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Optimal dividend and capital injection problem with a random time horizon and a ruin penalty in the dual model

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Abstract. In the dual risk model, we consider the optimal dividend and capital injection problem, which involves a random time horizon and a ruin penalty. Both fixed and proportional costs from the transactions of capital injection are considered. The objective is to maximize the total value of the expected discounted dividends, and the penalized discounted both capital injections and ruin penalty during the horizon, which is described by the minimum of the time of ruin and an exponential random variable. The explicit solutions for optimal strategy and value function are obtained, when the income jumps follow a hyper-exponential distribution. Besides, some numerical examples are presented to illustrate our results.

§1 Introduction

In this paper, the company surplus is described as a dual jump-diffusion model

$$X_t = x - ct + S_t + \sigma B_t, \quad t \geq 0, \tag{1}$$

where $x \geq 0$ is the initial surplus, the constant $c > 0$ is the rate of expense, and the process $\{B_t\}$ is a standard Brownian motion and $\sigma > 0$. The income process $\{S_t = \sum_{i=1}^{N_t} Y_i\}$ is a compound Poisson process with parameter λ, and is independent of $\{B_t\}$. Let $p(y)(y \geq 0)$ and ν denote the probability density and the expectation of Y, respectively. The drift of the surplus per unit of time is denoted by $\mu = \lambda\nu - c$. The dual model seems appropriate for companies with continuous expenses and stochastic gains. For companies such as pharmaceutical or petroleum companies, the jump can be interpreted as the net present value of future income from an invention or discovery.

Dividend payment and capital injection are two important approaches to control the asset process. In the dual model, Avanzi et al.(2007), and Avanzi and Gerber (2008) studied how to

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calculate the expectation of the discounted dividends until ruin. When dividends are maximized, ruin is usually certain. In some cases, it may be profitable to rescue the company by injecting some capitals. This idea goes back to Borch (1974). Yao et al. (2010, 2011) considered the optimal dividend and capital injection problem in the dual model. Avanzi et al. (2011) discussed the same problem in the dual model with diffusion. For the general dual model (spectrally positive Lévy model), the optimal dividend and capital injection problems were studied in Bayraktar et al. (2013, 2014), Yin et al. (2014) and Zhao et al. (2014). In addition, transaction cost, which usually includes two part - proportional cost and fixed cost -, is also an important factor in business activities. In Avanzi et al. (2011), proportional costs were involved into the optimal dividend problem. In Yao et al (2010, 2011) and Zhao et al. (2014), both proportional and fixed costs on capital injection were considered.

Similar to Albrecher and Thonhauser (2012), and Zhao et al. (2014), we assume that the surplus process is killed by an exponential random time with a parameter γ in this paper. We consider a ruin penalty $P \in \mathbb{R}$ if ruin occurs before stopping the business, as in Thonhauser and Albrecher (2007), Liang and Young (2012). In addition, both proportional and fixed transaction costs on capital injection are considered. The the optimal strategy and the value function are obtained by stochastic control theory. when the compound Poisson positive jumps follow a hyper-exponential distribution, we derive the closed form solution to value function. When the random time horizon tends to infinity (or γ tends to zero) and the penalty for ruin tends to zero, the optimal problem is simplified into the usual optimal problem such as that in Yao et al. (2011), Peng et al. (2012) and so on. Furthermore if the fixed transaction costs tend to zero, the optimal problem becomes that in Avanzi et al. (2011). Although Zhao et al. (2014) discussed the more general dual model, we apply the more simple method by the merit of compound Poisson hyper-exponential jumps to solve the optimal problem, i.e., we use the roots of characteristic equation and the properties of polynomial instead of the complicated fluctuation theory of Lévy process. In addition, the ruin penalty was not considered in Zhao et al. (2014).

