Appl. Math. J. Chinese Univ.
2016, 31(1): 103-111

Some properties of the psi function and evaluations of γ

QIU Song-liang ZHAO Xue

Abstract. In this paper, the authors show some monotonicity and concavity of the classical psi function, by which several known results are improved and some new asymptotically sharp estimates are obtained for this function. In addition, applying the new results to the psi function, the authors improve the well-known lower and upper bounds for the approximate evaluation of Euler's constant γ.

§1 Introduction and notation

For real and positive values of x, the real Euler gamma and psi functions are defined as

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\ \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}, \tag{1}$$

respectively. For their extensions to complex variables and for their basic properties, the reader is referred to [1,6,15]. It is well known that these functions have many applications to various fields of mathematics [1,6,15] as well as to some other disciplines. During the past several decades, many authors have obtained various properties and inequalities for these functions (see [2-18] and bibliographies in these papers). Keeping with this tradition, we here present some monotonicity and concavity of the psi function, from which its asymptotically sharp estimates follow. Applying our new results to the psi function, we improve the known approximate evaluations of Euler's constant γ.

It is well known that Euler's constant [1,6.1.3 & 6.3.2]

$$\gamma=\lim_{n\to\infty} d_n,\ d_n-\gamma=\psi(n+1)-\log n \tag{2}$$

for $n\in\mathbb{N}$, where $d_n=\sum_{k=1}^n(1/k)-\log n$. S.R. Tims and J.A. Tyrrell [17] obtained the bounds

$$\frac{1}{2(n+1)}<d_n-\gamma<\frac{1}{2(n-1)},\ n\geq 2,$$

which was improved by R.M. Young [18] as

$$\frac{1}{2(n+1)}<d_n-\gamma<\frac{1}{2n},\ n\in\mathbb{N}. \tag{3}$$

Received: 2014-03-09.
MR Subject Classification: 33E05.
Keywords: Psi function, gamma function, monotonicity, concavity, inequality, Euler's constant.
Digital Object Identifier(DOI): 10.1007/s11766-016-3272-8.
Supported by the National Natural Science Foundation of China (11171307).

In 1995, G.D. Anderson et al.[5] proved

$$\frac{1-\gamma}{n} < d_n - \gamma < \frac{1}{2n},\ n \in \mathbb{N}, \tag{4}$$

while H.Alzer [4,Theorem 3] improved (4) to the following double inequality

$$\frac{1}{2(n+\alpha)} \le d_n - \gamma < \frac{1}{2(n+\beta)},\ n \in \mathbb{N}, \tag{5}$$

where $\alpha = 1/[2(1-\gamma)]$ and $\beta = 1/6$.

In [16,Theorem 2.1], it is proved that for $x \ge 1$,

$$\log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) \le \log x - \frac{1}{2x} - \frac{2\gamma-1}{2x^2} \tag{6}$$

from which the following estimates follow [16,Corollary 2.13]

$$\frac{1}{2n} - \frac{\alpha}{n^2} < d_n - \gamma \le \frac{1}{2n} - \frac{\beta}{n^2},\ n \in \mathbb{N}, \tag{7}$$

with equality if $n = 1$, where the constants $\alpha = 1/12$ and $\beta = \gamma - (1/2)$ are best possible.

In the sequel,let $\zeta(x)$ stand for the Riemann zeta function. We now state our results.

Theorem 1.1. *(1) Let* $\alpha = (\pi^2/6) + \gamma - 2 = 0.2221\cdots$, $\beta = (\pi^2/6) + 1 - 2\zeta(3) = 0.2408\cdots$ *and* $f_1(x) \equiv x^2\psi'(x+1) - x\psi(x+1) + \log\Gamma(x+1)$. *Define the function* f_2 *on* $(0,\infty)$ *by*

$$f_2(x) = [f_1(x) - \alpha]/\log x \text{ for } x \ne 1,$$
$$f_2(1) = \lim_{x\to 1} f_2(x) = f_1'(1) = \beta.$$

Then f_1 *and* f_2 *are strictly increasing on* $(0,\infty)$ *with ranges* $f_1((0,\infty)) = (0,\infty)$ *and* $f_2((0,\infty)) = (0,1/2)$. *However,* f_1 *is neither convex nor concave on* $(0,\infty)$. *In particular,*

$$\alpha + \beta\log x \le f_1(x) \le \alpha + \frac{1}{2}\log x \tag{8}$$

for $x \in [1,\infty)$, *and*

$$\max\{0, \alpha + \beta\log x\} \le f_1(x) \le \alpha \tag{9}$$

for $x \in (0,1]$. *Equality holds in each instance if and only if* $x = 1$.

