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Boundedness of parammetric Marcinkiewicz integrals on weighted Hardy spaces

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Abstract. In this paper, several new results on the boundedness of parammetric Marcinkiewicz integrals on the weighted Hardy spaces and the weak weighted Hardy spaces are established.

§1 Introduction and results

Let $n \geq 2$. S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Assume that $\Omega \in L^1(S^{n-1})$ satisfies the following

$$\Omega(\lambda x) = \Omega(x), \quad \forall \lambda > 0, \tag{1.1}$$

$$\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0, \tag{1.2}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$. In 1960, L. Hörmander [7] studied the following parametric Marcinkiewicz integral operator μ_Ω^ρ

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty |F_{\Omega,t}^\rho(x)|^2 \frac{\mathrm{d}t}{t^{2\rho+1}} \right)^{1/2}, \tag{1.3}$$

where $0 < \rho < n$ and

$$F_{\Omega,t}^\rho(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-\rho}} f(y)\mathrm{d}y. \tag{1.4}$$

When $\rho = 1$, we shall denote μ_Ω^1 simply by μ_Ω.

For the case where $\rho = 1$, the Marcinkiewicz integral μ_Ω was first introduced by E. M. Stein in [12]. He proved that if $\Omega \in Lip_\alpha(S^{n-1})(0 < \alpha \leq 1)$, then μ_Ω is an operator of type (p,p) for $1 < p \leq 2$ and weak type (1,1). In 1962, A. Benedek, A. P. Calderón and R. Panzone [1] showed the type (p,p) boundedness of μ_Ω for $1 < p < \infty$. In 1999, A. Torchinsky and S. L. Wang [13] considered the weighted case. They proved that if $\Omega \in Lip_\alpha(S^{n-1})(0 < \alpha \leq 1)$, then for all $1 < p < \infty$ and $w \in A_p$, μ_Ω is bounded on $L_w^p(\mathbf{R}^n)$. In 1999, S. Sato [10] gave the $L_w^p(\mathbf{R}^n)$ boundedness of $\mu_\Omega^\rho(0 < \rho < n)$, when $\Omega \in L^\infty(S^{n-1})$ and $w \in A_p, 1 < p < \infty$.

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In 2009, X. F. Shi and Y. S. Jiang [11] studied the $L^p_w(\mathbf{R}^n)$ boundedness of μ^ρ_Ω for the case where $\Omega \in L^q(S^{n-1})(1 < q < \infty)$. Precisely, they proved the following

Theorem A. *Let $0 < \rho < n$. Suppose that $\Omega \in L^q(S^{n-1})(1 < q < \infty)$ and satisfies (1.2). If $w^{q'} \in A_p$, where $1 < p < \infty$. Then there exists a constant $C > 0$ independent of f such that*

$$\|\mu^\rho_\Omega(f)\|_{L^p_w} \leq C\|f\|_{L^p_w},$$

where $q' = \frac{q}{q-1}$.

We say that the function Ω satisfies the L^q-Dini condition if $\Omega \in L^q(S^{n-1}), q \geq 1$, and

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta < \infty,$$

where $\omega_q(\delta)$ denotes the integral modulus of continuity of order q of Ω defined by

$$\omega_q(\delta) = \sup_{|\gamma|<\delta}\left(\int_{S^{n-1}} |\Omega(\gamma x') - \Omega(x')|^q \mathrm{d}\sigma(x')\right)^{\frac{1}{q}}$$

and γ is a rotation in $\mathbf{R}^n$ with $|\gamma| = \|\gamma - I\|$.

In 2003, Y. Ding and M. Y. Lee [2] studied the boundedness of Marcinkiewicz integral on weighted Hardy spaces. They got the following

Theorem B. *Suppose that Ω satisfies (1.1), (1.2) and L^q-Dini condition, $1 < q < \infty$. If $w^{q'} \in A_1$, then there exists a constant $C > 0$ independent of f such that*

$$\|\mu_\Omega(f)\|_{L^1_w} \leq C\|f\|_{H^1_w}.$$

In 2002, Y. Ding, S. Z. Lu and Q. Y. Xue [4] considered the boundedness of Marcinkiewicz integral on weak Hardy space. They obtained the following

Theorem C. *Suppose that Ω satisfies (1.1), (1.2) and*

$$\int_0^1 \frac{\omega_1(\delta)}{\delta}(1 + |\log \delta|)^\sigma \mathrm{d}\delta < \infty, \text{ for some } \sigma > 1. \tag{1.5}$$

Then there exists a constant $C > 0$ such that

$$\|\mu_\Omega(f)\|_{WL^1} \leq C\|f\|_{WH^1}. \tag{1.6}$$

In 2014, Y. Hu and Y. S. Wang [8] considered the boundedness of Marcinkiewicz integral on weak weighted Hardy space. They established the following

Theorem D. *Suppose that Ω satisfies (1.1), (1.2) and*

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}(1 + |\log \delta|)^\sigma \mathrm{d}\delta < \infty, \text{ for some } \sigma > 1, \tag{1.7}$$

where $1 < q < \infty$. If $w^{q'} \in A_1$, then there exists a constant $C > 0$ such that

$$\|\mu_\Omega(f)\|_{WL^1_w} \leq C\|f\|_{WH^1_w}.$$

In this paper, we discuss the boundedness of parametric Marcinkiewicz integrals $\mu^\rho_\Omega(0 < \rho < n)$ on the weighted Hardy and weak weighted Hardy spaces (see Section 2 for the definitions). We established the following results.

