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Bi-f-harmonic map equations on singly warped product manifolds

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Abstract. Bi-f-harmonic maps are the critical points of bi-f-energy functional. This class of maps tends to integrate bi-harmonic maps and f-harmonic maps. In this paper, we show that bi-f-harmonic maps are not only an extension of f-harmonic maps but also an extension of bi-harmonic maps, and that there should exist many examples of proper bi-f-harmonic maps. In order to find some concrete examples of proper bi-f-harmonic maps, we study the basic properties of bi-f-harmonic maps from two directions which are conformal maps between the same dimensional manifolds and some special maps from or into a warped product manifold.

§1 Introduction

Harmonic maps as a generalization of important concepts of geodesics, minimal surfaces and harmonic functions, have been studied extensively with tremendous progress in the past forty years. There are volumes of books and literatures on the beautiful theory. More recently, the research in the field of Riemannian geometry was partly characterized by the study of some fourth order partial differential equations. Whether the origin of these equations is analytic, as in the case of the Paneitz operator, or geometric, as in the case of Willmore surfaces, they represent a generalization of the concept of harmonic map. A natural extension of harmonic maps, called bi-harmonic maps, was suggested by Eells and Sampson in [13]. Some results in this field were obtained by Jiang in 1986 (see [14]). Since 2000, bi-harmonic maps have been receiving a growing attention and have become a popular subject of study with many progresses (e.g., [2, 3, 5], etc.).

On the other hand, f-harmonic maps between Riemannian manifolds (as a generalization of harmonic maps) were first introduced and studied by Lichnerowicz in [16] (see also Section 10.20 in Eells-Lemaire's report [12]). The study of f-harmonic maps comes from a physical

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motivation, since in physics, f-harmonic map can be viewed as stationary solution of the inhomogeneous Heisenberg spin system (see [15]). First motivated by the physical interpretation of f-harmonic map (see [19], [15], [21]), we borrowed from the method for studying bi-harmonic maps in [5, 22] to investigate the behaviors of f-harmonic maps from or into doubly warped product manifold (WPM). We derived some characteristic equations for f-harmonicity and also constructed some non-trivial examples [17]. However, we found that f-tension field does not involve the Riemannian curvature tensor $\bar{R}$ on warped product manifold. So we were suggested to formulate a new type of tension field which contains the Rimannian curvature component similar to bi-tension field. Naturally, we focused on constructing a field so-called bi-f-tension field or f-bi-tension field by combining the benefits of both f-tension field and bi-tension field.

There are two ways to formalize such a link between bi-harmonic maps and f-harmonic maps. The first formalization is that by mimicking the theory for bi-harmonic maps, we extend bi-energy functional to bi-f-energy functional and obtain a new type of harmonic maps called bi-f-harmonic maps. This idea was already considered by Ouakkas, Nasri and Djaa [20]. They used the terminology "f-bi-harmonic maps" for the critical point of bi-f-energy functional. As parallel to "bi-harmonic maps", we think that it is more reasonable to call them "bi-f-harmonic maps". The second formalization is that by following the definition of f-harmonic map, we extend f-energy functional to f-bi-energy functional and obtain another type of harmonic maps called f-bi-harmonic maps for the critical points of f-bi-energy functional.

In this paper, we will show that bi-f-harmonic maps generalize not only f-harmonic maps but also bi-harmonic maps. It is well known that for $n \neq 2$, the harmonicity and f-harmonicity of a map $\phi : (M, g) \to (N, h)$ are related via a conformal change of the domain metric. Therefore, in general, bi-f-harmonic map does not generalize to the case of the relationship between bi-harmonic map and f-harmonic map, but very interestingly, we have the following result: A map $(M^2, g) \to (N^n, h)$ is a bi-f-harmonic map if and only if $\phi : (M^2, \frac{1}{f}g) \to (N^n, h)$ is an f-harmonic map with certain assumption. In addition, although the authors in [20] gave the first variation of bi-f-energy functional, they did not give a single example of proper bi-f-harmonic map which is neither harmonic nor f-harmonic map. So far as we know, such concrete examples have not been reported. Thus our main objective in this paper is to construct examples of proper bi-f-harmonic maps. Firstly, noting that the progress in the topic on bi-f-harmonic map such as [7–9], together with [1], we consider some examples for bi-f-harmonic maps with conformal dilation, see examples 4.2 and 4.3. Secondly, we characterize the behaviors of bi-f-harmonic maps from or into a warped product manifold so as to seek for some interesting and complicated examples, see section 5.

§2 Preliminaries

2.1 Harmonic, bi-harmonic and f-harmonic maps

Recall that the energy of a smooth map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is defined by integral $E(\phi) = \int_\Omega e(\phi) dv_g$, for every compact domain $\Omega \subset M$, where

$e(\phi) = \frac{1}{2}|d\phi|^2$ is energy density and ϕ is called *harmonic* if it's a critical point of energy. From the first variation formula for the energy, the Euler-Lagrange equation is given by the vanishing of *tension field* $\tau(\phi) = \mathrm{Tr}_g \nabla d\phi$ (see [13]). As two generalizations of harmonic maps, we now recall the concepts of bi-harmonic maps and f-harmonic maps.

Definition 2.1. (i) Bi-harmonic maps $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are the critical points of bi-energy functional

$$E_2(\phi) = \frac{1}{2}\int_\Omega |\tau(\phi)|^2 dv_g, \tag{1}$$

for any compact domain $\Omega \subset M$. The Euler-Lagrange equation of this functional gives the vanishing of bi-tension field ([14])

$$\tau_2(\phi) = -\mathrm{Tr}_g\big(\nabla^\phi_{\cdot}\nabla^\phi_{\cdot}\tau(\phi) - \nabla^\phi_{{}^M\nabla_{\cdot}\cdot}\tau(\phi)\big) - \mathrm{Tr}_g(\,{}^N R(d\phi, \tau(\phi))d\phi)), \tag{2}$$

where ${}^N R$ denotes the curvature operator of (N, h) defined by

$${}^N R(X, Y)Z = [{}^N\nabla_X, {}^N\nabla_Y]Z - {}^N\nabla_{[X,Y]}Z.$$

(ii) An f-harmonic map with a positive function $f \in C^\infty(M)$ is a critical point of f-energy

$$E_f(\phi) = \frac{1}{2}\int_{\Omega \Subset M} f|d(\phi)|^2 dv_g. \tag{3}$$

The Euler-Lagrange equation gives the vanishing of f-tension field (see [10], [20], [19])

$$\tau_f(\phi) = f\tau(\phi) + d\phi(\mathrm{grad} f). \tag{4}$$

2.2 Connection and Riemannian curvature tensor on singly WPM

First we refer to [23] and give the definition of doubly/singly warped product manifold (WPM).

Definition 2.2. Let (M, g) and (N, h) be Riemannian manifolds of dimensions m and n, respectively and let $\lambda : M \to (0, +\infty)$ and $\mu : N \to (0, +\infty)$ be smooth functions. A *doubly warped product manifold (WPM)* $\bar{G} = M \times_{(\mu,\lambda)} N$ is the product manifold $M \times N$ endowed with doubly warped product metric $\bar{g} = \mu^2 g \oplus \lambda^2 h$ defined by

$$\bar{g}(X, Y) = (\mu \circ \pi_1)^2 g(d\pi_1(X), d\pi_1(Y)) + (\lambda \circ \pi_2)^2 h(d\pi_2(X), d\pi_2(Y))$$

for all $X, Y \in T_{(x,y)}(M \times N)$, where $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ are the canonical projections. The functions λ and μ are called the *warping functions*.

If either $\mu = 1$ or $\lambda = 1$ but not both we obtain a *singly WPM*. If both $\mu = 1$ and $\lambda = 1$ then we have an *usual product manifold*. If neither μ nor λ is constant, then we have a *non-trivial doubly WPM*.

We have known that preciously the formulas about Riemann curvature and Ricci curvature are split into several parts according to the horizontal lift or vertical lift of the tangent vectors attached to the initial space M or target space N. For this we first introduce the unified connection and unified Riemannian curvature on WPM $\bar{G}$ (cf. [6], [5]) by introducing a new notation of lift vector.