(2) Let $f_3(x) = 2[x\psi(x+1) - \log\Gamma(x+1)] - x^2\psi'(x+1)$, $c = 2\zeta(3)/3 = 0.8013\cdots$, $C = f_3(1) = 2(1-\gamma) + 1 - (\pi^2/6) = 0.2006\cdots$ *and* $D = f_3'(1) = -\psi''(2) = 2[\zeta(3)-1] = 0.4041\cdots$. *Then* f_3 *is convex on* $(0,\infty)$, $f_4(x) \equiv f_3(x)/\log(x+1)$ *is strictly increasing from* $(0,\infty)$ *onto* $(0,\infty)$, *while* $f_5(x) \equiv f_3(x)/x^3$ *is strictly decreasing from* $(0,\infty)$ *onto* $(0,c)$. *In particular, for all* $x > 0$,

$$P(x) \le f_3(x) \le Q(x), \tag{10}$$

where $P(x) = Cx^3$ $(C + D(x-1))$ *and* $Q(x) = \min\{cx^3, Dx\}$ $(\min\{Cx^3, C+x-1\}$ *for* $0 < x < 1$ $(1 \le x < \infty$, *respectively), with equality in each instance if and only if* $x = 1$.

(3) Let $f_6(x) = [f_3(x) - C]/\log x$ *for* $x \ne 1$ *and* $f_6(1) = \lim_{x\to 1} f_6(x) = D$. *Then* f_6 *is strictly increasing on* $(0,\infty)$ *with* $f_6((0,1)) = (0,D)$ *and* $f_6([1,\infty)) = [D,\infty)$. *In particular,*

$$x^2\psi'(x+1) + \max\{0, C + D\log x\} \le 2[x\psi(x+1) - \log\Gamma(x+1)] \le C + x^2\psi'(x+1) \tag{11}$$

for $x \in (0,1]$, *and*

$$2[x\psi(x+1) - \log\Gamma(x+1)] \ge C + D\log x + x^2\psi'(x+1) \tag{12}$$

for $x \in [1,\infty)$. *Equality holds in each instance if and only if* $x = 1$.

(4) Let $f_7(x) = x^2\psi'(x+1)/[x\psi(x+1) - \log\Gamma(x+1)]$. *Then for* $x \in (0,\infty)$,

$$1 = f_7(\infty) = \inf\{f_7(x) : 0 < x < \infty\} < f_7(x) < \sup\{f_7(x) : 0 < x < \infty\} = f_7(0^+) = 2. \quad (13)$$

□

Our following result improves (6) and (7).

Theorem 1.2. *(1) The function* $f_8(x) \equiv x^4[\psi(x) - \log x + 1/(2x) + 1/(12x^2)]$ *is strictly increasing from* $(0,\infty)$ *onto* $(0, 1/120)$. *Moreover,* f_8 *is neither convex nor concave on* $(0,\infty)$, *and* $f_9(x) \equiv f_8(x)/x^2$ *is strictly decreasing from* $(0,\infty)$ *onto* $(0, 1/12)$.

(2) The function $f_{10}(x) \equiv x^2[(1/120) - f_8(x)]$ *is strictly increasing from* $(0,\infty)$ *onto* $(0, 1/252)$. *In particular,*

$$\log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} < \psi(x) < \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} \quad (14)$$

for $x \in (0,\infty)$.