§2 General optimal control problem

2.1 Definitions

We assume a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ satisfies the usual conditions, such that $\{X_t\}$ in (1) is adapted to the filtration $\{\mathcal{F}_t\}$. Let $\{L_t\}$ denote the dividend process. The capital injection process $\{G_t = \sum_{n=1}^{\infty} I_{\{\tau_n \le t\}} \xi_n\}$ is described by a sequence of increasing stopping times $\{\tau_n, n = 1, 2, \cdots\}$ and a sequence of random variables $\{\xi_n, n = 1, 2, \cdots\}$, which are associated with the timings and the amounts of capital injection. With a control strategy $\pi = (L^\pi; G^\pi) = (L^\pi; \tau_1^\pi, \cdots, \tau_n^\pi \cdots; \xi_1^\pi, \cdots, \xi_n^\pi \cdots)$, the dynamics of the controlled surplus process $X^\pi = \{X_t^\pi\}$ are given by

$$X_t^\pi = x - ct + S_t + \sigma B_t - L_t^\pi + \sum_{\tau_n \le t} \xi_n^\pi, \quad t \ge 0. \tag{2}$$

Definition 2.1. *A strategy π is said to be admissible if*

(i) $\{L_t^\pi\}$ is an increasing, $\{\mathcal{F}_t\}$-adapted cádlág process, and $\Delta L_t^\pi \le X_{t-}^\pi$;

(ii) τ_n^π is a stopping time with respect to $\{\mathcal{F}_t\}$ and $0 \le \tau_1^\pi < \tau_2^\pi < \cdots < \tau_n^\pi < \cdots$ a.s.;

(iii) ξ_n^π is nonnegative and measurable with respect to $\mathcal{F}_{\tau_n^\pi}$, $n = 1, 2, \cdots$;

(iv) $P(\lim_{n\to\infty} \tau_n^\pi \le T) = 0$, $\forall T \ge 0$.

Π denotes the set of admissible control strategies. The time ruin for X^π is defined by $T^\pi = \inf\{t \ge 0 : X_t^\pi \le 0\}$. When capital injection occurs, the fixed and proportional transaction costs are considered. We assume that $(\phi-1)\xi$ and $K > 0$ are respectively proportional costs and fixed costs to meet the capital injection of amount ξ, where $\phi > 1$. To reflect the damage done by ruin, we incorporate a monetary penalty P into the performance function if ruin occurs. When $P > 0$, we think it as a penalty for ruin, the monetary cost when an insurer ruins. If $P = 0$, it is in the case of no penalty. If $P < 0$, we think it as the salvage value of the insurer; for example, an insurer's brand name or agency network might be of value to a potential buyer of the insurer. In addition, we assume that the time horizon is an exponentially distributed random variable $\zeta \sim Exp(\gamma)$ independent of $\{S_t\}$ and $\{B_t\}$. In other words, we apply exponential killing at a constant rate γ to the risk reserve. Furthermore, we assume that the remaining at the stopping time is paid out as dividends if $\zeta > T^\pi$. Under these assumptions, the performance function associated with a strategy $\pi \in \Pi$ is defined by

$$V(x;\pi) = E^x\left[\int_0^{\zeta\wedge T^\pi} e^{-\delta s} dL_s^\pi + e^{-\delta\zeta} X_\zeta^\pi \mathbf{1}_{\{\zeta < T^\pi\}} - Pe^{-\delta T^\pi}\mathbf{1}_{\{\zeta \ge T^\pi\}} - \sum_{\tau_n^\pi \le \zeta\wedge T^\pi} e^{-\delta\tau_n^\pi}(K + \phi\xi_n^\pi)\right],$$

where E^x is the conditional expectation given the initial surplus x, and the force of interest $\delta > 0$ reflects the time preference of investors. By the similar arguments in Albrecher and Thonhauser (2012), we can rewrite $V(x;\pi)$ as

$$V(x;\pi) = E^x\left[\int_0^{T^\pi} e^{-(\delta+\gamma)s} dL_s^\pi + \int_0^{T^\pi} \gamma e^{-(\delta+\gamma)s} X_s^\pi ds - Pe^{-(\delta+\gamma)T^\pi} - \sum_{\tau_n^\pi \le T^\pi} e^{-(\delta+\gamma)\tau_n^\pi}(K + \phi\xi_n^\pi)\right]. \tag{3}$$

Our objective is to find the value function defined by

$$V(x) = \sup_{\pi\in\Pi} V(x;\pi), \tag{4}$$

and the optimal strategy $\pi^* \in \Pi$ such that $V(x) = V(x;\pi^*)$.