Theorem 1.1. *Let $0 < \rho < n$. Suppose that Ω satisfies (1.1), (1.2) and L^q-Dini condition,*

$1 < q < \infty$. *If* $w^{q'} \in A_1$, *then there exists a constant* $C > 0$ *independent of* f *such that*

$$\|\mu_\Omega^\rho(f)\|_{L^1_w} \le C\|f\|_{H^1_w}.$$

Theorem 1.2. *Let* $0 < \rho < n$. *Suppose that* Ω *satisfies (1.1), (1.2) and*

$$\int_0^1 \frac{\omega_q(\delta)}{\delta}(1 + |\log \delta|)\mathrm{d}\delta < \infty, \tag{1.8}$$

where $1 < q < \infty$. *If* $w^{q'} \in A_1$, *then there exists a constant* $C > 0$ *independent of* f *such that*

$$\|\mu_\Omega^\rho(f)\|_{WL^1_w} \le C\|f\|_{WH^1_w}.$$

Remark 1.1. Theorem 1.1 is a generalization of Theorem B.

Remark 1.2. Condition (1.8) is weaker than condition (1.7). So Theorem 1.2 improved and generalized Theorem D. By a similar discussion we can see that in Theorem C condition (1.5) can be replaced by the following weaker condition

$$\int_0^1 \frac{\omega_1(\delta)}{\delta}(1 + |\log \delta|)\mathrm{d}\delta < \infty. \tag{1.9}$$

Precisely, we have the following

Theorem 1.3. *Suppose that* Ω *satisfies (1.1), (1.2) and (1.9). Then (1.6) holds.*

Throughout this paper, the letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same one at each occurrence but is independent of the essential variables.

§2 Preliminaries and Lemmas

A non-negative locally integrable function is called a weight function.

Definition 2.1. Let w be a weight function, $1 < p < \infty$. If there is a constant $C > 0$, such that for every cube $Q \subseteq \mathbf{R}^n$,

$$\Big(\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\mathrm{d}x\Big)^{p-1} \le C,$$

then we say $w \in A_p$. We say $w \in A_1$, if there is a constant $C > 0$, such that for every cube $Q \subseteq \mathbf{R}^n$,

$$\frac{1}{|Q|}\int_Q w(x)\mathrm{d}x \le C \operatorname*{essinf}_{x\in Q} w(x),$$

where and throughout this paper, Q denotes the cube with sides parallel to the axes.

A weight function $w \in A_\infty$ if it satisfies the A_p condition for some $1 < p < \infty$. The smallest constant satisfying the fomulas above is called A_p constant of w, we denote it by $[w]_{A_p}$. It is well-known that if $w \in A_p$ for $1 < p < \infty$, then $w \in A_r$ for all $r > p$ and $w \in A_q$ for some $1 < q < p$. We thus use $q_w := \inf\{q > 1 : w \in A_q\}$ to denote the critical index of w.

As usual, we denote $\|f\|_{L^p(w)} = (\int_{\mathbf{R}^n} |f(x)|^p w(x)\mathrm{d}x)^{\frac{1}{p}}$, for $1 < p < \infty$, and $p = \infty$, $\|f\|_{L^\infty(w)} = \|f\|_{L^\infty}$. $\|f\|_{WL^p_w} = \sup_{\lambda>0} \lambda w(\{x : |f(x)| > \lambda\})^p < \infty$. Let Q be a cube in $\mathbf{R}^n$, write $w(Q) = \int_Q w(x)\mathrm{d}x$.

Definition 2.2. [5] Let $0 < p \le 1 \le q \le \infty, p \ne q$. Assume that $w \in A_q$ and $[x]$ denotes the greatest integer that is not greater than x. For $s \in \mathbb{Z}$ satisfying $s \ge s_0 = [n(q_w/p - 1)]$, a real-valued function $a(x)$ is called $\omega(p, q, s)$ atom centered at x_0 with respect to w(or w-(p,q,s)atom), if

(1) $a \in L^q_w(\mathbf{R}^n)$ and is supported in cube Q centered at x_0;
(2) $\|a\|_{L^q(w)} \le w(Q)^{\frac{1}{q}-\frac{1}{p}}$;
(3) $\int_{\mathbf{R}^n} x^\alpha a(x)\mathrm{d}x = 0, 0 \le |\alpha| \le s$.