(3) For $n \in \mathbb{N}$,

$$\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{b}{n^6} < d_n - \gamma \le \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{a}{n^6}, \quad (15)$$

where the constants $a = f_{10}(1) = \gamma - (23/40) = 0.00221\cdots$ *and* $b = 1/252 = 0.00396\cdots$ *are best possible. The equality holds if and only if* $n = 1$. □

§2 Preliminary results

In this section, we establish a technical lemma needed for the proofs of our main results. In the sequels, we shall often apply the expression [1,6.3.21]

$$\psi(x) = \log x - \frac{1}{2x} - 2\int_0^\infty \frac{t}{(t^2+x^2)(\mathrm{e}^{2\pi t}-1)}dt \quad (16)$$

for $x \in (0,\infty)$, by which the following formulas hold:

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + 4x\int_0^\infty \frac{t}{(t^2+x^2)^2(\mathrm{e}^{2\pi t}-1)}dt, \quad (17)$$

$$\psi''(x) = -\frac{1}{x^2} - \frac{1}{x^3} + 4\int_0^\infty \frac{t(t^2-3x^2)}{(t^2+x^2)^3(\mathrm{e}^{2\pi t}-1)}dt \quad (18)$$

and

$$\psi'''(x) = \frac{2}{x^3} + \frac{3}{x^4} + 48x\int_0^\infty \frac{t(x^2-t^2)}{(t^2+x^2)^4(\mathrm{e}^{2\pi t}-1)}dt. \quad (19)$$

Lemma 2.1. *(1) For each* $a \in (0,1)$, *the function* $F_1(x) \equiv x(e^{x/a} - e^{ax})/[(e^{ax}-1)(e^{x/a}-1)]$ *is strictly decreasing from* $(0,\infty)$ *onto* $(0, (1-a^2)/a)$.

(2) The function $F_2(x) \equiv x^2[\psi'(x+1) + x\psi''(x+1)]$ *is strictly increasing from* $[0,\infty)$ *onto* $[0, 1/2)$. *However, the function* $F_3(x) \equiv F_2(x)/x$ *is not monotone on* $(0,\infty)$.

Proof. (1) By l'Hpital Rule, one can obtain $F_1(0) = (1-a^2)/a$. It is easy to get $F_1(\infty) = 0$.

Set $t = \mathrm{e}^{x/a} \in (1,\infty), b = a^2, F_4(t) = (t - t^b)\log t, F_5(t) = (t^b - 1)(t-1)$ and $F_6(t) = [b(1-b)\log t + t^{1-b} + 1 - 2b]/[(1+b)t + 1 - b]$. Then $F_4(1) = F_5(1) = F_4'(1) = F_5'(1) =$

$F_6(1)-(1-b)=F_6(\infty)=0$, $F_1(x)=aF_4(t)/F_5(t)$, and

$$\frac{F_4'(t)}{F_5'(t)}=\frac{(1-bt^{b-1})\log t-t^{b-1}+1}{(1+b)t^b-bt^{b-1}-1},\ \frac{F_4''(t)}{F_5''(t)}=\frac{1}{b}F_6(t), \tag{20}$$

$$[(1+b)t-b+1]^2F_6'(t)=F_7(t), \tag{21}$$

where

$$\begin{aligned}F_7(t) &= (1-b)[(1+b)t-b+1](t^{-b}+bt^{-1})-(1+b)[t^{1-b}+b(1-b)\log t+1-2b]\\ &= -b(1+b)t^{1-b}+(1-b)^2t^{-b}+b(1-b)^2t^{-1}-b(1-b^2)\log t-(1+b)(1-b+b^2).\end{aligned}$$

It is easy to obtain the limiting values $F_7(1)=0$ and $F_7(\infty)=-\infty$. Since

$$t^2F_7'(t)=-b(1-b)(t^{1-b}+1)[(1+b)t+(1-b)]<0$$

for all $t\in(1,\infty)$ and $a\in(0,1)$, F_7 is strictly decreasing from $(1,\infty)$ onto $(-\infty,0)$. Hence the monotonicity of F_1 follows from (20), (21) and the so-called Monotone l'Hôpital's Rule [6, Theorem 1.25].

