{1+C(A_\infty m_H(u))^{\beta\lambda}}+4\frac{\ln[(2\ln(A_\infty m_H(u)))^{1/2H}]}{\ln(A_\infty m_H(u))}\right]\right)\\
\leq\;&C\exp\left((\ln(A_\infty m_H(u)))\left[\beta\lambda+o(1)\right]\right)\to 0,\quad u\to\infty
\end{aligned}$$

uniformly for $x\in[A_0,A_\infty]$, since $\lambda<0$, thus the proof is complete. □

Proof of Theorem 2.1. Case $H\in(0,1/2]$: First note that by the stationarity of $Z_H(\tau,s)$ with respect to the second component, Theorems 1.1-1.3 and the definition on K_x we have

$$\begin{aligned}
\prod_{k=1}^{K_x}P\left(\max_{(\tau,s)\in\mathbf{I}_k}Z_H(\tau,s)\leq u\right)\;&\sim\;\exp\left(-K_xP\left(\max_{(\tau,s)\in\mathbf{I}_1}Z_H(\tau,s)>u\right)\right)\\
&\sim\;\exp\left(-K_x(L-\delta)w_H(u)\right)\\
&\to\;\exp(-x),\quad\delta\downarrow 0,\quad u\to\infty,
\end{aligned}$$

uniformly for $x \in [A_0, A_\infty]$. Since further by Lemmas 3.1-3.5 as $u \to \infty$

$$\begin{aligned} P\left(\max_{(\tau,s)\in[0,1]\times[0,n_x]} Z_H(\tau,s) \le u\right) &\sim P\left(\max_{(\tau,s)\in[\kappa,1]\times[0,n_x]} Z_H(\tau,s) \le u\right) \\ &\sim P\left(\max_{(\tau,s)\in\cup_k \mathbf{I}_k} Z_H(\tau,s) \le u\right) \\ &\sim P\left(\max_{(\tau,s)\in\cup_k \mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \le u\right) \\ &\sim \prod_{k=1}^{K_x} P\left(\max_{(\tau,s)\in \mathbf{I}_k\cap\mathcal{R}} Z_H(\tau,s) \le u\right) \end{aligned}$$

uniformly for $x \in [A_0, A_\infty]$, the claim follows.

Case $H \in (1/2, 1)$: First, noting that $Z_H(1,s) = B_H(s+1) - B_H(s)$ is a stationary Gaussian process with correlation function $r(t)$ satisfying

$$\begin{aligned} r(t) &= \frac{1}{2}[|t+1|^{2H} + |t-1|^{2H} - 2|t|^{2H}] \\ &= 1 - |t|^{2H} + o(|t|^{2H}) \end{aligned}$$

as $t \to 0$, and

$$r(t) < 1, \quad \text{for} \quad t \ne 0.$$

Thus, by Pickands theorem (see e.g., [16,18,19]), we have for any fixed $h > 0$

$$P\left(\sup_{s\in[0,h]} Z_H(1,s) > u\right) = h\mathcal{H}_{2H} u^{1/H}\Psi(u)(1+o(1)), \tag{22}$$

as $u \to \infty$. Furthermore, it is easy to check that

$$|r(t)| \le C|t|^{2H-2},$$

as $t \to \infty$. So by Lemma 4.3 of [2], we have for each $0 < A_0 < A_\infty < \infty$

$$P\left(\sup_{s\in[0,xm_H(u)]} Z_H(1,s) \le u\right) \to e^{-x}$$

uniformly for $x \in [A_0, A_\infty]$, as $u \to \infty$. So, by Lemma 3.1, if we can show that

$$\left|P\left(\sup_{\tau\in[\kappa,1],s\in[0,xm_H(u)]} Z_H(\tau,s) \le u\right) - P\left(\sup_{s\in[0,xm_H(u)]} Z_H(1,s) \le u\right)\right| \to 0$$

holds uniformly for $x \in [A_0, A_\infty]$, as $u \to \infty$, then Theorem 2.1 is proved. Note that