Lemma 2.1. [5] *Let $0 < p \le 1 \le q \le \infty, p \ne q$. Assume that $w \in A_q$. For each $f \in H^p_w(\mathbb{R}^n)$, there exists a sequence $\{a_i\}$ of $w-(p, q, [n(q_w/p-1)])$-atoms and a sequence $\{\lambda_i\}$ of real numbers with $\sum_i |\lambda_i|^p \le C\|f\|^p_{H^p_w}$ such that $f = \sum_i \lambda_i a_i$ both in the sense of distributions and in the H^p_w norm.*

Recall the definition of weak weighted Hardy space.

Let $w \in A_\infty, 0 < p \le 1$ and $N = [n(q_w/p - 1)]$. Define

$$\mathscr{A}_{N,w} = \{\varphi \in \mathscr{S}(\mathbf{R}^n) : \sup_{x\in\mathbb{R}^n} \sup_{|\alpha|\le N+1} (1 + |x|)^{N+n+1}|D^\alpha\varphi(x)| \le 1\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N}\cup\{0\})^n$, $|\alpha| = |\alpha_1|+\cdots+|\alpha_n|$ and $D^\alpha\varphi = \partial^{|\alpha|}\varphi/(\partial x_1^{\alpha_1}, \cdots, \partial x_n^{\alpha_n})$.

For fixed $f \in \mathscr{S}'(\mathbf{R}^n)$ the grand maximal function of f is defined by

$$G_w(f)(x) = \sup_{\varphi\in\mathscr{A}_{N,w}} \sup_{|x-y|<t} |(\varphi_t * f)(y)|.$$

Then we can define the weighted weak Hardy space $WH^p_w(\mathbf{R}^n)$ by

$$WH^p_w(\mathbf{R}^n) = \{f \in \mathscr{S}'(\mathbf{R}^n) : G_w f \in WL^p_w(\mathbf{R}^n)\}.$$

Moreover, we set $\|f\|_{WH^p_w} = \|G_w f\|_{WL^p_w}$.

Lemma 2.2. [9] *Let $0 < p \le 1$ and $w \in A_\infty$. For every $f \in WH^p_w(\mathbf{R}^n)$, there exists a sequence of bounded measurable functions $\{f_k\}_{k=-\infty}^\infty$ such that*

(i) $f = \sum_{k=-\infty}^\infty f_k$ in the sense of distributions.

(ii) Each f_k can be further decomposed into $f_k = \sum_i b^k_i$, where $\{b^k_i\}$ satisfies

(a) Each b^k_i is supported in a cube Q^k_i with $\sum_i w(Q^k_i) \le c2^{-kp}$ and $\sum_i \chi_{Q^k_i} \le c$. Here χ_E denotes the characteristic function of the set E and $c \sim \|f\|_{WH^p_w}$;

(b) $\|b^k_i\|_{L^\infty} \le C2^k$, where $C > 0$ is independent of i, k;

(c) $\int_{\mathbf{R}^n} b^k_i x^\alpha \mathrm{d}x = 0$ for every multi-index $|\alpha| \le [n(q_w/p - 1)]$.

Conversely, if $f \in \mathscr{S}'(\mathbf{R}^n)$ has a decomposition satisfying (i) and (ii), then $f \in WH^p_w(\mathbf{R}^n)$. Moreover, we have $\|f\|_{WH^p_w} \sim c$.

Lemma 2.3. [6] *Let $w \in A_p$ with $p \ge 1$. Then, for any cube Q, there exists an absolute constant $C > 0$ such that*

$$w(2Q) \le Cw(Q).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda Q) \le C\lambda^{np}w(Q),$$

where $C > 0$ does not depend on Q and λ.

Lemma 2.4. [6] *Let $w \in A_q$ with $q > 1$. Then, for all $r > 0$, there exists an constant $C > 0$ such that*

$$\int_{|x|\geq r} \frac{w(x)}{|x|^{nq}}\mathrm{d}x \leq Cr^{-nq}w(Q(0,2r)).$$

Lemma 2.5. [3] *Let $0 < \rho < n$ and $q > 1$. Suppose that Ω satisfies (1.1) and the L^q-Dini condition. Then, given $R > 0$, $0 < a_0 < 1$ and $|y| < a_0 R$, we have*

$$\Big(\int_{R<x<2R}\Big|\frac{\Omega(x-y)}{|x-y|^{n-\rho}} - \frac{\Omega(x)}{|x|^{n-\rho}}\Big|^q \mathrm{d}x\Big)^{1/q} \leq CR^{\frac{n}{q}-n+\rho}\Big(\frac{|y|}{R} + \int_{|y|/2R}^{|y|/R}\frac{\omega_q(t)}{t}\mathrm{d}t\Big),$$

where C is independent of R and y, and may depend on a_0, n, q, ρ and ω.