(2) Since [1, 6.3.5],

$$\psi(x+1)=\psi(x)+1/x, \tag{22}$$

it follows from (17) and (18) that

$$F_2(x)=x^2\left\{\left[\psi'(x)-\frac{1}{x^2}\right]+x\left[\psi''(x)+\frac{2}{x^3}\right]\right\}=\frac{1}{2}+8x^3\int_0^\infty\frac{t(t^2-x^2)}{(t^2+x^2)^3(\mathrm{e}^{2\pi t}-1)}dt.$$

Putting $t=xu$, we have

$$\begin{aligned}F_2(x) &= \frac{1}{2}+8\int_0^\infty\frac{xu(u^2-1)}{(1+u^2)^3(\mathrm{e}^{2\pi xu}-1)}du\\ &= \frac{1}{2}-8\int_0^1\frac{xu(1-u^2)}{(1+u^2)^3(\mathrm{e}^{2\pi xu}-1)}du+8\int_1^\infty\frac{xu(u^2-1)}{(1+u^2)^3(\mathrm{e}^{2\pi xu}-1)}du\\ &= \frac{1}{2}-8\int_0^1\frac{xu(1-u^2)}{(1+u^2)^3(\mathrm{e}^{2\pi xu}-1)}du+8\int_0^1\frac{xu(1-u^2)}{(1+u^2)^3(\mathrm{e}^{2\pi x/u}-1)}du\\ &= \frac{1}{2}-8\int_0^1\frac{xu(1-u^2)(\mathrm{e}^{2\pi x/u}-\mathrm{e}^{2\pi xu})}{(1+u^2)^3(\mathrm{e}^{2\pi xu}-1)(\mathrm{e}^{2\pi x/u}-1)}du\\ &= \frac{1}{2}-\frac{4}{\pi}\int_0^1\frac{u(1-u^2)}{(1+u^2)^3}F_1(2\pi x)du,\end{aligned}$$

where F_1 is as in part (1) with $a=u$, so that the monotonicity of F_2 follows from part (1).

By part (1),

$$F_2(\infty)=\frac{1}{2},\ F_2(0^+)=\frac{1}{2}-\frac{4}{\pi}I,\ \text{where } I=\int_0^1\frac{(1-u^2)^2}{(1+u^2)^3}du.$$

Set $u=\tan v$. Then, $(1-u^2)^2/(1+u^2)^3=(\cos^4 v)(2\cos^2 v-1)$, $du=(\cos^{-2}v)dv$ and

$$I=2\int_0^{\pi/2}\cos^4 v dv-\int_0^{\pi/2}\cos^2 v dv=\frac{\pi}{8},$$

so that $F_2(0^+)=0$.

Finally, the conclusion for F_3 follows from the fact that $F_3(0)=F_3(\infty)=0$. □

3 Proof of the main results

In this section, we prove the main theorems stated in Section 1.

Proof of Theorem 1.1. (1) Clearly, $f_1(0) = 0$. By the asymptotic properties of $\Gamma(x)$ and $\psi(x)$ (see [1, 6.4.12, 6.3.18 & 6.1.41]), one can obtain $f_1(\infty) = \infty$.

Differentiation gives

$$f_1'(x) = x[\psi'(x+1) + x\psi''(x+1)] = F_3(x),$$

where F_3 is as in Lemma 2.1(2). Hence, by Lemma 2.1(2), f_1 is strictly increasing from $(0, \infty)$ onto $(0, \infty)$, and is neither convex nor concave on $(0, \infty)$.

Clearly, $f_1(1) = \alpha$. Let $G_1(x) = f_1(x) - \alpha$ and $G_2(x) = \log x$. Then $G_1(1) = G_2(1) = 0$, $f_2(x) = G_1(x)/G_2(x)$ and $G_1'(x)/G_2'(x) = F_2(x)$, where F_2 is as in Lemma 2.1(2). Hence the monotonicity of f_2 follows from Lemma 2.1(2) and [6, Theorem 1.25]. The remaining conclusions in part (1) are clear.

(2) Clearly, $f_3(0) = 0$. By [1, 6.3.18, 6.1.41 & 6.4.12], one can obtain the limiting value $f_3(\infty) = \infty$. Differentiation and [1, 6.4.1 & 6.4.10] give

$$f_3'(x) = -x^2\psi''(x+1) = 2x^2 \sum_{n=1}^{\infty} \frac{1}{(n+x)^3} = \int_0^{\infty} \frac{u/x}{\mathrm{e}^{u/x} - 1} u\mathrm{e}^{-u} du, \tag{23}$$

which is clearly strictly increasing from $(0, \infty)$ onto $(0, 1)$, and hence the convexity of f_3 follows.