Proposition 2.1. *Let ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on the Riemannian product manifold $M \times N$ and WPM $\bar{G} = M \times_\lambda N$, respectively. Then Levi-Civita connection of $\bar{G}$ is given by*

$$\begin{aligned} &\bar{\nabla}_X Y \\ = \;& \nabla_X Y + \tfrac{1}{2\lambda^2} X_1(\lambda^2)(0, Y_2) + \tfrac{1}{2\lambda^2} Y_1(\lambda^2)(0, X_2) - \tfrac{1}{2} h(X_2, Y_2)(grad\, \lambda^2, 0) \\ = \;& \big({}^M\nabla_{X_1} Y_1 - \tfrac{1}{2} h(X_2, Y_2) grad\, \lambda^2, 0\big) + \big(0, {}^N\nabla_{X_2} Y_2 + \tfrac{1}{2\lambda^2} X_1(\lambda^2) Y_2 + \tfrac{1}{2\lambda^2} Y_1(\lambda^2) X_2\big), \end{aligned} \tag{5}$$

for any $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \mathscr{X}(\bar{G})$, where $X_1, Y_1 \in \mathscr{X}(M)$ and $X_2, Y_2 \in \mathscr{X}(N)$. If R and $\bar{R}$ denote the curvature operators of $M \times N$ and $\bar{G}$, respectively, then we have

$$\begin{aligned} \bar{R}_{XY} - R_{XY} = \frac{1}{2\lambda^2}\Big\{&\Big({}^M\nabla_{Y_1} grad_g \lambda^2 - \frac{1}{2\lambda^2} Y_1(\lambda^2) grad_g \lambda^2, 0\Big) \wedge_{\bar{g}} (0, X_2) \\ &- \Big({}^M\nabla_{X_1} grad_g \lambda^2 - \frac{1}{2\lambda^2} X_1(\lambda^2) grad_g \lambda^2, 0\Big) \wedge_{\bar{g}} (0, Y_2) \\ &- \frac{1}{2\lambda^2} |grad_g \lambda^2|^2 (0, X_2) \wedge_{\bar{g}} (0, Y_2)\Big\}, \end{aligned} \tag{6}$$

where the wedge product $(X \wedge_{\bar{g}} Y)Z = \bar{g}(Y, Z)X - \bar{g}(X, Z)Y$, for all $X, Y, Z = (Z_1, Z_2) \in \mathscr{X}(\bar{G})$.

For the detailed proof, see [18]. From (6), we easily obtain the following results.

Proposition 2.2. *Let ${}^M R$ and ${}^N R$ be the Riemannian curvature tensors of M and N, respectively. With the same notations as in Proposition 2.1, (1,3)-type Riemannian curvature tensor $\bar{R}$ on $\bar{G}$ can be expressed as*

$$\begin{aligned} \bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1, Z_2) = \;& ({}^M R_{X_1 Y_1} Z_1, {}^N R_{X_2 Y_2} Z_2) \\ &+ \frac{1}{2} h(X_2, Z_2)\big({}^M\nabla_{Y_1} \text{grad } \lambda^2 - \frac{1}{2\lambda^2} Y_1(\lambda^2) \text{grad } \lambda^2, 0\big) \\ &- \frac{1}{2} h(Y_2, Z_2)\big({}^M\nabla_{X_1} \text{grad } \lambda^2 - \frac{1}{2\lambda^2} X_1(\lambda^2) \text{grad } \lambda^2, 0\big) \\ &+ \Big(0, \frac{1}{2\lambda^2} g\big({}^M\nabla_{X_1} \text{grad } \lambda^2 - \frac{1}{2\lambda^2} X_1(\lambda^2) \text{grad } \lambda^2, Z_1\big) Y_2\Big) \\ &- \Big(0, \frac{1}{2\lambda^2} g\big({}^M\nabla_{Y_1} \text{grad } \lambda^2 - \frac{1}{2\lambda^2} Y_1(\lambda^2) \text{grad } \lambda^2, Z_1\big) X_2\Big) \\ &+ \Big(0, \frac{1}{4\lambda^2} \mid \text{grad } \lambda^2 \mid^2 h(X_2, Z_2) Y_2\Big) \\ &- \Big(0, \frac{1}{4\lambda^2} \mid \text{grad } \lambda^2 \mid^2 h(Y_2, Z_2) X_2\Big). \end{aligned} \tag{7}$$

Proposition 2.2 can be rewritten as the following simple form with Laplacian operator and Hessian operator.

Corollary 2.1. *We have*

$$\begin{aligned} \bar{R}_{(X_1,X_2)(Y_1,Y_2)}(Z_1, Z_2) = \;& ({}^M R_{X_1 Y_1} Z_1, {}^N R_{X_2 Y_2} Z_2) \\ &+ \lambda h(X_2, Z_2)\big({}^M\nabla_{Y_1} \text{grad } \lambda, 0\big) - \lambda h(Y_2, Z_2)\big({}^M\nabla_{X_1} \text{grad } \lambda, 0\big) \\ &+ \frac{1}{\lambda} \text{Hess}(\lambda)(X_1, Z_1)(0, Y_2) - \frac{1}{\lambda} \text{Hess}(\lambda)(Y_1, Z_1)(0, X_2) \\ &+ \mid \text{grad } \lambda \mid^2 h(X_2, Z_2)(0, Y_2) - \mid \text{grad } \lambda \mid^2 h(Y_2, Z_2)(0, X_2). \end{aligned} \tag{8}$$

§3 Some characterizations of bi-f-harmonic maps

According to Definition 2.1, there are two ways to formalize a link between bi-harmonic maps and f-harmonic maps. Both of them motivate the following definitions.

Definition 3.1. (i) Bi-f-energy functional of smooth map $\phi : (M, g) \to (N, h)$ is defined by

$$E_{f,2}(\phi) = \frac{1}{2}\int_{\Omega} |\tau_f(\phi)|^2 dv_g, \tag{9}$$

for every compact domain $\Omega \subset M$. A map ϕ is called bi-f-harmonic map if it is the critical point of bi-f-energy functional.

(ii) f-Bi-energy functional of smooth map $\phi : (M, g) \to (N, h)$ is defined by

$$E_{2,f}(\phi) = \frac{1}{2}\int_{\Omega} f|\tau(\phi)|^2 dv_g, \tag{10}$$

for every compact domain $\Omega \subset M$. A map ϕ is called f-bi-harmonic map if it is the critical point of f-bi-energy functional.

Remark 3.1. Note that in [20], the terminology "f-bi-harmonic maps" was used as the critical points of the functional (9). Here, since we consider that the functional (9) is parallel to "bi-energy functional" (c.f. Definition 2.1), we shall say "bi-f-energy functional" and "bi-f-harmonic map" instead of "f-bi-energy functional" and "f-bi-harmonic map" in [20].

Now we mainly focus on the first class of generalized harmonic maps called bi-f-harmonic maps. The following proposition for Euler-Lagrange equation of bi-f-harmonic maps originates from [20].

Proposition 3.1. *Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Then in terms of Euler-Lagrange equation, ϕ is a bi-f-harmonic map if and only if its bi-f-tension field*

$$\tau_{f,2}(\phi) = \Delta_f^2 \tau_f(\phi) - f\mathrm{Tr}_g {}^N R(\tau_f(\phi), d\phi)d\phi \tag{11}$$

vanishes, where

$$\Delta_f^2 \tau_f(\phi) = -\mathrm{Tr}_g\big(\nabla^\phi f \nabla^\phi \tau_f(\phi) - f\nabla^\phi_{{}^M\nabla_{\cdot\cdot}} \tau_f(\phi)\big)$$

and $\tau_f(\phi)$ is the f-tension field defined by (4). For an orthonormal frame $\{e_i\}_{i=1}^m$, we have

$$\begin{aligned}\mathrm{Tr}_g\big(\nabla^\phi f \nabla^\phi \tau_f(\phi) - f\nabla^\phi_{\nabla_{\cdot\cdot}} \tau_f(\phi)\big) &= \sum_{i=1}^m \big(\nabla^\phi_{e_i} f \nabla^\phi_{e_i} \tau_f(\phi) - f\nabla^\phi_{{}^M\nabla_{e_i}e_i} \tau_f(\phi)\big)\\ &= f\sum_{i=1}^m \Big(\nabla^\phi_{e_i}\nabla^\phi_{e_i} \tau_f(\phi) - \nabla^\phi_{{}^M\nabla_{e_i}e_i} \tau_f(\phi)\Big) + \nabla^\phi_{\mathrm{grad}\, f} \tau_f(\phi).\end{aligned} \tag{12}$$

Proof. See the proof of Proposition 6 in [20]. □

From (11) and (12), $\tau_{f,2}(\phi)$ can simplified as

$$\tau_{f,2}(\phi) = -f\mathrm{Tr}_g\Big(\nabla^\phi\nabla^\phi \tau_f(\phi) - \nabla^\phi_{{}^M\nabla_{\cdot\cdot}} \tau_f(\phi)) - {}^N R\big(d\phi, \tau_f(\phi)\big)d\phi\Big) - \nabla^\phi_{\mathrm{grad}\, f} \tau_f(\phi). \tag{13}$$

Clearly, it is observed from (13) that bi-f-harmonic map is a much wider generalization of harmonic map, because it is not only a generalization of f-harmonic map (as $f \neq 1$ and $\tau_f(\phi) = 0$) but also a generalization of bi-harmonic map (as $f = 1$). Therefore, it would be interesting to know whether there is any non-trivial or proper bi-f-harmonic map which is neither harmonic map nor f-harmonic map with $f \neq$ const.. In addition, bi-f-harmonic maps have the following property.