2.2 Property of the value function

Proposition 2.1. *The value function $V(x)$ defined by (4) is increasing for $x \ge 0$ with*

$$x - y \le V(x) - V(y) \le \phi(x-y) + K, \quad 0 \le y \le x, \tag{5}$$

and satisfies the following conditions

$$x - P \le V(x) \le x + \frac{\lambda\nu}{\delta+\gamma} + |P|\mathbf{1}_{\{P\le 0\}}, \quad x \ge 0. \tag{6}$$

Proof. By the optimality of the value function, it is easy to prove the inequalities in (5). To prove the inequalities in (6), we construct the strategy $\bar{\pi}$ as follows: the total initial reserve is paid out immediately as dividends, and then the ruin occurs. Given the initial reserve x, the associated performance function is $V(x;\bar{\pi}) = x - P$. Hence, the first inequity of (6) holds.

For the uncontrolled surplus process $\{X_t\}$ in (1), we have

$$E^x\left[\int_0^t e^{-(\delta+\gamma)s} dX_s\right] \le E^x\left[\int_0^t e^{-(\delta+\gamma)s} dS_s\right] \le \frac{\lambda\nu}{\delta+\gamma}.$$

By Itô formula, we get, for $\pi \in \Pi$,

$$e^{-(\delta+\gamma)(t\wedge T^\pi)} X^\pi_{t\wedge T^\pi} = x - (\delta+\gamma)\int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s} X^\pi_s ds + \int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s} dX^\pi_s.$$

Taking expectation on both sides in the above equation yields

$$E^x\left[-\int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s} dX^\pi_s + \int_0^{t\wedge T^\pi} \gamma e^{-(\delta+\gamma)s} X^\pi_s ds\right] \le x.$$

Hence, by Fatou Lemma,

$$\begin{aligned} V(x;\pi) \le \limsup_{t\to\infty} E^x\Bigg[&\int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s} dX_s - \int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s} dX^\pi_s + \int_0^{t\wedge T^\pi} \gamma e^{-(\delta+\gamma)s} X^\pi_s ds \\ &- P e^{-(\delta+\gamma)(t\wedge T^\pi)}\Bigg] \le x + \frac{\lambda\nu}{\delta+\gamma} + |P|\mathbf{1}_{\{P\le 0\}}. \end{aligned}$$

The second inequality of (6) is obtained by (4). The proof is completed. □

2.3 Comparison Theorem

By the standard stochastic optimal control theory (see, e.g., Fleming and Soner (2006)), if the value function $V(x)$ is sufficiently smooth, it satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\max\{\mathcal{L}v(x), 1 - v(x), \mathcal{M}v(x) - v(x)\} = 0, \tag{7}$$

where the operators $\mathcal{L}$ and $\mathcal{M}$ are given by, respectively,

$$\begin{aligned} \mathcal{L}v(x) &= \frac{\sigma^2}{2} v''(x) - cv'(x) - (\lambda+\delta+\gamma)v(x) + \lambda\int_0^\infty v(x+y)p(y)dy + \gamma x, \\ \mathcal{M}v(x) &= \sup_{y\ge 0}\{v(x+y) - K - \phi y\}. \end{aligned}$$

Theorem 2.1. (Comparison Theorem) *Let $v(x) \in \mathcal{C}^2((0,\infty))$ be an increasing and concave function satisfying*

$$\max\{\mathcal{L}v(x), 1 - v(x), \mathcal{M}v(x) - v(x)\} \le 0, \quad x > 0, \tag{8}$$

with $v(0) \ge -P$. Then we have $v(x) \ge V(x)$.