By (23), we have

$$\frac{f_3'(x)}{\frac{d}{dx}[\log(x+1)]} = (x+1)f_3'(x) = (x+1)\int_0^{\infty} \frac{u/x}{\mathrm{e}^{u/x} - 1} u\mathrm{e}^{-u} du, \tag{24}$$

which is strictly increasing on $(0, \infty)$, and hence the monotonicity of f_4 follows from [6,Theorem 1.25].

Clearly, $f_4(0^+) = 0$, and by [1,6.4.13 & 6.4.14], $f_4(\infty) = \infty$.

It follows from (23) that

$$\frac{f_3'(x)}{\frac{d}{dx}(x^3)} = -\frac{1}{3}\psi''(x+1) = \frac{2}{3}\sum_{n=1}^{\infty} \frac{1}{(n+x)^3}, \tag{25}$$

which is clearly strictly decreasing from $(0, \infty)$ onto $(0, c)$, and hence the monotonicity of f_5 follows from [6,Theorem 1.25]. By (25), the l'Hôpital's Rule and [1,6.4.2 & 6.4.13], $f_5(0^+) = c$ and $f_5(\infty) = -(1/3)\lim_{x\to\infty} \psi''(x+1) = 0$.

By the monotonicity of f_3', we have

$$0 < f_3'(x) < f_3'(1) = D, \text{ for } x \in (0, 1), \tag{26}$$

and

$$D \le f_3'(x) < 1, \text{ for } x \in [1, \infty). \tag{27}$$

Integrating (26) from 0 to x, we obtain

$$0 < f_3(x) \le Dx, \text{ for } x \in (0, 1). \tag{28}$$

Similarly, integrating (27) from 1 to x, we obtain

$$D(x-1) \le f_3(x) - f_3(1) \le x - 1, \text{ for } x \in [1, \infty). \tag{29}$$

On the other hand, it follows from the monotonicity of f_5 that

$$Cx^3 < f_3(x) < cx^3, \text{ for } x \in (0,1), \tag{30}$$

and

$$0 < f_3(x) \le Cx^3, \text{ for } x \in [1,\infty). \tag{31}$$

The double inequality (10) now follows from (28)–(31). The equality case is clear.

(3) By differentiation and (23),

$$\frac{\frac{d}{dx}[f_3(x)-C]}{\frac{d}{dx}(\log x)} = x\int_0^\infty \frac{u/x}{e^{u/x}-1} u e^{-u} du, \tag{32}$$

which is strictly increasing on $(0,\infty)$, and hence so is f_6 by [6,Theorem 1.25]. The remaining conclusions in part (3) are clear.

(4) Applying l'Höpital's Rule and [1, 6.4.12 & 6.4.13], we have

$$f_7(0^+) = 2+\lim_{x\to 0}\frac{x\psi''(x+1)}{\psi'(x+1)} = 2,\ f_7(\infty) = 2+\lim_{x\to\infty}\frac{x\psi''(x+1)}{\psi'(x+1)} = 1. \tag{33}$$

By part (1),

$$x^2\psi'(x+1) > x\psi(x+1) - \log\Gamma(x+1),\ x\in(0,\infty). \tag{34}$$

(13) now follows from (33), (34) and (10). □

In [2, Lemma 2.1], it was proved that

$$\sum_{n=1}^{\infty}\frac{n-x}{(n+x)^3} > 0 \text{ for } x\in[1,4),$$

which can now been improved to the following conclusion by Theorem 1.1(1).

Corollary 3.1. *For all $x>0$,*

$$0 < \sum_{n=1}^{\infty}\frac{n-x}{(n+x)^3} < \frac{1}{2x^2}. \tag{35}$$

Proof. By [1, 6.4.10],

$$xf_1'(x) = x^2[\psi'(x+1)+x\psi''(x+1)] = F_2(x) = x^2\sum_{n=1}^{\infty}\frac{n-x}{(n+x)^3},$$

and hence (35) follows from Lemma 2.1(2). □

Remark 3.2. Theorem 1.1(1) also improves Lemma 2.6 in [2] and simplifies its proof.

Proof of Theorem 1.2. (1) By (22),

$$f_8(0^+) = \lim_{x\to 0} x^4\left[\psi(x+1)-\frac{1}{x}-\log x+\frac{1}{2x}+\frac{1}{12x^2}\right] = 0,$$

while $f_8(\infty) = 1/120$ by [1,6.3.18].