Proposition 3.2. *If $\phi : (M, g) \to (N, h)$ is a bi-f-harmonic map ($f \neq 1$) from a compact Riemannian manifold M into a Riemannian manifold N with non-positive curvature satisfying*

$$f \mathrm{Tr}_g \nabla^\phi \nabla^\phi \tau_f(\phi) - \nabla^\phi_{\mathrm{grad} f} \tau_f(\phi) \geq 0.$$

Then ϕ is f-harmonic map.

Proof. Similar to the proof of Theorem 3.1 in [8]. □

Remark 3.2. Proposition 3.2 implies that there is no proper bi-f-harmonic map ϕ from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying $f \mathrm{Tr}_g \nabla^\phi \nabla^\phi \tau_f(\phi) - \nabla^\phi_{\mathrm{grad} f} \tau_f(\phi) \geq 0$.

It is well known that for $m \neq 2$, the harmonicity and f-harmonicity of a map $\phi : (M, g) \to (N, h)$ are related via a conformal change of the domain metric, that is

Proposition 3.3. *[16] A map $\phi : (M^m, g) \to (N^n, h)$, $m \geq 3$ is an f-harmonic map if and only if $\phi : (M^m, f^{\frac{2}{m-2}} g) \to (N^n, h)$ is a harmonic map.*

Therefore, the study of proper bi-f-harmonic maps would be very interesting whether there exist many non-f-harmonic maps. However, since the f-tension field $\tau_f(\phi) = f\tau(\phi) + d\phi(\mathrm{grad} f)$ is much more complicated than the tension field $\tau(\phi) = \mathrm{Tr}_g \nabla d\phi$, it is doomed to be very hard to find an example of proper bi-f-harmonic map. This is probably the reason that a single example was not given in [20]. To the best of our knowledge, up to now, a concrete non-trivial example has not been successfully constructed. Therefore, it is very interesting to find some examples for proper bi-f-harmonic map.

In order to solve the difficulty for calculating the terms $\tau_f(\phi)$ and $\nabla^\phi_{\mathrm{grad} f} \tau_f(\phi)$, in the subsequent sections we are going to discuss bi-f-harmonic maps from two directions.

§4 Properties and examples of bi-f-harmonic maps with dilation

In order to get a better understanding of bi-f-harmonic maps, we characterize some properties on conformal map between equi-dimensional manifolds. Meanwhile, we try to construct some examples of proper bi-f-harmonic maps.

Proposition 4.1. *([20]) Let $\phi : (M^n, g) \to (N^n, h)$ be a conformal map with dilation λ, i.e., $\phi^* h = \lambda^2 g$. Then ϕ is a bi-f-harmonic map if and only if*

$$\begin{aligned} 0 =& (n-2) f^2 d\phi\big(\mathrm{grad}_g (\Delta \log \lambda)\big) - (n-2)^2 f^2 \nabla_{\mathrm{grad}_g \log \lambda} d\phi(\mathrm{grad}_g \log \lambda) \\ &+ 4(n-2) f \nabla_{\mathrm{grad}_g f} d\phi(\mathrm{grad}_g \log \lambda) + (n-2) f d\phi(\mathrm{grad}_g \lambda) \Delta f - f d\phi\big(\mathrm{grad}_g (\Delta f)\big) \\ &+ 2(n-2) f^2 \langle \nabla d\phi, \nabla d \log \lambda \rangle - 2 \langle \nabla d\phi, \nabla df \rangle + (n-2) |\mathrm{grad}_g f|^2 d\phi(\mathrm{grad}_g \log \lambda) \\ &+ 2(n-2) f^2 d\phi\big({}^M\mathrm{Ric}(\mathrm{grad}_g \log \lambda)\big) - \nabla_{\mathrm{grad}_g f} d\phi(\mathrm{grad}_g f) - f d\phi\big({}^M\mathrm{Ric}(\mathrm{grad}_g \lambda)\big), \end{aligned} \tag{14}$$

for a local orthonormal frame $\{e_i\}_{i=1,\ldots,n}$ *on* M*, where* ${}^M\mathrm{Ric}(X)=\sum\limits_{i=1}^{n}{}^MR(X,e_i)e_i$*, and*

$$\langle\nabla d\phi,\nabla d\log\lambda\rangle=\mathrm{Tr}_g\nabla d\phi(*,\nabla_*\mathrm{grad}_g\log\lambda)=\sum_{i=1}^{n}\nabla d\phi(e_i,{}^M\nabla_{e_i}\mathrm{grad}_g\log\lambda)\qquad(15)$$

is defined as same as Lemma 3 in [11].

Proof. By the assumption of ϕ and (4), the f-tension field of ϕ is given by

$$\tau_f(\phi)=(2-n)fd\phi(\mathrm{grad}_g\log\lambda)+d\phi(\mathrm{grad}_gf).\qquad(16)$$

Using (11), a long deduction (for detail, see [20], pp.22-24) gives

$$\begin{aligned}\tau_{f,2}(\phi)&=(n-2)f^2d\phi\big(\mathrm{grad}_g(\Delta\log\lambda)\big)-(n-2)^2f^2\nabla_{\mathrm{grad}_g\log\lambda}d\phi(\mathrm{grad}_g\log\lambda)\\&+4(n-2)f\nabla_{\mathrm{grad}_gf}d\phi(\mathrm{grad}_g\log\lambda)+(n-2)fd\phi(\mathrm{grad}_g\lambda)\Delta f-fd\phi\big(\mathrm{grad}_g(\Delta f)\big)\\&+2(n-2)f^2\langle\nabla d\phi,\nabla d\log\lambda\rangle-2\langle\nabla d\phi,\nabla df\rangle+(n-2)|\mathrm{grad}_gf|_g^2d\phi(\mathrm{grad}_g\log\lambda)\\&+2(n-2)f^2d\phi\big({}^M\mathrm{Ric}(\mathrm{grad}_g\log\lambda)\big)-\nabla_{\mathrm{grad}_gf}d\phi(\mathrm{grad}_gf)-fd\phi\big({}^M\mathrm{Ric}(\mathrm{grad}_g\lambda)\big).\end{aligned}\qquad(17)$$

Thus the necessary and sufficient condition for bi-f-harmonic map of ϕ is clear. □

Observe that if $f=\lambda$, then (16) is of the form

$$\tau_f(\phi)=(3-n)\lambda d\phi(\mathrm{grad}_g\log\lambda).$$

Hence, when $n\geq 4$, we obtain

Corollary 4.1. *([20]) Let* $\phi:(M^n,g)\to(N^n,g)$ $(n\geq 4)$ *be a conformal map with dilation* $\lambda=f$*. Then* ϕ *is a bi-*λ*-harmonic map if and only if*

$$\begin{aligned}&d\phi\big({}^M\mathrm{Ric}(\mathrm{grad}_g\log\lambda)\big)+\mathrm{grad}_g(\log\lambda)-(\Delta\log\lambda)\mathrm{grad}_g\log\lambda\\&+\tfrac{9-n}{2}\mathrm{grad}_g(|\mathrm{grad}_g\lambda|_g^2)+(7-n)|\mathrm{grad}_g\lambda|_g^2\mathrm{grad}_g\log\lambda=0.\end{aligned}\qquad(18)$$

Following the method of Examples A and B in [7, P107], and noting that the function $f:M\times N\to(0,+\infty)$ and $f_\phi:M\to(0,+\infty)$ which differs from our discussing function $f:M\to N$ in this paper, we give two relatively simple examples.

Example 4.2. *Let* $\phi:\mathbb{R}\to(N^2,h)$ *be a conformal map with constant dilation* λ*. If* $f_\phi=f_N\circ\phi=\gamma:\mathbb{R}^2\to\mathbb{R}$ *is a smooth function, where, at* $(x,y)\in\mathbb{R}^2$*,* $f_N:N\to(0,+\infty)$ *defined by* $f_N(y)=f(x,y)$ *for all* $y\in N$*, then* ϕ *is bi-*f*-harmonic if and only if*

$$\begin{cases}\frac{\partial\gamma}{\partial x}\frac{\partial^2\gamma}{\partial x^2}+\frac{\partial\gamma}{\partial y}\frac{\partial^2\gamma}{\partial x\partial y}=0,\\\frac{\partial\gamma}{\partial y}\frac{\partial^2\gamma}{\partial y^2}+\frac{\partial\gamma}{\partial x}\frac{\partial^2\gamma}{\partial x\partial y}=0.\end{cases}\qquad(19)$$

Example 4.3. *Let* $\phi:(M^2,g)\to(N^2,h)$ *be a conformal map with constant dilation* λ*. If* $f_\phi=f_N\circ\phi=\log\lambda$ *is a smooth function, then* ϕ *is bi-*f*-harmonic if and only if* λ *satisfies*

$$\mathrm{grad}_g\big(\big|\mathrm{grad}_g\log\lambda\big|^2\big)=0.\qquad(20)$$

§5 Bi-f-harmonic map equations on singly WPM

In this section, we will use several special maps from or into WPM to study bi-f-harmonicity by following the method in [5, 17, 22]. To our knowledge, some interesting and complicated examples of proper f-bi-harmonic maps could be found.