§3 Proof of the Theorems

Proof of Theorem 1.1. It is sufficient to show that there exists a constant $C > 0$, such that for any $w-(1,\infty,0)$-atom a, $\|\mu_\Omega^\rho(a)\|_{L_w^1} \leq C$.

Assume that w is a $w-(1,\infty,0)$-atom, let $Q^* = 2\sqrt{n}Q$. Denote by x_0 and d the center and the side length of cube Q, we see

$$\|\mu_\Omega^\rho(a)\|_{L_w^1} = \int_{Q^*}|\mu_\Omega^\rho a(x)|w(x)\mathrm{d}x + \int_{(Q^*)^c}|\mu_\Omega^\rho a(x)|w(x)\mathrm{d}x := I + II.$$

Next we estimate I. Since $w^{q'} \in A_1$, then $w \in A_1$. By Hölder's inequality, Theorem A and Lemma 2.3, we have

$$\begin{aligned} I &\leq (\int_{Q^*}|\mu_\Omega^\rho a(x)|^q w(x)\mathrm{d}x)^{\frac{1}{q}}(\int_{Q^*} w(x)\mathrm{d}x)^{1-\frac{1}{q}} \\ &\leq C(\int_Q |a(x)|^q w(x)\mathrm{d}x)^{\frac{1}{q}}(\int_{Q^*} w(x)\mathrm{d}x)^{1-\frac{1}{q}} \\ &\leq C\|a\|_{L_w^\infty} w(Q)^{\frac{1}{q}} w(Q^*)^{1-\frac{1}{q}} \\ &\leq C. \end{aligned}$$

To estimate II, we see

$$\begin{aligned} II &\leq \int_{(Q^*)^c}\Big(\int_0^{|x-x_0|+\sqrt{n}d}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\Big)^{\frac{1}{2}}w(x)\mathrm{d}x \\ &+ \int_{(Q^*)^c}\Big(\int_{|x-x_0|+\sqrt{n}d}^{\infty}\Big|\int_{|x-y|\leq t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}a(y)\mathrm{d}y\Big|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\Big)^{\frac{1}{2}}w(x)\mathrm{d}x \\ &:= II_1 + II_2. \end{aligned}$$

For $y \in Q$ and $x \in (Q^*)^c$, we have $|x-y| \sim |x-x_0| \sim |x-x_0| + \sqrt{n}d$. Thus

$$\Big|\frac{1}{|x-y|^{2\rho}} - \frac{1}{(|x-x_0|+\sqrt{n}d)^{2\rho}}\Big| \leq C\frac{d}{|x-y|^{2\rho+1}}.$$

Appling Minkowski's inequality, we have

$$\begin{aligned} II_1 &\leq \int_{(Q^*)^c}\int_Q \frac{\Omega(x-y)}{|x-y|^{n-\rho}}|a(y)|\Big(\int_{|x-y|}^{|x-x_0|+\sqrt{n}d}\frac{\mathrm{d}t}{t^{2\rho+1}}\Big)^{1/2}\mathrm{d}y w(x)\mathrm{d}x \\ &\leq Cd^{\frac{1}{2}}\int_Q\int_{(Q^*)^c}\frac{|\Omega(x-y)|}{|x-y|^{n+\frac{1}{2}}}w(x)\mathrm{d}x|a(y)|\mathrm{d}y \end{aligned}$$

$$\begin{aligned}
&\leq Cd^{\frac{1}{2}}\int_Q\Big(\int_{(Q^*)^c}\frac{|\Omega(x-y)|^q}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{(Q^*)^c}\frac{w(x)^{q'}}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\Big)^{\frac{1}{q'}}|a(y)|\mathrm{d}y\\
&\leq Cd^{\frac{1}{2}}\int_Q\Big(\sum_{k=0}^{\infty}\int_{2^{k+1}Q^*\backslash 2^kQ^*}\frac{|\Omega(x-y)|^q}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{(Q^*)^c}\frac{w(x)^{q'}}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\Big)^{\frac{1}{q'}}|a(y)|\mathrm{d}y.
\end{aligned}$$

Since $w^{q'}\in A_1$, then $w\in A_1\subseteq A_{1+\frac{1}{2n}}$. It follows from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned}
&\left(\int_{Q^c}\frac{w(x)^{q'}}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\right)^{1/q'}\\
&\leq C(d)^{-\frac{n}{q'}-\frac{1}{2q'}}(w^{q'}(Q))^{1/q'}\\
&\leq C(d)^{-\frac{n}{q'}-\frac{1}{2q'}}w^{q'}(Q)^{1/q'}\\
&\leq C(d)^{-\frac{1}{2q'}}\inf_{x\in Q}w(x).
\end{aligned}\tag{3.1}$$