Differentiation gives

$$f_8'(x) = 4x^3[\psi(x)-\log x] + x^4\left[\psi'(x)-\frac{1}{x}\right] + \frac{3}{2}x^2+\frac{x}{6}. \tag{36}$$

Hence, by (16) and (17), it follows from (36) that

$$H_1(x) \equiv \frac{1}{x}f_8'(x) = \frac{1}{6} - 4\int_0^\infty \frac{t}{e^{2\pi t}-1}H_2(x,t)dt, \tag{37}$$

where $H_2(x,t) = x^2(x^2+2t^2)/(t^2+x^2)^2$. Clearly, $H_1(0^+) = 1/6$. Since [1, 6.4.1]

$$\psi^{(n)}(x) = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-u}}dt, \tag{38}$$

$$\begin{aligned} H_1(\infty) &= \frac{1}{6} - 4\int_0^\infty \frac{t}{e^{2\pi t}-1}dt = \frac{1}{6} - \frac{1}{\pi^2}\int_0^\infty \frac{ue^{-u}}{1-e^{-u}}du \\ &= \frac{1}{6} - \frac{1}{\pi^2}\psi'(1) = \frac{1}{6} - \frac{1}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2} = 0, \end{aligned}$$

where $u = 2\pi t$. It is easy to show that H_2 is strictly increasing in x on $(0,\infty)$ so that H_1 is strictly decreasing from $(0,\infty)$ onto $(0,1/6)$. The monotonicity of f_8 now follows from (37).

It is easy to obtain the limiting values $f_8'(0^+) = f_8'(\infty) = 0$ by (36). Hence f_8' is not monotonic on $(0,\infty)$, so that f_8 is neither convex nor concave on $(0,\infty)$.

Since

$$\frac{f_8'(x)}{\frac{d}{dx}(x^2)} = \frac{1}{2x}f_8'(x) = \frac{1}{2}H_1(x),$$

the result for f_9 follows from [6, Theorem 1.25] and the monotonicity of H_1.

(2) Let $y = 1/x$, $H_3(y) = (1/120) - f_8(1/y)$, and $H_4(y) = y^2$. Then $H_3(0^+) = H_4(0) = 0$, $f_{10}(x) = H_3(y)/H_4(y)$, and

$$\frac{H_3'(y)}{H_4'(y)} = \frac{1}{2y^3}f_8'\left(\frac{1}{y}\right) = \frac{1}{2}H_5(x) \equiv \frac{1}{2}x^3 f_8'(x). \tag{39}$$

By (37),

$$H_5(x) = x^4 H_1(x) = \frac{1}{6}x^4 - 4\int_0^\infty \frac{t}{e^{2\pi t}-1}H_6(x,t)dt,$$

where $H_6(x,t) = x^4 H_2(x,t) = x^6(x^2+2t^2)/(t^2+x^2)^2$. Let $H_7(x) = H_5(\sqrt{x})$. Then

$$\frac{1}{x}H_7'(x) = H_8(x) \equiv \frac{1}{3} - 8\int_0^\infty \frac{t}{e^{2\pi t}-1}H_9(x,t)dt, \tag{40}$$

where $H_9(x,t) = x(3t^4+3xt^2+x^2)/(t^2+x)^3$. Since

$$\frac{\partial H_9}{\partial x} = \frac{3t^6}{(t^2+x)^4},$$

H_9 is strictly increasing in x on $(0,\infty)$ so that H_8 is strictly decreasing on $(0,\infty)$. Clearly, $H_8(0^+) = 1/3$. On the other hand, we have

$$H_8(\infty) = \frac{1}{3} - 8\int_0^\infty \frac{t}{e^{2\pi t}-1}dt = \frac{1}{3} - \frac{2}{\pi^2}\int_0^\infty \frac{ue^{-u}}{1-e^{-u}}dt$$

with $u = 2\pi t$, and hence by (38),

$$H_8(\infty) = \frac{1}{3} - \frac{2}{\pi^2}\psi'(1) = \frac{1}{3} - \frac{2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2} = 0.$$

Therefore, by (40), H_7 is strictly increasing on $(0,\infty)$, and so is H_5. This yields the monotonicity of f_{10} by (39) and [6,Theorem 1.25].