However, if we consider doubly WPM to discuss bi-f-harmonic maps by using the method of [17] , then we will encounter a dreadful trouble [18]. For example, we consider the inclusion map

$$i_{y_0} : (M, g) \to M \times_{(\mu,\lambda)} N, \quad i_{y_0}(x) = (x, y_0),$$

since by Equations (7) and (8) in [17], we know that

$$\tau(i_{y_0}) = -\frac{m}{2}(0_1, \mathrm{grad}_h \mu^2)\Big|_{i_{y_0}}, \tag{21}$$

$$\tau_f(i_{y_0}) = -\frac{m}{2} f(0_1, \mathrm{grad}_h \mu^2) + (\mathrm{grad}_g f, 0_2). \tag{22}$$

Since $\tau_f(i_{y_0})$ contains $f = f(x)$, in order to calculate $\tau_{f,2}(i_{y_0})$, we must compute the term "$\overline{\nabla}_{(e_j,0)}\overline{\nabla}_{(e_j,0)} f(0_1, \mathrm{grad}_h\mu^2)$" (see (13)). By using (5), we first obtain

$$\overline{\nabla}_{(e_j,0)} f(0_1, \mathrm{grad}_h\mu^2) = -e_j(f)(0_1, \mathrm{grad}_h\mu^2) + \frac{1}{2\mu^2}|\mathrm{grad}_h\mu^2|_h^2\ f(e_j, 0_2).$$

If again we fall into exponential growth terms, to avoid this trouble is that we should restrict the term with f to occur and let $\mu = 1$. This implies that we'd better consider singly WPM but not doubly WPM.

5.1 Bi-f-harmonicity of the inclusion maps

We present some existence results for bi-f-harmonicity of inclusion maps i_{y_0} from M and i_{x_0} from N into a singly WPM. Firstly, we consider the inclusion map $i_{y_0} : (M, g) \to M \times_\lambda N$, $i_{y_0}(x) = (x, y_0)$ for any $y_0 \in N$.

Theorem 5.1. *The inclusion map*

$$i_{y_0} : (M, g) \to M \times_\lambda N$$

is a proper bi-f-harmonic map if and only if λ and f simultaneously satisfy

$$2f\big(\mathrm{Tr}_g\, {}^M\nabla^2 \mathrm{grad}_g f + {}^M Ric(\mathrm{grad}_g f)\big) + \mathrm{grad}_g(|\mathrm{grad}_g f|^2) = 0, \tag{23}$$

where $f : M \to \mathbb{R}$ is a smooth positive and non-constant function.

Proof. Let $\{e_j\}_{j=1}^m$ be an orthonormal frame on M. Then from (13), bi-f-tension field of i_{y_0} is

$$\begin{aligned}\tau_{f,2}(i_{y_0}) &= -f\Big(Tr_g(\nabla^{i_{y_0}})^2\tau_f(i_{y_0}) + Tr_g\bar{R}(di_{y_0}, \tau_f(i_{y_0}))di_{y_0}\Big) - \nabla^{i_{y_0}}_{grad_g f}\tau_f(i_{y_0}) \\ &= -f\sum_{j=1}^m\Big(\big(\nabla^{i_{y_0}}_{e_j}\nabla^{i_{y_0}}_{e_j} - \nabla^{i_{y_0}}_{{}^M\nabla_{e_j}e_j}\big)\tau_f(i_{y_0}) \\ &\quad +\bar{R}\big(\tau_f(i_{y_0}), (e_j, 0_2)\big)(e_j, 0_2)\Big) - \overline{\nabla}_{(\mathrm{grad}_g f, 0)}\tau_f(i_{y_0}).\end{aligned} \tag{24}$$

Since (21) and (22) with $\mu = 1$ give

$$\tau(i_{y_0}) = 0, \quad \tau_f(i_{y_0}) = (\mathrm{grad}_g f, 0_2),$$

using (5) and (7), we have

$$\nabla^{i_{y_0}}_{e_j}\tau_f(i_{y_0}) = \overline{\nabla}_{(e_j,0_2)}(\mathrm{grad}_g f, 0_2) = ({}^M\nabla_{e_j}\mathrm{grad}_g f, 0_2),$$

$$\nabla^{i_{y_0}}_{e_j}\nabla^{i_{y_0}}_{e_j}\tau_f(i_{y_0}) = \overline{\nabla}_{(e_j,0)}({}^M\nabla_{e_j}\mathrm{grad}_g f, 0_2) = ({}^M\nabla_{e_j}{}^M\nabla_{e_j}\mathrm{grad}_g f, 0_2),$$

$$\overline{\nabla}_{(\mathrm{grad}_g f,0_2)}\tau_f(i_{y_0}) = ({}^M\nabla_{\mathrm{grad}_g f}\mathrm{grad}_g f, 0_2) = (\frac{1}{2}\mathrm{grad}_g(|\mathrm{grad}_g f|^2), 0_2),$$

$$\nabla^{i_{y_0}}_{{}^M\nabla_{e_j}e_j}(\tau_f(i_{y_0})) = \overline{\nabla}_{({}^M\nabla_{e_j}e_j,0_2)}(\mathrm{grad}_g f, 0_2) = ({}^M\nabla_{{}^M\nabla_{e_j}e_j}\mathrm{grad}_g f, 0_2),$$

$$\sum_{j=1}^m \overline{R}\big(\tau_f(i_{y_0}),(e_j,0_2)\big)(e_j,0_2) = \big(\sum_{j=1}^m {}^M R(\mathrm{grad}_g f, e_j)e_j, 0_2\big) = \big({}^M Ric(\mathrm{grad}_g f), 0_2\big).$$

Thus we obtain

$$\tau_{f,2}(i_{y_0}) = -f\big(\mathrm{Tr}_g\, {}^M\nabla^2\mathrm{grad}_g f + {}^M Ric(\mathrm{grad}_g f), 0_2\big) - \tfrac{1}{2}\mathrm{grad}_g(|\mathrm{grad}_g f|^2), 0_2), \tag{25}$$

from which, we conclude that i_{y_0} is a proper bi-f-harmonic map if and only if

$$2f\big(\mathrm{Tr}_g\, {}^M\nabla^2\mathrm{grad}_g f + {}^M Ric(\mathrm{grad}_g f)\big) + \mathrm{grad}_g(|\mathrm{grad}_g f|^2) = 0. \tag{26}$$

□

Remark 5.2. If PDE (23) has a solution besides $f = const.$, then we really find a proper bi-f-harmonic map which is usual harmonic ($\tau(i_{y_0}) = 0$) but not f-harmonic ($\tau_f(i_{y_0}) \neq 0$). This is a very strange phenomenon that bi-f-harmonic map is only an extension of f-harmonic map but not harmonic map.

Note that $i_{x_0} : N, h) \to M \times_\lambda N$, $i_{x_0}(y) = (x_0, y)$ for any $x_0 \in M$ is no longer harmonic like i_{y_0}. Now we shall first give bi-f-tension field of i_{x_0}.

Theorem 5.3. *Let $f : (N,h) \to (0,+\infty)$ be a smooth function. The bi-f-tension field of the inclusion map $i_{x_0} : (N,h) \to M \times_\lambda N$ is given by*

$$\begin{aligned}\tau_{f,2}(i_{x_0}) &= \Big(\big(\tfrac{n+2}{2}f(y)\Delta_N f(y) + \tfrac{n+1}{2}|\mathrm{grad}_h f(y)|^2\big)\mathrm{grad}_g\lambda^2 - \tfrac{n^2}{8}f^2(y)\mathrm{grad}_g(|\mathrm{grad}_g\lambda^2|^2), 0_2\Big)\\ &\quad + \Big(0_1, \tfrac{3n+1}{4\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2\mathrm{grad}_h f(y) - f(y){}^N Ric(\mathrm{grad}_g f(y))\Big)\Big|_{i_{x_0}}.\end{aligned} \tag{27}$$

Proof. Let $\{\bar{e}_\alpha\}_{\alpha=1}^n$ be an orthonormal frame on N. Then from (13), bi-f-tension field of i_{x_0} is

$$\begin{aligned}\tau_{f,2}(i_{x_0}) &= -f\sum_{\alpha=1}^n\Big(\big(\nabla^{i_{x_0}}_{\bar{e}_\alpha}\nabla^{i_{x_0}}_{\bar{e}_\alpha} - \nabla^{i_{x_0}}_{{}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha}\big)\tau_f(i_{x_0})\\ &\quad + \bar{R}\big(\tau_f(i_{x_0}),(0_1,\bar{e}_\alpha)\big)(0_1,\bar{e}_\alpha)\Big) - \overline{\nabla}_{(\mathrm{grad}_g f,0)}\tau_f(i_{x_0}).\end{aligned} \tag{28}$$