Proof. For a policy $\pi \in \Pi$, we define $\Lambda_t = \{s \le t : L^\pi_{s-} \ne L^\pi_s\}$ and $\Lambda'_t = \{s \le t : G^\pi_{s-} \ne G^\pi_s\} =$

$\{\tau_i^\pi : \tau_i^\pi \le t, i = 1, 2, \cdots\}$. By Itô formula for semimartingales, we have

$$\begin{aligned} & e^{-(\delta+\gamma)(t\wedge T^\pi)} v(X^\pi_{t\wedge T^\pi}) \\ =& v(x) + \int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s}(\mathcal{L}v(X_s^\pi) - \gamma X_s^\pi) ds + \sum_{\tau_n^\pi \le t\wedge T^\pi} e^{-(\delta+\gamma)\tau_n^\pi}[v(X_{s-}^\pi + \xi_n^\pi) - v(X_{s-}^\pi)] \\ & - \int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s} v'(X_{s-}^\pi) dL_s^\pi + \sum_{s\in\Lambda_{t\wedge T^\pi}} e^{-(\delta+\gamma)s}[v(X_s^\pi) - v(X_{s-}^\pi) - v'(X_{s-}^\pi)\Delta X_s^\pi] + D_{t\wedge T^\pi}, \end{aligned} \tag{9}$$

where $\{D_{t\wedge T^\pi}\}$ is a martingale with respect to $\{\mathcal{F}_t\}$ and $E^x[D_{t\wedge T^\pi}] = 0$. Since the function v is increasing and concave, by the mean value theorem, we have $v(y) - v(x) - (y-x)v'(x) \le 0$ for $x, y \ge 0$. Noting that $\mathcal{L}v(x) \le 0$, $v'(x) \ge 1$, $v(x+y) - v(x) \le \phi y + K$ and $v(0) \ge -P$, we obtain, by (9)

$$\begin{aligned} -Pe^{-(\delta+\gamma)(t\wedge T^\pi)} \le & v(x) - \int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s}\gamma X_s^\pi ds - \int_0^{t\wedge T^\pi} e^{-(\delta+\gamma)s} dL_s^\pi \\ & + \sum_{\tau_n^\pi \le t\wedge T^\pi} e^{-(\delta+\gamma)\tau_n^\pi}(\phi\xi_n^\pi + K) + D_{t\wedge T^\pi}. \end{aligned}$$

Taking expectations and letting $t \to \infty$ on both sides of the above inequality, for any $\pi \in \Pi$, we have $v(x) \ge V(x;\pi)$, and so $v(x) \ge V(x)$. □

Theorem 2.2. *If $P \le -\frac{\mu}{\delta+\gamma}$, then $V(x) = x - P$ for $x \ge 0$ and the optimal strategy is immediately to pay all the surplus as dividends and to declare ruin, and then to claim the salvage value $-P$.*

Proof. Let $v(x) = x - P$ for $x \ge 0$, and then v is increasing, concave and satisfies $\max\{1 - v(x), \mathcal{M}v(x) - v(x)\} \le 0$ with $v(0) \ge -P$. By for all $x \ge 0$

$$\mathcal{L}v(x) = -c + \lambda\nu - \delta x + P(\delta+\gamma) = \mu - \delta x + P(\delta+\gamma) \le 0, \tag{10}$$

we have $V(x) \le v(x)$ by Theorem 2.1. Considering the strategy to pay all the surplus as dividends and to declare ruin, and then to claim the salvage value $-P$, we obtain $V(x) \ge v(x)$ for $x \ge 0$. The result is proved. □

Remark 2.1. *The above theorem shows that the business is not profitable if $P \le -\frac{\mu}{\delta+\gamma}$. It had better pay all surplus as dividends immediately, and stop the business, and then claim the salvage. This strategy is called* **take-the-money-and-run**. *Conversely, if $v(x) = x - P$ is the value function defined in (4), it satisfies the HJB equation (7). The inequity in (10) shows that $P \le -\frac{\mu}{\delta+\gamma}$. Therefore,*

$$V(x) = x - P \text{ if and only if } P \le -\frac{\mu}{\delta+\gamma}.$$

Particularly, when $P = 0$, $V(x) = x$ if and only if $\mu \le 0$.

In order to ensure that the optimal strategy is non-trivial, we assume $P > -\frac{\mu}{\delta+\gamma}$ in the following sections.