By $\Omega\in L^q(S^{n-1})$, we obtain

$$\int_{2^{k+1}Q^*\backslash 2^kQ^*}\frac{|\Omega(x-y)|^q}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\leq\int_{2^k\sqrt{n}d}^{2^{k+1}\sqrt{n}d}\int_{S^{n-1}}\frac{|\Omega(x')|^q}{r^{n+\frac{1}{2}}}r^{n-1}\mathrm{d}\sigma(x')\mathrm{d}r\leq C2^{-\frac{k}{2}}d^{-\frac{1}{2}}\|\Omega\|^q_{L^q(S^{n-1})}.\tag{3.2}$$

Using Hölder's inequality and combining (3.1) and (3.2), we get

$$\begin{aligned}
&\int_{(Q^*)^c}\frac{|\Omega(x-y)|}{|x-y|^{n+\frac{1}{2}}}w(x)\mathrm{d}x\\
&\leq\Big(\int_{(Q^*)^c}\frac{|\Omega(x-y)|^q}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\Big)^{\frac{1}{q}}\Big(\int_{(Q^*)^c}\frac{w(x)^{q'}}{|x-y|^{n+\frac{1}{2}}}\mathrm{d}x\Big)^{\frac{1}{q'}}\\
&\leq Cd^{-\frac{1}{2}}\inf_{x\in Q}w(x).
\end{aligned}\tag{3.3}$$

It follows from (3.3) that

$$II_1\leq Cd^{\frac{1}{2}}\int_Q d^{-\frac{1}{2}}\inf_{x\in Q}w(x)|a(y)|\mathrm{d}y\leq C\|a\|_{L^\infty_w}\int_Q w(y)\mathrm{d}y\leq C.\tag{3.4}$$

Now we estimate II_2. Since $t>|x-x_0|+\sqrt{n}d\sim|x-x_0|$ for $x\in(Q^*)^c$, then $Q\subseteq\{y:|y-x|<t\}$. By the vanishing moment condition of a and Lemma 2.5, we get

$$\begin{aligned}
II_2&\leq C\int_{(Q^*)^c}\int_Q\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right||a(y)|\left(\int_{|x-x_0|+\sqrt{n}d}^{\infty}\frac{dt}{t^{2\rho+1}}\right)^{\frac{1}{2}}\mathrm{d}yw(x)\mathrm{d}x\\
&\leq C\int_{(Q^*)^c}\int_Q\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|\frac{|a(y)|}{|x-x_0|^{\rho}}\mathrm{d}yw(x)\mathrm{d}x\\
&\leq C\int_Q\sum_{j=0}^{\infty}\int_{2^j\sqrt{n}d\leq|x|<2^{j+1}\sqrt{n}d}\frac{1}{(2^j\sqrt{n}d)^{\rho}}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right||a(y)|w(x)\mathrm{d}x\mathrm{d}y\\
&\leq C\int_Q|a(y)|\sum_{j=0}^{\infty}\frac{1}{(2^j\sqrt{n}d)^{\rho}}\left(\int_{2^j\sqrt{n}d\leq|x|<2^{j+1}\sqrt{n}d}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_0)}{|x-x_0|^{n-\rho}}\right|^q\mathrm{d}x\right)^{\frac{1}{q}}\\
&\quad\times\left(\int_{2^j\sqrt{n}d\leq|x|<2^{j+1}\sqrt{n}d}w(x)^{q'}\mathrm{d}x\right)^{\frac{1}{q'}}\mathrm{d}y\\
&\leq C\int_Q|a(y)|\sum_{j=0}^{\infty}\frac{1}{(2^j\sqrt{n}d)^{\rho}}(2^j\sqrt{n}d)^{\frac{n}{q}-n+\rho}\left\{\frac{|y-x_0|}{2^j\sqrt{n}d}\right.
\end{aligned}$$

$$+\int_{\frac{|y-x_0|}{2^{j+1}\sqrt{n}d}\le\delta<\frac{|y-x_0|}{2^j\sqrt{n}d}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big\}(w^{q'}(2^{j+2}\sqrt{n}Q))^{\frac{1}{q'}}\mathrm{d}y$$

Since $w^{q'}\in A_1$, and then by Lemma 2.3, we have

$$\begin{aligned}(w^{q'}(2^{j+2}\sqrt{n}Q))^{\frac{1}{q'}} &\le C[(2^{j+2}\sqrt{n})^n w^{q'}(Q)]^{\frac{1}{q'}}\\ &\le C(2^{j+2}\sqrt{n})^{\frac{n}{q'}}|Q|^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_Q w(x)^{q'}\mathrm{d}x\right)^{\frac{1}{q'}}\\ &\le C2^{\frac{jn}{q'}}|Q|^{\frac{1}{q'}}\inf_{x\in Q}w(x).\end{aligned}\tag{3.5}$$