Since

$$\begin{aligned}\tau(i_{x_0}) &= \mathrm{Tr}_h\nabla di_{x_0} = \sum_{\alpha=1}^n\{\big(\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha) - (0_1, {}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)\big)\\ &= (-\frac{n}{2}\mathrm{grad}_g\lambda^2, 0_2)\circ i_{x_0}, \qquad \text{(by (5))}\end{aligned} \tag{29}$$

then (4) gives

$$\tau_f(i_{x_0}) = -\frac{n}{2}f(y)(\mathrm{grad}_g\lambda^2, 0_2)\circ i_{x_0} + (0_1, \mathrm{grad}_h f(y))\circ i_{x_0}. \tag{30}$$

Thus by (5) we have

$$\begin{aligned}
\nabla^{i_{x_0}}_{\bar e_\alpha}\tau_f(i_{x_0}) &= -\tfrac{n}{2}\overline{\nabla}_{(\bar e_\alpha,0_2)}f(y)(\mathrm{grad}_g\lambda^2,0_2)\circ i_{x_0}+\overline{\nabla}_{(\bar e_\alpha,0_2)}(0_1,\mathrm{grad}_h f(y))\circ i_{x_0}\\
&= -\tfrac{n+1}{2}\bar e_\alpha(f(y))(\mathrm{grad}_g\lambda^2,0_2)-\tfrac{n}{4\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(0_1,\bar e_\alpha)\\
&\quad +(0_1,\ {}^N\nabla_{\bar e_\alpha}grad_h f)\big|_{i_{x_0}},\\
\nabla^{i_{x_0}}_{{}^N\nabla_{\bar e_\alpha}\bar e_\alpha}\tau_f(i_{x_0}) &= -\tfrac{n+1}{2}{}^N\nabla_{\bar e_\alpha}\bar e_\alpha(f(y))(\mathrm{grad}_g\lambda^2,0_2)\big|_{i_{x_0}}\\
&\quad -\tfrac{n}{4\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(0_1,{}^N\nabla_{\bar e_\alpha}\bar e_\alpha)+(0_1,\ {}^N\nabla_{{}^N\nabla_{\bar e_\alpha}\bar e_\alpha}grad_h f)\big|_{i_{x_0}},\\
\nabla^{i_{x_0}}_{\mathrm{grad}_h f(y)}\tau_f(i_{x_0}) &= -\tfrac{n+1}{2}\mathrm{grad}_h f(y)(f(y))(\mathrm{grad}_g\lambda^2,0_2)\\
&\quad -\tfrac{n}{4\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(0_1,grad_h f(y))+(0_1,\ {}^N\nabla_{\mathrm{grad}_h f(y)}\mathrm{grad}_h f(y))\big|_{i_{x_0}}\\
&= -\tfrac{n+1}{2}|grad_h f(y)|^2(\mathrm{grad}_g\lambda^2,0_2)+(0_1,\tfrac12\mathrm{grad}_h(|\mathrm{grad}_h f(y)|^2)\\
&\quad -\tfrac{n}{4\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(0_1,\bar e_\alpha(\mathrm{grad}_h f(y)\bar e_\alpha)\big|_{i_{x_0}},\\
\nabla^{i_{x_0}}_{\bar e_\alpha}\nabla^{i_{x_0}}_{\bar e_\alpha}\tau_f(i_{x_0}) &= \Big(-\tfrac{n+1}{2}\bar e_\alpha\big(\bar e_\alpha(f(y))\big)-\tfrac12{}^N\mathrm{Hess}(f(y))(\bar e_\alpha,\bar e_\alpha)\Big)(\mathrm{grad}_g\lambda^2,0_2)\\
&\quad +\tfrac{n}{8\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(\mathrm{grad}_g\lambda^2,0_2)-\tfrac{2n+1}{4\lambda^2}|\mathrm{grad}_g\lambda^2|^2(0_1,\bar e_\alpha(f(y))\bar e_\alpha)\\
&\quad -\tfrac{n}{4\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(0_1,{}^N\nabla_{\bar e_\alpha}\bar e_\alpha)+(0_1,\ {}^N\nabla_{\bar e_\alpha}grad_h f)\big|_{i_{x_0}},
\end{aligned}\tag{31}$$

$$\begin{aligned}
\mathrm{Tr}_h(\nabla^{i_{x_0}})^2\tau_f(i_{x_0}) &= -\tfrac{n+2}{2}\big(\Delta_N f(y)(\mathrm{grad}_g\lambda^2,0_2\big)+\tfrac{n^2}{8\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(\mathrm{grad}_g\lambda^2,0_2)\\
&\quad -\tfrac{2n+1}{4\lambda^2}|\mathrm{grad}_g\lambda^2|^2\big(0_1,\mathrm{grad}_h f(y)\big)\big|_{i_{x_0}}.
\end{aligned}\tag{32}$$

On the other hand, by (7) we have

$$\begin{aligned}
&\sum_{\alpha=1}^n\bar R\big(\tau_f(i_{x_0}),(0_1,\bar e_\alpha)\big)(0_1,\bar e_\alpha)\\
&=\tfrac{n^2}{4}f(y)\Big({}^M\nabla_{\mathrm{grad}_g\lambda^2}\mathrm{grad}_g\lambda^2-\tfrac{1}{2\lambda^2}\mathrm{grad}_g\lambda^2(\lambda^2)\mathrm{grad}\ \lambda^2,0_2\Big)\\
&\quad +\Big(0_1,\sum_{\alpha=1}^n{}^N R(\mathrm{grad}_g f(y),\bar e_\alpha)\bar e_\alpha\Big)\big|_{i_{x_0}}\\
&=\tfrac{n^2}{8}f(y)\big(\mathrm{grad}_g(|\mathrm{grad}_g\lambda^2|^2),0_2\big)-\tfrac{n^2}{8\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2(\mathrm{grad}_g\lambda^2,0_2)\\
&\quad +\big(0_1,{}^N Ric(\mathrm{grad}_g f(y))\big)\big|_{i_{x_0}}.
\end{aligned}\tag{33}$$

Substituting (31)-(33) into (28), we obtain

$$\begin{aligned}
\tau_{f,2}(i_{y_0}) &= \Big(\big(\tfrac{n+2}{2}f(y)\Delta_N f(y)+\tfrac{n+1}{2}|\mathrm{grad}_h f(y)|^2\big)\mathrm{grad}_g\lambda^2-\tfrac{n^2}{8}f^2(y)\mathrm{grad}_g(|\mathrm{grad}_g\lambda^2|^2),0_2\Big)\\
&\quad +\Big(0_1,\tfrac{3n+1}{4\lambda^2}f(y)|\mathrm{grad}_g\lambda^2|^2\mathrm{grad}_h f(y)-f(y){}^N Ric(\mathrm{grad}_g f(y))\Big)\big|_{i_{x_0}},
\end{aligned}$$

as claimed. □

Corollary 5.4. *Let $f:(N,h)\to(0,+\infty)$ be a smooth function. The inclusion map $i_{x_0}:(N,h)\to M\times_\lambda N$ is bi-f-harmonic map if and only if λ and f satisfy*

$$\begin{cases}\Big(4(n+2)f(y)\Delta_N f(y)+4(n+1)|\mathrm{grad}_h f(y)|^2\Big)\mathrm{grad}_g\lambda^2-n^2f^2(y)\mathrm{grad}_g(|\mathrm{grad}_g\lambda^2|^2\big|_{i_{x_0}}=0,\\ (3n+1)f(y)|\mathrm{grad}_g\lambda^2|^2\mathrm{grad}_h f(y)-4f(y)\lambda^2{}^N Ric(\mathrm{grad}_g f(y))\big|_{i_{x_0}}=0.\end{cases}\tag{34}$$

Corollary 5.5. *If x_0 is a critical point of $\mathrm{grad}_g\lambda^2$ but not a critical point of λ^2 and, ${}^N Ric(\mathrm{grad}_g f(y))=0$ but $\mathrm{grad}_g f(y)\neq 0$, then the inclusion map $i_{x_0}:(N,h)\to M\times_\lambda N$ is a proper bi-f-harmonic map.*

5.2 Bi-f-harmonicity of the projection maps

In this subsection, we attempt two methods, so-called projection maps related to singly WPM to discuss bi-f-harmonic maps. We first give two lemmas.