§3 Two suboptimal control problems

3.1 Optimal problem without capital injection

We consider the optimal problem without capital injection. Let $\Pi_p = \{\pi_p : \pi_p = (L^{\pi_p}; 0) \in \Pi\} \subset \Pi$ denote the set of admissible strategies for this suboptimal problem. The associated value function is defined by

$$\begin{aligned} V_p(x) &= \sup_{\pi_p \in \Pi_p} V(x; \pi_p) \\ &= \sup_{\pi_p \in \Pi_p} E^x \left[\int_0^{T^{\pi_p}} e^{-(\delta+\gamma)s} dL_s^{\pi_p} + \int_0^{T^{\pi_p}} \gamma e^{-(\delta+\gamma)s} X_s^{\pi_p} ds - P e^{-(\delta+\gamma)T^{\pi_p}} \right]. \end{aligned} \tag{11}$$

We will try to find the value function $V_p(x)$ and the corresponding optimal strategy $\pi_p^* \in \Pi_p$ such that $V_p(x) = V(x; \pi_p^*)$. If the value function $V_p(x)$ is sufficiently smooth, it satisfies the following HJB equation, for $x > 0$,

$$\max\{\mathcal{L}v(x), 1 - v(x)\} = 0, \quad \text{with} \quad v(0) = -P. \tag{12}$$

Theorem 3.1. *If an increasing and concave function $g(x) \in \mathcal{C}^2((0,\infty))$ satisfies the HJB equation (12), we have*
(i) For any strategy $\pi_p \in \Pi_p$, $g(x) \geq V_p(x)$;
(ii) Moreover, if there exists a point $x_p^* > 0$ such that

$$\mathcal{L}g(x) = 0, x \in (0, x_p^*], \qquad g(x) = x - x_p^* + g(x_p^*), x \in (x_p^*, \infty), \tag{13}$$

then $g(x) = V_p(x) = V(x; \pi_p^*)$, where $\pi_p^* = (L^{\pi_p^*}, 0) \in \Pi_p$ is the optimal strategy such that

$$\begin{aligned} X_t^{\pi_p^*} &= x - ct + \sigma B_t + S_t - L_t^{\pi_p^*}, \\ dL_t^{\pi_p^*} &= (X_{t-}^{\pi_p^*} - x_p^*)\mathbf{1}_{\{X_{t-}^{\pi_p^*} > x_p^*\}} + dM_t^{\pi_p^*}, \\ M_t^{\pi_p^*} &= \int_0^t \mathbf{1}_{\{X_s^{\pi_p^*} = x_p^*\}} dM_s^{\pi_p^*}, \end{aligned} \tag{14}$$

here $(X_{t-}^{\pi_p^*} - x_p^*)\mathbf{1}_{\{X_{t-}^{\pi_p^*} > x_p^*\}}$ represents the dividend distributed at time t if the surplus process jumps above the barrier x_p^*, and $M_t^{\pi_p^*}$ is the local time of the process $\{X_t^{\pi_p^*}\}$ at the barrier x_p^* representing dividends due to oscillations of the Brownian Motion when the surplus is at the barrier.

Proof. The proof of (i) is similar to that of Theorem 2.1. Now we only give the proof of (ii). Under the strategy π_p^* defined by (14), (9) can be written as

$$\begin{aligned} e^{-(\delta+\gamma)(t\wedge T^{\pi_p^*})} g(X_{t\wedge T^{\pi_p^*}}^{\pi_p^*}) &= g(x) + \int_0^{t\wedge T^{\pi_p^*}} e^{-(\delta+\gamma)s} (\mathcal{L}g(X_s^{\pi_p^*}) - \gamma X_s^{\pi_p^*}) \mathbf{1}_{\{0 \leq X_s^{\pi_p^*} \leq x_p^*\}} ds \\ &+ \sum_{s \in \Lambda_{t\wedge T^{\pi_p^*}}} e^{-(\delta+\gamma)s} [g(X_s^{\pi_p^*}) - g(X_{s-}^{\pi_p^*}) - g'(X_{s-}^{\pi_p^*}) \Delta X_s^{\pi_p^*}] \mathbf{1}_{\{X_{s-}^{\pi_p^*} > x_p^*, X_s^{\pi_p^*} = x_p^*\}} \\ &- \int_0^{t\wedge T^{\pi_p^*}} e^{-(\delta+\gamma)s} g'(X_{s-}^{\pi_p^*}) \mathbf{1}_{\{X_{s-}^{\pi_p^*} \geq x_p^*\}} dL_s^{\pi_p^*} + D_{t\wedge T^{\pi_p^*}}. \end{aligned}$$