It follows from (3.5) and Lemma 1 that

$$II_2\le C\int_Q|a(y)|(C+\int_0^1\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta)\inf_{x\in Q}w(x)\mathrm{d}y\le C\|a\|_{L^\infty_w}\int_Q w(y)\mathrm{d}y\le C.\tag{3.6}$$

By combining the estimates of (3.4) and (3.6), we get $II\le C$. Thus from the inequality $I\le C$, we conclude that $\|\mu^\rho_\Omega a\|_{L^1_w}\le C$. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. For any given $\gamma>0$, we may choose $k_0\in\mathbb{Z}$ such that $2^{k_0}\le\gamma<2^{k_0+1}$. For every $f\in WH^1_w$, we write

$$f=\sum_{k=-\infty}^{\infty}f_k=\sum_{k=-\infty}^{k_0}f_k+\sum_{k=k_0+1}^{\infty}f_k:=F_1+F_2,$$

where $F_1=\sum_{k=-\infty}^{k_0}\sum_i b_i^k$, $F_2=\sum_{k=k_0+1}^{\infty}\sum_i b_i^k$, and $\{b_i^k\}$ satisfied (a)-(c) in Lemma 2.2. We see

$$\begin{aligned}\gamma w\{x\in\mathbf{R}^n:|\mu^\rho_\Omega(f)(x)|>\gamma\} &\le \gamma w\{x\in\mathbf{R}^n:|\mu^\rho_\Omega(F_1)(x)|>\frac{\gamma}{2}\}\\ &\quad+\gamma w\{x\in\mathbf{R}^n:|\mu^\rho_\Omega(F_2)(x)|>\frac{\gamma}{2}\}\\ &:= P_1+P_2.\end{aligned}$$

To estimate P_1, first we claim

$$\|F_1\|_{L^2_w}\le C\gamma^{1-\frac{1}{2}}\|f\|^{\frac{1}{2}}_{WH^1_w}$$

holds. Since $\|b_i^k\|_{L^\infty}\le C2^k$ and by Lemma 2.2, we have

$$\begin{aligned}\|F_1\|_{L^2_w} &\le \sum_{k=-\infty}^{k_0}\sum_i\|b_i^k\|_{L^2_w}\\ &\le \sum_{k=-\infty}^{k_0}\sum_i\|b_i^k\|_{L^\infty}w(Q_i^k)^{\frac{1}{2}}\\ &\le C\sum_{k=-\infty}^{k_0}2^k(\sum_i w(Q_i^k))^{\frac{1}{2}}\\ &\le C\sum_{k=-\infty}^{k_0}2^{k(1-\frac{1}{2})}\|f\|^{\frac{1}{2}}_{WH^1_w}\\ &\le C\sum_{k=-\infty}^{k_0}2^{(k-k_0)(1-\frac{1}{2})}\gamma^{1-\frac{1}{2}}\|f\|^{\frac{1}{2}}_{WH^1_w}\\ &\le C\gamma^{1-\frac{1}{2}}\|f\|^{\frac{1}{2}}_{WH^1_w}.\end{aligned}\tag{3.7}$$

Since $w^{q'}\in A_1$, then $w^{q'}\in A_2$. It follows from (3.7) and Theorem A that

$$P_1\le\gamma\frac{4}{\gamma^2}\|\mu^\rho_\Omega(F_1)\|^2_{L^2_w}\le C\gamma^{-1}\|F_1\|^2_{L^2_w}\le C\|f\|_{WH^1_w}.\tag{3.8}$$

Now we estimate P_2. Denote $A_{k_0}=\cup_{k=k_0+1}^{\infty}\cup_i\tilde{Q}_i^k$, where $\tilde{Q}_i^k=Q(x_i^k,(\frac{3}{2})^{\frac{k-k_0}{n}}\sqrt{n}d_i^k)$. Then we see

$$P_2 \le \gamma\omega\Big\{x \in A_{k_0}, |\mu_\Omega^\rho(F_2)(x)| > \frac{\gamma}{2}\Big\} + \gamma\omega\Big\{x \in (A_{k_0})^c, |\mu_\Omega^\rho(F_2)(x)| > \frac{\gamma}{2}\Big\} := P_2' + P_2''.$$

Since $w \in A_1$ and by Lemma 2.2, Lemma 2.3, we get

$$\begin{aligned} P_2' &\le \gamma\sum\nolimits_{k=k_0+1}^{\infty}\sum\nolimits_i w(\tilde{Q}_i^k) \\ &\le C\gamma\sum\nolimits_{k=k_0+1}^{\infty}(\frac{3}{2})^{k-k_0}\sum\nolimits_i w(Q_i^k) \\ &\le C\gamma\sum\nolimits_{k=k_0+1}^{\infty}(\frac{3}{2})^{k-k_0}2^{-k}\|f\|_{WH_w^1} \\ &\le C\sum\nolimits_{k=k_0+1}^{\infty}(\frac{3}{4})^{k-k_0}\|f\|_{WH_w^1} \\ &\le C\|f\|_{WH_w^1}. \end{aligned} \tag{3.9}$$