Clearly, $f_{10}(0^+) = 0$. By [1.6.3.18],

$$f_{10}(\infty) = -\lim_{x\to\infty} x^6\left[\psi(x) - \log x + \frac{1}{2x} + \frac{1}{12x^2} - \frac{1}{120x^4}\right] = \frac{1}{252}.$$

The double inequality (14) is clear.

(3) Since

$$\begin{aligned} f_{10}(n) &= n^6 \left\{ \frac{1}{120n^4} - \left[\psi(n) - \log n + \frac{1}{2n} + \frac{1}{12n^2} \right] \right\} \\ &= n^6 \left\{ \frac{1}{120n^4} - \left[\psi(n+1) - \log n - \frac{1}{2n} + \frac{1}{12n^2} \right] \right\} \\ &= n^6 \left[\frac{1}{120n^4} - (d_n - \gamma) + \frac{1}{2n} - \frac{1}{12n^2} \right] \end{aligned}$$

by (2) and (22), we have

$$a = f_{10}(1) \le f_{10}(n) < b, \ n \in \mathbb{N}$$

by part (2). Hence

$$\frac{a}{n^6} \le \frac{1}{120n^4} - (d_n - \gamma) + \frac{1}{2n} - \frac{1}{12n^2} < \frac{b}{n^6}$$

for $n \in \mathbb{N}$, with equality if and only if $n = 1$. This yields the double inequality (15) and its equality case. □

References

[1] M Abramowitz, I A Stegun (Eds). *Handbook of Mathematical Functions: With Formulas, Graphs and Mathematical Tables*, New York, Dover, 1965: 253-294.

[2] G D Aderson, S L Qiu. *A monotonicity property of the gamma function*, Proc Amer Math Soc, 1997, 125(11): 3355-3362.

[3] H Alzer. *Gamma function inequalities*, Numer Algorithms, 2008, 49: 53-84.

[4] H Alzer. *Inequalities for the gamma and polygamma functions*, Abh Math Sem Univ Hamburg,1998, 68: 363-372.

[5] G D Anderson, R W Barnard, K C Richards, M K Vamanamurthy and M Vuorinen. *Inequalities for zero-balanced hypergeometric functions*, Trans Amer Math Soc, 1995, 347: 1713-1723.

[6] G D Anderson, M K Vamanamurthy, M Vuorinen. *Conformal Invariants, Inequalities, and Quasiconformal Maps*, New York, John Wiley & Sons, 1997: 32-47.

[7] N Batir. *Inequalities for the gamma function*, Archiv der Mathematik, 2008, 91: 554-563.

[8] C P Chen, C R Mortici. *New sequence converging towards the Euler-Mascheroni constant*, Comput Math Appl, 2012, 64(4): 391-398.

[9] Á Elbert, A Laforgia. *On some properties of the gamma function*, Amer Math Soc, 2000, 128(9): 2667-2673.

[10] Q Feng. *Bounds for the ratio of two Gamma functions*, J Inequal Appl, 2010, Article ID 493058, 84 pages.

[11] C Mortici. *A quicker convergeces toward the gamma constant with the logarithm term involving the constant*, Carpathian J Math, 2010, 26(1): 86-91.

[12] C Mortici. *Estimating gamma function by digamma function*, Math Comput Model, 2010, 52(5-6): 924-946.

[13] C Mortici. *Fast convergeces towards the Euler-Mascheroni constant*, Comput Appl Math, 2010, 29(3): 479-491.

[14] E Neuman. *Some inequalities for the gamma function*, Math Comput, 2011, 218: 4349-4352.

[15] S L Qiu, M Vuorinen. *Special function in geometric function theory*, In: *Handbook of Complex Analysis: Geometric Function Theory*, North Holland, Elsevier, 2005: 621-659.

[16] S L Qiu, M Vuorinen. *Some properties of the gamma and psi functions, with applications*, Math Comput, 2005, 74: 723-742.

[17] S R Tims, J A Tyrrell. *Approximate evaluation of Euler's constant*, Math Gaz, 1971, 55: 65-67.

[18] R M Young. *Euler's constant*, Math Gaz, 1991, 75: 187-190.

School of Science, Zhejiang Sci-Tech University, Hangzhou 310018, China.

Email: sl_qiu@zstu.edu.cn