$$\begin{aligned} &\left|P\left(\sup_{\tau\in[\kappa,1],s\in[0,xm_H(u)]} Z_H(\tau,s) \le u\right) - P\left(\sup_{s\in[0,xm_H(u)]} Z_H(1,s) \le u\right)\right| \\ \le\ &\left|P\left(\sup_{\tau\in[\kappa,1],s\in[0,xm_H(u)]} Z_H(\tau,s) > u\right) - K_x P\left(\sup_{\tau\in[\kappa,1],s\in[0,L-\delta)} Z_H(\tau,s) > u\right)\right| \\ &+ \left|K_x P\left(\sup_{\tau\in[\kappa,1],s\in[0,L-\delta)} Z_H(\tau,s) > u\right) - K_x P\left(\sup_{s\in[0,L-\delta)} Z_H(1,s) > u\right)\right| \\ &+ \left|P\left(\sup_{s\in[0,xm_H(u)]} Z_H(1,s) > u\right) - K_x P\left(\sup_{s\in[0,L-\delta)} Z_H(1,s) > u\right)\right| \\ =:\ & P_1 + P_2 + P_3. \end{aligned}$$

For P_1 we have

$$\begin{aligned} P_1 &\le \left| P\left(\sup_{\tau\in[\kappa,1], s\in[0,xm_H(u)]} Z_H(\tau,s) > u \right) - P\left(\sup_{(\tau,s)\in\cup \mathbf{I}_k} Z_H(\tau,s) > u \right) \right| \\ &\quad + \left| P\left(\sup_{(\tau,s)\in\cup \mathbf{I}_k} Z_H(\tau,s) > u \right) - K_x P\left(\sup_{\tau\in[\kappa,1], s\in[0,L-\delta)} Z_H(\tau,s) > u \right) \right| =: P_{11} + P_{12}. \end{aligned}$$

By Lemma 3.2, we have $P_{11} = o(1)$ uniformly for $x \in [A_0, A_\infty]$, as $u \to \infty$. To bound P_{12}, define the Gaussian random field

$$Y(\tau, s, \tau', s') = Z_H(\tau, s) + Z_H(\tau', s').$$

It is easy to see that the Gaussian random field has the variance function on $\mathbf{I}_k \times \mathbf{I}_l$

$$\begin{aligned} \mathrm{Var}(Y(\tau,s,\tau',s')) &= \tau + \tau' + 2r(\tau,s,\tau',s') \le 2 + 2r(\tau,s,\tau',s') \\ &= 4 - (1+o(1))(|s+\tau-s'-\tau'|^{2H} + |s-s'|^{2H}). \end{aligned}$$

By the stationarity, we have

$$\begin{aligned} P_{12} &\le \sum_{k\neq l} P\left(\sup_{(\tau,s)\in \mathbf{I}_k} Z_H(\tau,s) > u, \sup_{(\tau,s)\in \mathbf{I}_l} Z_H(\tau,s) > u \right) \\ &\le \sum_{k\neq l} P\left(\sup_{(\tau,s,\tau',s')\in \mathbf{I}_k\times \mathbf{I}_l} Z_H(\tau,s) + Z_H(\tau',s') > 2u \right) \\ &= \sum_{|k-l|=1} P(\cdot) + \sum_{|k-l|>1} p(\cdot). \end{aligned}$$

Note that $\mathrm{Var}(Y(\tau,s,\tau',s')) \le 4 - 2(1-\varepsilon)\delta$ for some $\varepsilon > 0$ when $|k-l| = 1$. Thus, by Piterbarg Theorem (see e.g., Theorem 8.1 in [19]), we have

$$\begin{aligned} \sum_{|k-l|=1} P(\cdot) &\le CK_x u^{2/H} \Psi(2u/\sqrt{4-2(1-\varepsilon)\delta}) \\ &\le CK_x u^{2/H} \Psi(u) e^{-\frac{1}{4}(1-\varepsilon)\delta u^2} \\ &\le Cn_x u^{1/H} \Psi(u) o(1) \\ &\le A_\infty o(1), \end{aligned}$$