Lemma 5.1. *For given projection map*

$$\bar{\pi}_1 : M \times_\lambda N \to M, \quad \bar{\pi}_1(x,y) = x,$$

let $f : M \times_\lambda N \to (0,+\infty)$ be smooth function. Then bi-f-bi-tension field of $\bar{\pi}_1$ is

$$\begin{aligned} \tau_{f,2}(\bar{\pi}_1) &= -f\mathrm{Tr}_g({}^M\nabla^2)f \cdot \mathrm{grad}_g \log(f\lambda^n) - f\,{}^M\mathrm{Ric}\big(\mathrm{grad}_g f \ \log(f\lambda^n)\big) \\ &\quad -fn\,{}^M\nabla_{\mathrm{grad}_g \log\lambda} f \cdot \mathrm{grad}_g \log(f\lambda^n) - {}^M\nabla_{\mathrm{grad}_g f} f \cdot \mathrm{grad}_g \log(f\lambda^n). \end{aligned} \tag{35}$$

Proof. Let $\{e_j\}_{j=1}^m$ and $\{\bar{e}_\alpha\}_{\alpha=1}^n$ be local orthonormal frame fields on (M,g) and (N,h), respectively. Then $\{(e_j,0_2),(0_1,\frac{1}{\lambda}\bar{e}_\alpha)\}_{j=1,\ldots,m,\alpha=1,\ldots,n}$ is a local orthonormal frame on $M \times_\lambda N$. By a similar calculation as (29) and (30), we have

$$\begin{aligned} \tau(\bar{\pi}_1) &= Tr_{\bar{g}}\nabla d\bar{\pi}_1 = \sum_{j=1}^m \big({}^M\nabla_{d\bar{\pi}_1(e_j,0_2)} d\bar{\pi}_1(e_j,0_2) - d\bar{\pi}_1(\bar{\nabla}_{(e_j,0_2)}(e_j,0_2))\big) \\ &\quad + \tfrac{1}{\lambda^2}\sum_{\alpha=1}^n \big({}^M\nabla_{d\bar{\pi}_1(0_1,\bar{e}_\alpha)} d\bar{\pi}_1(0_1,\bar{e}_\alpha) - d\bar{\pi}_1(\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha))\big) \\ &= \tfrac{n}{2\lambda^2}\mathrm{grad}_g \lambda^2 \mid \bar{\pi}_1 = n\mathrm{grad}_g \log\lambda \mid \bar{\pi}_1, \end{aligned} \tag{36}$$

$$\tau_f(\bar{\pi}_1) = fn\mathrm{grad}_g \log\lambda \circ \bar{\pi}_1 + d\bar{\pi}_1(\mathrm{grad}_g f, \frac{1}{\lambda^2}\mathrm{grad}_h f) = f\mathrm{grad}_g \log(\lambda^n f) \mid \bar{\pi}_1. \tag{37}$$

Thus we have

$$\nabla^{\bar{\pi}_1}_{(e_j,0_2)} \tau_f(\bar{\pi}_1) = {}^M\nabla_{d\bar{\pi}_1(e_j,0)} \tau_f(\bar{\pi}_1) = {}^M\nabla_{e_j} f \cdot \mathrm{grad}_g \log(f\lambda^n),$$

$$\nabla^{\bar{\pi}_1}_{\bar{\nabla}_{e_j,0_2)}(e_j,0_2)} \tau_f(\bar{\pi}_1) = \nabla^{\bar{\pi}_1}_{({}^M\nabla_{e_j} e_j,0_2)} \tau_f(\bar{\pi}_1) = {}^M\nabla_{{}^M\nabla_{e_j} e_j} f \cdot \mathrm{grad}_g \log(f\lambda^n),$$

$$\nabla^{\bar{\pi}_1}_{(e_j,0_2)} \nabla^{\bar{\pi}_1}_{e_j,0_2)} \tau_f(\bar{\pi}_1) = \nabla^{\bar{\pi}_1}_{(e_j,0_2)} {}^M\nabla_{e_j} f \cdot \mathrm{grad}_g \log(f\lambda^n) = {}^M\nabla_{e_j} {}^M\nabla_{e_j} f \cdot \mathrm{grad}_g \log(f\lambda^n),$$

$$\nabla^{\bar{\pi}_1}_{\mathrm{grad}_{\bar{g}} f} \tau_f(\bar{\pi}_1) = \nabla^{\bar{\pi}_1}_{(\mathrm{grad}_g f,0_2)+(0_1,\frac{1}{\lambda^2}\mathrm{grad}_h f)} \tau_f(\bar{\pi}_1) = {}^M\nabla_{\mathrm{grad}_g f} f \cdot \mathrm{grad}_g \log(f\lambda^n),$$

$$\nabla^{\bar{\pi}_1}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)} \tau_f(\bar{\pi}_1) = {}^M\nabla_{0_1} \tau_f(\bar{\pi}_1) = 0,$$

$$\begin{aligned} \nabla^{\bar{\pi}_1}_{\bar{\nabla}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)}(0_1,\frac{1}{\lambda}\bar{e}_\alpha)} \tau_f(\bar{\pi}_1) &= \tfrac{1}{\lambda^2}\nabla^{\bar{\pi}_1}_{(0_1,{}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)-\frac{1}{2}(\mathrm{grad}_g\lambda^2,0_2)} \tau_f(\bar{\pi}_1) \\ &= -{}^M\nabla_{\mathrm{grad}_g \log\lambda} f \cdot \mathrm{grad}_g \log(f\lambda^n), \end{aligned}$$

$$\nabla^{\bar{\pi}_1}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)} \nabla^{\bar{\pi}_1}_{(0_1,\frac{1}{\lambda}\bar{e}_\alpha)} \tau_f(\bar{\pi}_1) = 0,$$

which imply that

$$Tr_{\bar{g}}(\nabla^{\bar{\pi}_1})^2 \tau_f(\bar{\pi}_1) = Tr_g({}^M\nabla^2)f \cdot \mathrm{grad}_g \log(f\lambda^n) + n\,{}^M\nabla_{\mathrm{grad}_g \log\lambda} f \cdot \mathrm{grad}_g \log(f\lambda^n). \tag{38}$$

On the other hand, since

$$\begin{aligned} Tr_{\bar{g}}{}^M R(d\bar{\pi}_1,\tau_f(\bar{\pi}_1))d\bar{\pi}_1 &= \sum_{j=1}^m {}^M R\big(d\pi_1(e_j,0_2),\tau_f(\bar{\pi}_1)\big)d\pi_1(e_j,0_2) \\ &\quad + \sum_{\alpha=1}^n {}^M R\big(d\pi_1(0_1,\tfrac{1}{\lambda}\bar{e}_\alpha),\tau_f(\bar{\pi}_1)\big)d\pi_1(0_1,\tfrac{1}{\lambda}\bar{e}_\alpha) \\ &= \sum_{j=1}^m {}^M R\big(e_j,\mathrm{grad}_g f \ \log(f\lambda^n)\big)e_j = -{}^M\mathrm{Ric}\big(\mathrm{grad}_g f \ \log(f\lambda^n)\big), \end{aligned} \tag{39}$$

we get

$$\begin{aligned}\tau_{f,2}(\bar\pi_1) &= -f\big(\mathrm{Tr}_{\bar g}(\nabla^{\bar\pi_1})^2\tau_f(\bar\pi_1)+f\mathrm{Tr}_{\bar g}{}^{M}R(d\bar\pi_1,\tau_f(\bar\pi_1))d\bar\pi_1\big)-\nabla^{\bar\pi_1}_{\mathrm{grad}_{\bar g}f}\tau_f(\bar\pi_1)\\ &= -f\mathrm{Tr}_g({}^{M}\nabla^2)f\cdot\mathrm{grad}_g\log(f\lambda^n)-f{}^{M}\mathrm{Ric}\big(\mathrm{grad}_g f\ \ \log(f\lambda^n)\big)\\ &\quad -fn\,{}^{M}\nabla_{\mathrm{grad}_g\log\lambda}f\cdot\mathrm{grad}_g\log(f\lambda^n)-{}^{M}\nabla_{\mathrm{grad}_g f}f\cdot\mathrm{grad}_g\log(f\lambda^n).\end{aligned}$$

Thus we complete the proof. □

For another projection map $\bar\pi_2 : M\times_\lambda N\to N$, note that at this time there are some differences between $\bar\pi_1$ and $\bar\pi_2$, that is, $\tau(\bar\pi_2)$ and $\tau_f(\bar\pi_2)$ respectively satisfy

$$\tau(\bar\pi_2)=0,\qquad \tau_f(\bar\pi_2)=\frac{1}{\lambda^2}\mathrm{grad}_h f\mid\bar\pi_2,$$

we have

Lemma 5.2. *Given a projection map $\bar\pi_2 : (M\times_\lambda N\to(N,h)$, $\bar\pi_2(x,y)=y$, let $f : M\times_\lambda N\to(0,+\infty)$ be smooth function. Then bi-f-tension field of $\bar\pi_2$ is*

$$\tau_{f,2}(\bar\pi_2) = -\tfrac{f}{\lambda^2}Tr_h\,{}^{N}\nabla^2\mathrm{grad}_h f-\tfrac{f}{\lambda^2}{}^{N}\mathrm{Ric}\big(\mathrm{grad}_h f\big)-\tfrac{1}{2\lambda^2}\mathrm{grad}_h f(|\mathrm{grad}_h f|^2). \tag{40}$$

From Lemmas 5.1 and 5.2, we easily conclude that

Corollary 5.6. *(i) If λ and f are non-constant functions and $\mathrm{grad}_g\log(f\lambda^n)\circ\bar\pi_1=0$, then $\bar\pi_1$ is a proper bi-f-harmonic map.*
(ii) Suppose λ and f are non-constant functions. If $\mathrm{grad}_h f$ is non-zero constant and ${}^{N}\mathrm{Ric}\big(\mathrm{grad}_h f\big)=0$, then $\bar\pi_2$ is a proper bi-f-harmonic map.