Using Chebyshev's inequality, we have

$$P_2'' \le C\int_{(A_{k_0})^c}|\mu_\Omega^\rho(F_2)(x)|w(x)\mathrm{d}x \le C\sum_{k=k_0+1}^{\infty}\sum_i\int_{(A_{k_0})^c}|\mu_\Omega^\rho(b_i^k)(x)|w(x)\mathrm{d}x.$$

Denote $J = \int_{(A_{k_0})^c}|\mu_\Omega^\rho(b_i^k)(x)|w(x)\mathrm{d}x$. We get

$$\begin{aligned} J \;\le\; & \int_{(A_{k_0})^c}\bigg(\int_0^{|x-x_i^k|+\sqrt{n}d_i^k}\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}b_i^k(y)\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\bigg)^{1/2}w(x)\mathrm{d}x \\ + & \int_{(A_{k_0})^c}\bigg(\int_{|x-x_i^k|+\sqrt{n}d_i^k}^{\infty}\bigg|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-\rho}}b_i^k(y)\mathrm{d}y\bigg|^2\frac{\mathrm{d}t}{t^{2\rho+1}}\bigg)^{1/2}w(x)\mathrm{d}x \\ := \; & J_1 + J_2. \end{aligned}$$

Since $y \in Q_i^k, x \in \tilde{Q}_i^k$, then $|x-y| \sim |x-x_i^k| \sim |x-x_i^k| + \sqrt{n}d_i^k$. Thus

$$\frac{1}{|x-y|^{2\rho}} - \frac{1}{(|x-x_i^k|+\sqrt{n}d_i^k)^{2\rho}} \le C\frac{d_i^k}{|x-y|^{2\rho+1}}.$$

Using Minkowski's inequality, we have

$$\begin{aligned} J_1 &\le \int_{(A_{k_0})^c}\int_{Q_i^k}\frac{|\Omega(x-y)|}{|x-y|^{n-\rho}}|b_i^k(y)|\bigg(\int_{|x-y|}^{|x-x_i^k|+\sqrt{n}d_i^k}\frac{\mathrm{d}t}{t^{2\rho+1}}\bigg)^{1/2}\mathrm{d}yw(x)\mathrm{d}x \\ &\le {d_i^k}^{\frac{1}{2}}\int_{Q_i^k}\bigg(\int_{(A_{k_0})^c}\frac{|\Omega(x-y)|}{|x-y|^{n+\frac{1}{2}}}w(x)\mathrm{d}x\bigg)|b_i^k(y)|\mathrm{d}y. \end{aligned}$$

Similar to the estimate of II_1 in Theorem 1.1, we have

$$J_1 \le C2^k(d_i^k)^{\frac{1}{2}}((\frac{3}{2})^{\frac{k-k_0}{n}}d_i^k)^{-\frac{1}{2}}\inf_{x\in Q_i^k}w(x)|Q_i^k| \le C2^k w(Q_i^k)(\frac{2}{3})^{\frac{k-k_0}{2n}}. \tag{3.10}$$

Select $R_j^k = 2^j(\frac{3}{2})^{\frac{k-k_0}{n}}\sqrt{n}$. Similar to the estimate of (3.5) in Theorem 1, we have

$$\bigg(w^{q'}(R_{j+1}^kQ_i^k)\bigg)^{1/q'} \le C(R_j^k)^{\frac{n}{q'}}(d_i^k)^{\frac{n}{q'}}\inf_{x\in Q_i^k}w(x). \tag{3.11}$$

Since $x \in (\tilde{Q}_i^k)^c$, $|x-x_i^k| \sim |x-x_i^k| + \sqrt{n}d_i^k$, then $Q_i^k \subseteq \{y : |x-y| < t\}$. By the vanishing moment condition of b_i^k and applying Minkowski's inequality, the estimate of (3.11), Lemma