using $n_x u^{\frac{1}{H}-1}\varphi(u) = O(1)$. When $|k-l| > 1$, the distance between $\mathbf{I}_k$ and $\mathbf{I}_l$ is at least L. So we have $\mathrm{Var}(Y(\tau,s,\tau',s')) \le 4 - 2(1-\varepsilon)L$ for some $\varepsilon > 0$. Applying Piterbarg Theorem again, by choosing the constant $L > \frac{2}{1-\varepsilon}$ we have

$$\begin{aligned} \sum_{|k-l|>1} P(\cdot) &\le CK_x^2 u^{2/H} \Psi(2u/\sqrt{4-2(1-\varepsilon)L}) \\ &\le CK_x^2 u^{2/H} \Psi(u) e^{-\frac{1}{4}(1-\varepsilon)Lu^2} \\ &\le CK_x u^{1/H} \Psi(u) K_x u^{1/H} e^{-\frac{1}{4}(1-\varepsilon)Lu^2} \\ &= CK_x u^{1/H} \Psi(u) u e^{[\frac{1}{2}-\frac{1}{4}(1-\varepsilon)L]u^2} \\ &\le n_x u^{1/H} \Psi(u) o(1) \\ &\le A_\infty o(1), \end{aligned}$$

using $n_x u^{\frac{1}{H}-1}\varphi(u) = O(1)$ again. Thus, $P_1 = o(1)$ uniformly for $x \in [A_0, A_\infty]$, as $u \to \infty$. By the same arguments, we can show that $P_3 = o(1)$ uniformly for $x \in [A_0, A_\infty]$, as $u \to \infty$. For P_2, by Lemma 3.1, Theorem 1.3 and (22), we have

$$\begin{aligned}
P_2 &\le K_x \left| P\left(\sup_{\tau\in[\kappa,1], s\in[0,L-\delta)} Z_H(\tau,s) > u \right) - (L-\delta)\mathcal{H}_{2H} u^{1/H}\Psi(u) \right| \\
&\quad + K_x \left| P\left(\sup_{s\in[0,L-\delta)} Z_H(1,s) > u \right) - (L-\delta)\mathcal{H}_{2H} u^{1/H}\Psi(u) \right| \\
&\le 2LK_x\mathcal{H}_{2H} u^{1/H}\Psi(u)o(1) \\
&\le 2A_\infty o(1).
\end{aligned}$$

Thus, $P_2 = o(1)$ uniformly for $x \in [A_0, A_\infty]$, as $u \to \infty$. □

3.2 Proof of Theorem 2.2

Since the proof of Theorem 2.2 is the same as that of Theorem 3.2 of [2], we omit the details.

3.3 Proof of Theorem 2.3

By Theorem 2.1, it is easy to check that

$$\lim_{T\to\infty} \sup_{x\in\mathbb{R}} \left| P\left(\alpha_T\left(M_H(T) - \beta_T\right) \le x\right) - \exp\{-e^{-x}\} \right| = 0$$

holds. Noting that the following convergence

$$\frac{\alpha_T}{\alpha_{\mathcal{T}_T}} \to 1, \quad \alpha_T(\alpha_{\mathcal{T}_T} - \beta_T) \to \ln \mathcal{T}$$

holds in probability, then it is easy to see that

$$\begin{aligned}
P\left(\alpha_T\left(M_H(\mathcal{T}_T) - \beta_T\right) \le x\right) &= P\left(\frac{\alpha_T}{\alpha_{\mathcal{T}_T}} \alpha_{\mathcal{T}_T}\left(M_H(\mathcal{T}_T) - \beta_{\mathcal{T}_T}\right) + \alpha_T(\beta_{\mathcal{T}_T} - \beta_T) \le x \right) \\
&\to P\left(\Lambda + \ln \mathcal{T} \le x\right),
\end{aligned}$$

where Λ is a random variable with Gumbel distribution function being further independent of $\mathcal{T}$, and thus the proof is complete. □