5.3 Bi-f-harmonicity of the product maps with harmonic factor

Now, we turn to consider a type of product map such as

$$\overline{\Phi}=\varphi_M\times\varphi_N : M\times_\lambda N\to(M\times N, g\oplus h)$$

defined by

$$\varphi_M\times\varphi_N(x,y)=(\varphi_M(x),\varphi_N(y)),$$

where $\varphi_M : M\to M$ and $\varphi_N : N\to N$ are smooth maps. In order to get some interesting results, we usually make some restrictions on φ_M and φ_N. Since in advance we observe that $\tau(\overline{\Phi})$ contains $\tau(\varphi_M)$ and $\tau(\varphi_N)$ (see below (43)), typically, φ_M and φ_N should be chosen as harmonic maps so that $\tau(\overline{\Phi}_M)$ has a simpler form. Thus we have

Proposition 5.1. *Suppose that $\varphi_M : (M,g)\to M$, $\varphi_N : N\to N$ are two harmonic maps. Let the product map $\overline{\Phi}=\varphi_M\times\varphi_N : M\times_\lambda N\to(M\times N, g\oplus h)$ be defined by $\overline{\Phi}(x,y)=(\varphi_M(x),\varphi_N(y))$ and $f : M\times_\lambda N\to\mathbb{R}$ smooth positive function. Then bi-f-tension field of $\overline{\Phi}$ is*

$$\tau_f(\overline{\Phi})=\big(d\varphi_M(\tau_{f,2}(\bar\pi_1)),d\varphi_N(\tau_{f,2}(\bar\pi_2))\big) \tag{41}$$

under some conventions below:

$$\begin{aligned}&d\varphi_L({}^{L}\nabla.):={}^{L}\nabla_{d\varphi_L(\cdot)},\quad L=M,N,\\ &d\varphi_L({}^{L}\nabla^2_{.}):={}^{L}\nabla_{d\varphi_L(\cdot)}{}^{L}\nabla_{d\varphi_L(\cdot)},\\ &d\varphi_L({}^{L}R(\tau_f(\bar\pi_i),\cdot)\cdot={}^{L}R(\tau_f(\bar\pi_i),d\varphi_L(\cdot))d\varphi_L(\cdot),\quad i=1,2.\end{aligned} \tag{42}$$

Proof. Since φ_M and φ_N are harmonic, we have $\tau(\varphi_M)=\tau(\varphi_N)=0$. As by the trick of [17], we have

$$\begin{aligned}\tau(\overline{\Phi}) &= \sum_{j=1}^{m}\big(\nabla_{d\overline{\Phi}(e_j,0_2)}d\overline{\Phi}(e_j,0_2)-d\overline{\Phi}(\bar{\nabla}_{(e_j,0_2)}(e_j,0_2))\big)\\ &\quad+\sum_{\alpha=1}^{n}\frac{1}{\lambda^2}\Big(\nabla_{d\varphi_M\times d\varphi_N(0_1,\bar{e}_\alpha)}d\varphi_M\times d\varphi_N(0_1,\bar{e}_\alpha)-d\varphi_M\times d\varphi_N(\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha))\Big)\\ &=(\tau(\varphi_M),0_2)+n(d\varphi_M(\mathrm{grad}_{\mu^2 g}\log\lambda),0_2)+\frac{1}{\lambda^2}(0_1,\tau(\varphi_N)))\\ &=(n\ d\varphi_M(\mathrm{grad}_g\log\lambda),0_2).\end{aligned}\tag{43}$$

So

$$\begin{aligned}\tau_f(\overline{\Phi}) &= fn(d\varphi_M(\mathrm{grad}_g\log\lambda),0_2)+d\varphi_M\times d\varphi_N(\mathrm{grad}_g f,\tfrac{1}{\lambda^2}\mathrm{grad}_h f)\\ &=(fnd\varphi_M(\mathrm{grad}_g\log\lambda)+d\varphi_M(\mathrm{grad}_g f),\tfrac{1}{\lambda^2}d\varphi_N(\mathrm{grad}_h f))\\ &=\big(f\ d\varphi_M(\mathrm{grad}_g\log(\lambda^n f)),\tfrac{1}{\lambda^2}d\varphi_N(\mathrm{grad}_h f)\big)\end{aligned}\tag{44}$$

Next we process $\tau_{f,2}(\overline{\Phi})$. To this end, we need to tackle two intricate terms by two steps:

Step 1 Consider $Tr_{\bar{g}}(\nabla^{\bar{\Phi}})^2\tau_f(\bar{\Phi})$. Since

$$\begin{aligned}\nabla^{\bar{\Phi}}_{(e_j,0_2)}\tau_f(\bar{\Phi}) &= ({}^M\nabla_{d\varphi_M(e_j)}f\ d\varphi_M(\mathrm{grad}_g\log(\lambda^n f)),\ 0_2),\\ \nabla^{\bar{\Phi}}_{(e_j,0_2)}\nabla^{\bar{\Phi}}_{(e_j,0_2)}\tau_f(\bar{\Phi}) &= ({}^M\nabla_{d\varphi_M(e_j)}{}^M\nabla_{d\varphi_M(e_j)}f\ d\varphi_M(\mathrm{grad}_g\log(\lambda^n f)),\ 0_2),\\ \nabla^{\bar{\Phi}}_{\bar{\nabla}_{(e_j,0_2)}(e_j,0_2)}\tau_f(\bar{\Phi}) &= \nabla^{\bar{\Phi}}_{({}^M\nabla_{e_j},0_2)}\tau_f(\bar{\Phi})=({}^M\nabla_{d\varphi_M({}^M\nabla_{e_j}e_j)}fd\varphi_M(\mathrm{grad}_g\log(\lambda^n f)),0_2),\\ \nabla^{\bar{\Phi}}_{(0_1,\bar{e}_\alpha)}\tau_f(\bar{\Phi}) &= \nabla_{(0_1,d\varphi_N(\bar{e}_\alpha)}\tau_f(\bar{\Phi})=(0_1,\ {}^N\nabla_{d\varphi_N(\bar{e}_\alpha)}\tfrac{1}{\lambda^2}d\varphi_N(\mathrm{grad}_h f),\\ \nabla^{\bar{\Phi}}_{(0_1,\bar{e}_\alpha)}\nabla^{\bar{\Phi}}_{(0_1,\bar{e}_\alpha)}\tau_f(\bar{\Phi}) &= (0_1,\ {}^N\nabla_{d\varphi_N(\bar{e}_\alpha)}{}^N\nabla_{d\varphi_N(\bar{e}_\alpha)}\tfrac{1}{\lambda^2}d\varphi_N(\mathrm{grad}_h f)),\\ \nabla^{\bar{\Phi}}_{\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha)}\tau_f(\bar{\Phi}) &= \nabla^{\bar{\Phi}}_{(0_1,{}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)-\frac{1}{2}(\mathrm{grad}_g\lambda^2,0_2)}\tau_f(\bar{\Phi})\\ &=(0_1,\ {}^N\nabla_{d\varphi_N({}^N\nabla_{\bar{e}_\alpha}\bar{e}_\alpha)}\tfrac{1}{\lambda^2}d\varphi_N(\mathrm{grad}_h f))\\ &\quad-(\lambda\,{}^M\nabla_{d\varphi_M(\mathrm{grad}_g\lambda)}f\ d\varphi_M(\mathrm{grad}_g\log(\lambda^n f)),\ 0_2),\end{aligned}$$

we have

$$\begin{aligned}\mathrm{Tr}_{\bar{g}}(\nabla^{\bar{\Phi}})^2\tau_f(\bar{\Phi}) &= \big(\mathrm{Tr}_g({}^M\nabla_{d\varphi_M})^2\ fd\varphi_M(\mathrm{grad}_g\log(\lambda^n f))\\ &\qquad+n\,{}^M\nabla_{d\varphi_M(\mathrm{grad}_g\log\lambda)}fd\varphi_M(\mathrm{grad}_g\log(\lambda^n f)),\ 0_2\big)\\ &\quad+\big(0_1,\tfrac{1}{\lambda^4}\mathrm{Tr}_h({}^N\nabla_{d\varphi_N)})^2d\varphi_N(\mathrm{grad}_h f)\big).\end{aligned}\tag{45}$$