2.5, we have

$$\begin{aligned} J_2 &\le \int_{(\tilde{Q}_i^k)^c}\int_{Q_i^k}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}-\frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-\rho}}\right||b_i^k(y)| \\ &\quad\times\Big(\int_{|x-x_i^k|+\sqrt{n}d_i^k}^{\infty}\frac{\mathrm{d}t}{t^{2\rho+1}}\Big)^{1/2}\mathrm{d}yw(x)\mathrm{d}x \\ &\le \int_{Q_i^k}\sum_{j=0}^{\infty}\frac{1}{(R_j^k d_i^k)^\rho}\int_{R_j^k d_i^k\le|x-x_i^k|<R_{j+1}^k d_i^k}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}\right. \\ &\quad\left.-\frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-\rho}}\right|w(x)\mathrm{d}x|b_i^k(y)|\mathrm{d}y \\ &\le \|b_i^k\|_{L^\infty}\int_{Q_i^k}\sum_{j=0}^{\infty}\frac{1}{(R_j^k d_i^k)^\rho}\Big(\int_{R_j^k d_i^k\le|x-x_i^k|<R_{j+1}^k d_i^k}\left|\frac{\Omega(x-y)}{|x-y|^{n-\rho}}\right. \\ &\quad\left.-\frac{\Omega(x-x_i^k)}{|x-x_i^k|^{n-\rho}}\right|^q\mathrm{d}x\Big)^{1/q}\Big(w^{q'}(R_{j+1}^k Q_i^k)\Big)^{1/q'}\mathrm{d}y \\ &\le \|b_i^k\|_{L^\infty}\int_{Q_i^k}\sum_{j=0}^{\infty}\frac{1}{(R_j^k d_i^k)^\rho}(R_j^k d_i^k)^{\frac{n}{q}-n+\rho}\Big(\frac{|y-x_i^k|}{R_j^k d_i^k}+\int_{\frac{|y-x_i^k|}{R_{j+1}^k d_i^k}}^{\frac{|y-x_i^k|}{R_j^k d_i^k}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big) \\ &\quad(R_j^k)^{\frac{n}{q'}}(d_i^k)^{\frac{n}{q'}}\inf_{x\in Q_i^k}w(x)\mathrm{d}y \\ &\le C2^k\int_{Q_i^k}\inf_{x\in Q_i^k}w(x)\mathrm{d}y\Big\{\sum_{j=0}^{\infty}\frac{1}{2^j}(\frac{2}{3})^{\frac{k-k_0}{n}}+\int_0^{(\frac{2}{3})^{\frac{k-k_0}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big\} \\ &\le C2^k w(Q_i^k)\Big\{(\tfrac{2}{3})^{\frac{k-k_0}{n}}+\int_0^{(\frac{2}{3})^{\frac{k-k_0}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big\}. \end{aligned} \tag{3.12}$$

It follows from (3.10) and (3.12) that

$$\begin{aligned} P_2'' &\le C\sum_{k=k_0+1}^{\infty}\sum_i 2^k w(Q_i^k)\Big\{(\frac{2}{3})^{\frac{k-k_0}{n}}+\int_0^{(\frac{2}{3})^{\frac{k-k_0}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta\Big\} \\ &\le C\|f\|_{WH_w^1}\sum_{k=k_0+1}^{\infty}(\frac{2}{3})^{\frac{k-k_0}{n}}+C\|f\|_{WH_w^1}\sum_{k=k_0+1}^{\infty}\int_0^{(\frac{2}{3})^{\frac{k-k_0}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta \\ &:= U+V. \end{aligned} \tag{3.13}$$

Next we estimate V. We see

$$\begin{aligned} V &= C\|f\|_{WH_w^1}\sum_{p=1}^{\infty}\int_0^{(\frac{2}{3})^{\frac{p}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta \\ &= C\|f\|_{WH_w^1}\Big(\int_{(\frac{2}{3})^{\frac{2}{n}}}^{(\frac{2}{3})^{\frac{1}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta+2\int_{(\frac{2}{3})^{\frac{3}{n}}}^{(\frac{2}{3})^{\frac{2}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta+\cdots\Big) \\ &= C\|f\|_{WH_w^1}\sum_{p=1}^{\infty}p\int_{(\frac{2}{3})^{\frac{p+1}{n}}}^{(\frac{2}{3})^{\frac{p}{n}}}\frac{\omega_q(\delta)}{\delta}\mathrm{d}\delta \\ &\le C\|f\|_{WH_w^1}\sum_{p=1}^{\infty}\int_{(\frac{2}{3})^{\frac{p+1}{n}}}^{(\frac{2}{3})^{\frac{p}{n}}}\frac{\omega_q(\delta)}{\delta}(1+|\log\delta|)\mathrm{d}\delta \\ &< C\|f\|_{WH_w^1}\int_0^1\frac{\omega_q(\delta)}{\delta}(1+|\log\delta|)\mathrm{d}\delta. \end{aligned} \tag{3.14}$$

By (3.13) and (3.14), we get

$$U+V<C\|f\|_{WH^1_w}+C\|f\|_{WH^1_w}\int_0^1\frac{\omega_q(\delta)}{\delta}(1+|\log\delta|)\mathrm{d}\delta\le C\|f\|_{WH^1_w}. \tag{3.15}$$

By combining the estimates of (3.8),(3.9) and (3.15), we get $\gamma w\{x\in\mathbf{R}^n:|\mu^\rho_\Omega(f)(x)|>\gamma\}<C\|f\|_{WH^1_w}$. This completes the proof of Theorem 1.2. □

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