Due to (13), the above equation becomes

$$e^{-(\delta+\gamma)(t\wedge T^{\pi_p^*})}g(X^{\pi_p^*}_{t\wedge T^{\pi_p^*}}) = g(x) - \int_0^{t\wedge T^{\pi_p^*}} e^{-(\delta+\gamma)s}\gamma X_s^{\pi_p^*}ds - \int_0^{t\wedge T^{\pi_p^*}} e^{-(\delta+\gamma)s}dL_s^{\pi_p^*} + D_{t\wedge T^{\pi_p^*}}. \tag{15}$$

Taking expectations and letting $t \to \infty$ on both sides of the above equation, and by $g(0) = -P$ and $E^x[D_{t\wedge T^{\pi_p^*}}] = 0$, we obtain $g(x) = V(x; \pi_p^*) = V_p(x)$. □

Corollary 3.1. *If an increasing and concave function $g(x) \in \mathcal{C}^2((0,\infty))$ satisfies (13) for some $x_p^* > 0$ with the boundary condition $g(0) = -P$, we have $g(x) = V_p(x) = V(x; \pi_p^*)$ where π_p^* is given by (14).*

Proof. Since $g(x) \in \mathcal{C}^2((0,\infty))$ and (13) holds, we have $g'(x_p^*) = 1$ and $g''(x_p^*) = 0$. Due to the concavity of g, we obtain $g'(x) \geq 1$ for $x \in (0, x_p^*]$. Letting $x = x_p^*$ in (13) yields that

$$\mathcal{L}g(x_p^*) = \mu - (\delta+\gamma)g(x_p^*) + \gamma x_p^* = 0. \tag{16}$$

Hence, for $x > x_p^*$, $\mathcal{L}g(x) = \mu - (\delta+\gamma)g(x_p^*) + rx_p^* - \delta(x - x_p^*) = -\delta(x - x_p^*) < 0$. The results are obtained by (ii) of Theorem 3.1. □

3.2 Optimal problem with forced capital injections to prevent ruin

In this subsection, we require that the company survives forever by forced capital injections. Let Π_q denote the set of admissible strategies of this suboptimal problem, i.e.,

$$\Pi_q = \{\pi_q = (L^{\pi_q}; G^{\pi_q}) : \pi_q \in \Pi \text{ such that } X_t^{\pi_q} > 0 \text{ for all } t \geq 0\}.$$

The value function $V_q(x)$ is defined by

$$\begin{aligned} V_q(x) &= \sup_{\pi_q\in\Pi_q} V(x;\pi_q) \\ &= \sup_{\pi_q\in\Pi_q} E^x\left[\int_0^\infty e^{-(\delta+\gamma)s}dL_s^{\pi_q} + \int_0^\infty \gamma e^{-(\delta+\gamma)s}X_s^{\pi_q}ds - \sum_{n=1}^\infty e^{-(\delta+\gamma)\tau_n^{\pi_q}}(K+\phi\xi_n^{\pi_q})\right]. \end{aligned} \tag{17}$$

We will search for the optimal strategy $\pi_q^* \in \Pi_q$ and the value function $V_q(x) = V(x;\pi_q^*)$. If the value function $V_q(x)$ is sufficiently smooth, it satisfies the following HJB equation, for $x > 0$,

$$\max\{\mathcal{L}v(x), 1 - v(x), \mathcal{M}v(x) - v(x)\} = 0, \quad \text{with} \quad \mathcal{M}v(0) \leq v(0). \tag{18}$$

Theorem 3.2. *If an increasing and concave function $h(x) \in \mathcal{C}^2((0,\infty))$ satisfies the HJB equation (18), we have*

(i) For any strategy $\pi_q \in \Pi_q$, $h(x) \geq V_q(x)$;