Step 2 Consider $\mathrm{Tr}_{\bar{g}}R\big(d\overline{\Phi},\tau_f(\bar{\Phi})d\overline{\Phi}$. Since

$$\begin{aligned}&\sum_{j=1}^{m}R\big(d\varphi_M(e_j),0_2),\tau_f(\bar{\Phi})\big)(d\varphi_M(e_j),0_2)\\ &=\Big(\sum_{j=1}^{m}{}^MR(d\varphi_M(e_j),d\varphi_M(f\ \mathrm{grad}_g\log(f\lambda^n)))d\varphi_M(e_j),0_2\Big)\\ &=\big(\mathrm{Tr}_g{}^MR(d\varphi_M,d\varphi_M(f\ \mathrm{grad}_g\log(f\lambda^n)))d\varphi_M,\ 0_2\big),\end{aligned}$$

$$\begin{aligned}&\sum_{\alpha=1}^{n}\frac{1}{\lambda^2}R\big((0_1,d\phi_N(\bar{e}_\alpha)),\tau_f(\bar{\Phi})\big)(0_1,d\phi_N(\bar{e}_\alpha))\\ &=\big(0_1,\tfrac{1}{\lambda^4}\mathrm{Tr}_h{}^NR(d\phi_N,\mathrm{grad}_h f)d\phi_N\big),\end{aligned}$$

we obtain

$$\begin{aligned}\mathrm{Tr}_{\bar{g}}R\big(d\overline{\Phi},\tau_f(\bar{\Phi})d\overline{\Phi} &= -\big(\mathrm{Tr}_g{}^MR(d\varphi_M(f\mathrm{grad}_g\log(f\lambda^n),d\varphi_M))d\varphi_M,\ 0_2\big)\\ &\quad-\big(0_1,\tfrac{1}{\lambda^2}\mathrm{Tr}_h{}^NR(d\phi_N(\mathrm{grad}_h f),d\phi_N)d\phi_N\big).\end{aligned}\tag{46}$$

Finally, note that

$$\begin{aligned}\nabla^{\bar{\Phi}}_{\mathrm{grad}_{\bar{g}}f}\tau_f(\bar{\Phi}) &= ({}^M\nabla_{d\varphi_M(\mathrm{grad}_g f)}f\ d\varphi_M(\mathrm{grad}_g\log(f\lambda^n)),0_2)\\ &\quad+(0_1,\tfrac{1}{\lambda^4}\,{}^M\nabla_{d\varphi_M(\mathrm{grad}_h f)}d\varphi_N(\mathrm{grad}_h f)).\end{aligned}\tag{47}$$

Putting (45)-(47) together, we have

$$\tau_{f,2}(\overline{\Phi}) = (A, B), \tag{48}$$

where A and B denote by

$$\begin{aligned} A = &-f\ \mathrm{Tr}_g({}^M\nabla_{d\varphi_M})^2\ f\ d\varphi_M(\mathrm{grad}_g \log(\lambda^n f)) \\ &-f\ \mathrm{Tr}_g{}^M R(d\varphi_M(f\ \mathrm{grad}_g \log(f\lambda^n)), d\varphi_M\ )d\varphi_M \\ &-\ {}^M\nabla_{d\varphi_M(\mathrm{grad}_g f)} f\ d\varphi_M(\mathrm{grad}_g \log(f\lambda^n) \\ &-nf\ {}^M\nabla_{d\varphi_M(\mathrm{grad}_g \log\lambda)} f\ d\varphi_M(\mathrm{grad}_g \log(f\lambda^n) \end{aligned} \tag{49}$$

and

$$\begin{aligned} B = &-\tfrac{f}{\lambda^2}\mathrm{Tr}_h({}^N\nabla_{d\varphi_N}))^2 d\varphi_N(\ \mathrm{grad}_h f) - \tfrac{f}{\lambda^2}\mathrm{Tr}_h{}^N R(d\phi_N(\mathrm{grad}_h f), d\phi_N,)d\phi_N \\ &-\tfrac{1}{\lambda^2}\ {}^M\nabla_{d\varphi_M(\mathrm{grad}_h f)} d\varphi_N(\ \mathrm{grad}_h f). \end{aligned} \tag{50}$$

Combining with Lemmas 5.1 and 5.2 above, under the notation conventions in (42), we can rewrite (48) as

$$\tau_{f,2}(\overline{\Phi}) = \big(d\varphi_M(\tau_{f,2}(\overline{\pi}_1)), d\varphi_N(\tau_{f,2}(\overline{\pi}_2))\big),$$

as claimed. □

When $\varphi_M = Id_M$ or $\varphi_N = Id_N$, we easily obtain the following propositions.

Proposition 5.2. *(i) The product map $\overline{\Phi}$ with $\varphi_M = Id_M$ is a bi-f-harmonic map if and only if the projection map $\overline{\pi}_1$ is bi-f-harmonic and $d\varphi_N(\tau_{f,2}(\overline{\pi}_2)) = 0$.*
(ii) $\overline{\Phi}$ with $\varphi_N = Id_N$ is a bi-f-harmonic map if and only if so is the projection map $\overline{\pi}_2$ and $d\varphi_N(\tau_{f,2}(\overline{\pi}_1)) = 0$.
(iii) $\overline{\Phi}$ with $\varphi_M = Id_M$ and $\varphi_N = Id_N$ is a bi-f-harmonic map if and only if so are both $\overline{\pi}_1$ and $\overline{\pi}_2$.

Remark 5.7. In the above propositions, neither $d\varphi_N(\tau_{f,2}(\overline{\pi}_2)) = 0$ nor $d\varphi_M(\tau_{f,2}(\overline{\pi}_1)) = 0$ implies $\tau_{f,2}(\overline{\pi}_2) \in \mathrm{Ker}(d\varphi_N)$ or $\tau_{f,2}(\overline{\pi}_1)) \in \mathrm{Ker}(d\varphi_M)$. Because $d\varphi_N(\tau_{f,2}(\overline{\pi}_2))$ and $d\varphi_M(\tau_{f,2}(\overline{\pi}_1))$ don't have the usual sense for differential map but only a kind of special notation, see already stipulations (42).

If we interchange the roles between the domain and codomain of $\overline{\Phi}$, we will obtain another type of product map such as

$$\widehat{\Psi} = \widehat{\varphi_M \times \varphi_N} : (M \times N, g \oplus h) \to M \times_\lambda N$$

defined by $\widehat{\Psi}(x, y) = (\varphi_M(x), \varphi_N(y))$.

In this case, although we finally expect that the operators $\bar{\nabla}$ and $\bar{R}$ will be fully applied more than previous cases, unfortunately $\tau_{f,2}(\widehat{\Psi})$ is hard to work out. More precisely, we hardly find a simple form like the previous cases. For instance, let $\varphi_M = Id_M$, then we can immediately get

$$\tau(\widehat{\Psi}) = -e(\varphi_N)(\mathrm{grad}_g\lambda^2, 0_2),$$

and

$$\tau_f(\widehat{\Psi}) = \big(-\ e(\varphi_N)f\ \mathrm{grad}_g\lambda^2 + \mathrm{grad}_g f, d\varphi_N(\mathrm{grad}_h f\big),$$

where $e(\varphi_N)$ is the energy density of φ_N, $e(Id_N) = \frac{n}{2}$. Next, we have to tackle the terms such as

$$\sum_{j=1}^{m}\big(\bar\nabla_{(e_j,0_2)}\bar\nabla_{(e_j,0_2)}-\bar\nabla_{{}^M\nabla_{e_j}e_j}\big)\tau_f(\widehat\Psi),$$

$$\sum_{\alpha=1}^{n}\big(\bar\nabla_{(0_1,d\varphi_N(\bar e_\alpha)}\bar\nabla_{(0_1,d\varphi_N(\bar e_\alpha)}-\bar\nabla_{{}^N\nabla_{d\varphi_N(\bar e_\alpha)}d\varphi_N(\bar e_\alpha)}\big)\tau_f(\widehat\Psi),$$

$$\sum_{j=1}^{m}\bar R\big((0_1,e_j),\tau_f(\widehat\Psi)\big)(0_1,e_j),$$

$$\sum_{\alpha=1}^{n}\bar R\big((0_1,d\varphi_N(\bar e_\alpha)),\tau_f(\widehat\Psi)\big)(0_1,d\varphi_N(\bar e_\alpha).$$

They produce much more sub-terms which are hard to integrate. Based on the disadvantage, we omit investigating the product map $\widehat\Psi$.

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