3.4 Proof of Theorem 2.4

Let $M([l,t]) = \sup_{\tau\in[0,1], s\in[l,t]} Z_{1/2}(\tau,s)$ and $\eta(t) = I(M([1,t]) \le u_t) - P(M([1,t]) \le u_t)$ with $u_t = \beta_t + \alpha_t^{-1}x$. Notice that $\eta(t)$ is a real-valued random process with continuous and bounded sample paths and $\mathrm{Var}(\eta(t)) \le 1$. First, we estimate $\mathrm{Var}(\int_1^T \frac{1}{t}\eta(t)dt)$. Clearly

$$\begin{aligned}
\mathrm{Var}\left(\int_1^T \frac{1}{t}\eta(t)dt \right) &\le \mathrm{E}\left(\int_1^T \frac{1}{t}\eta(t)dt \right)^2 \\
&= 2\int\int_{1\le l<t\le T} \frac{\mathrm{E}(\eta(l)\eta(t))}{lt} dtdl.
\end{aligned}$$

For $1 \le l < t \le T$, write

$$|\mathrm{E}(\eta(l)\eta(t))| = |\mathrm{Cov}(I(M([1,l]) \le u_l), I(M([1,t]) \le u_t))|$$

$$\begin{aligned}
&\leq \quad |\mathrm{Cov}(I(M([1,l]) \leq u_l), [I(M([1,t]) \leq u_t) - I(M([l+1,t]) \leq u_t)])| \\
&\quad\quad + |\mathrm{Cov}(I(M([1,l]) \leq u_s), I(M([l,t]) \leq u_t))| \\
&\leq \quad 2\,\mathrm{E}\,|I(M([1,t]) \leq u_t) - I(M([l+1,t]) \leq u_t)| \\
&\quad\quad + |\mathrm{Cov}(I(M([1,l]) \leq u_l), I(M([l+1,t]) \leq u_t))| \\
&=: \quad W_{t,1} + W_{t,2}.
\end{aligned}$$

Noting that $M([1,l])$ and $M([l+1,t])$ are independent, we have $W_{t,2} = 0$. For $W_{t,1}$, using the fact that for fixed s, $Z_{1/2}(\cdot, s)$ is a stationary process and by Theorem 1.1, we get that

$$\begin{aligned}
&\mathrm{E}\,|I(M([1,t]) \leq u_t) - I(M([l+1,t]) \leq u_t)| \\
&= P(M([l+1,t]) \leq u_t) - P(M([1,t]) \leq u_t) \\
&= P([l+1,t]) \leq u_t, M([1,t]) > u_t) \\
&\leq P(M([1,l+1)) > u_t) \\
&\leq lP(M([0,1]) > u_t) \quad \text{by the stationarity} \\
&\leq lw_{1/2}(u_t) \quad \text{by Theorem 1.1} \\
&= \frac{l}{t} t\mu(u_t) \\
&\leq C\frac{l}{t}, \quad \text{using the condition that } t\omega_{1/2}(u_t) \to e^{-x}.
\end{aligned}$$

Consequently

$$\begin{aligned}
W_{t,1} &\leq C \int\int_{1 \leq l < t \leq T} \frac{\frac{l}{t}}{lt} dl dt \\
&\leq C \ln T \leq C(\ln T)^2 (\ln\ln T)^{-(1+\varepsilon)}.
\end{aligned}$$

Hence

$$\mathrm{Var}\left(\int_1^T \frac{1}{t}\eta(t)dt\right) \leq C(\ln T)^2(\ln\ln T)^{-(1+\varepsilon)}.$$

From Theorem 1.1, we have

$$\lim_{t\to\infty} P(M([1,t]) \leq u_t) = \lim_{t\to\infty} P(M([0,t]) \leq u_t) = \exp\{-e^{-x}\}.$$

Clearly, this implies

$$\lim_{T\to\infty} \frac{1}{\ln T} \int_1^T \frac{1}{t} P(M([1,t]) \leq u_t)dt = \exp\{-e^{-x}\}. \tag{23}$$

Now, the result of the theorem follows from Lemma 3.1 in [24] and (23). □

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