(ii) Moreover, if there exists two points $x_q^* > \eta > 0$ such that

$$\mathcal{L}h(x) = 0, x \in (0, x_q^*], \quad h(x) = x - x_q^* + h(x_q^*), x \in (x_q^*, \infty), \tag{19}$$

and

$$h(0) = h(\eta) - \phi\eta - K, \quad h'(\eta) = \phi, \tag{20}$$

then $h(x) = V_q(x)$, *where* $\pi_q^* = (L^{\pi_q^*}, G^{\pi_q^*}) \in \Pi_q$ *is the associated optimal strategy such that*

$$\begin{aligned} &X_t^{\pi_q^*} = x - ct + \sigma B_t + S_t - L_t^{\pi_q^*} + G_t^{\pi_q^*}, \\ &dL_t^{\pi_q^*} = (X_{t-}^{\pi_q^*} - x_q^*)\mathbf{1}_{\{X_{t-}^{\pi_q^*} > x_q^*\}} + dM_t^{\pi_q^*}, \quad M_t^{\pi_q^*} = \int_0^t \mathbf{1}_{\{X_s^{\pi_q^*} = x_q^*\}} dM_s^{\pi_q^*}, \\ &\tau_1^{\pi_q^*} = \inf\{t \geq 0 : X_{t-}^{\pi_q^*} = 0\}, \quad \tau_n^{\pi_q^*} = \inf\{t > \tau_{n-1}^{\pi_q^*}; X_{t-}^{\pi_q^*} = 0\}, n = 2, 3, \cdots, \\ &\xi_n^{\pi_q^*} = \eta, n = 1, 2, \cdots. \end{aligned} \tag{21}$$

Proof. The proof of (i) is similar to that of Theorem 2.1. Now we only give the proof of (ii). For the strategy π_q^* given in (21), we have

$$\begin{aligned} \sum_{\tau_n^{\pi_q^*} \leq t} e^{-(\delta+\gamma)\tau_n^{\pi_q^*}} [h(X_{s-}^{\pi_q^*} + \xi_n^{\pi_q^*}) - h(X_{s-}^{\pi_q^*})] &= \sum_{\tau_n^{\pi_q^*} \leq t} e^{-(\delta+\gamma)\tau_n^{\pi_q^*}} [h(\xi_n^{\pi_q^*}) - h(0)] \\ &= \sum_{\tau_n^{\pi_q^*} \leq t} e^{-(\delta+\gamma)\tau_n^{\pi_q^*}} (\phi\xi_n^{\pi_q^*} + K), \end{aligned}$$

the second equality is due to (20). Similar to (15), equation (9) can be written as

$$\begin{aligned} e^{-(\delta+\gamma)t} h(X_t^{\pi_q^*}) =& h(x) - \int_0^t e^{-(\delta+\gamma)s} \gamma X_s^{\pi_q^*} ds - \int_0^t e^{-(\delta+\gamma)s} dL_s^{\pi_q^*} \\ &+ \sum_{\tau_n^{\pi_q^*} \leq t} e^{-(\delta+\gamma)\tau_n^{\pi_q^*}} (\phi\xi_n^{\pi_q^*} + K) + D_t. \end{aligned}$$

We have $\lim_{t\to\infty} e^{-(\delta+\gamma)t} h(X_t^{\pi_q^*}) = 0$ by $h(X_t^{\pi_q^*}) \leq h(x_q^*)$. Taking expectation and letting $t \to \infty$ on both sides of the above equation yield $h(x) = V(x; \pi_q^*) = V_q(x)$. □

Corollary 3.2. *If an increasing and concave function* $h(x) \in \mathcal{C}^2((0, \infty))$ *satisfies (19) and (20) for* $0 < \eta < x_q^*$, *we have* $h(x) = V_q(x) = V(x; \pi_q^*)$ *where* π_q^* *is given by (21).*

Proof. By the proof of Corollary 3.1, we know $\mathcal{L}h(x) < 0$ for $x \in (x_q^*, \infty)$ and $h'(x) \geq 1$ for $x \in (0, x_q^*]$. Since h is increasing and concave, the function $F(y) = h(x + y) - \phi y - K$ is increasing in $(0, \eta - x]$ and decreasing in $[\eta - x, \infty)$ for $0 \leq x \leq \eta$. Then

$$\mathcal{M}h(x) = \begin{cases} h(\eta) - \phi(\eta - x) - K, & 0 \leq x \leq \eta, \\ h(x) - K, & \eta < x < \infty, \end{cases} \tag{22}$$

which together with the boundary condition $h(0) = h(\eta) - \phi\eta - K$ yields that $\mathcal{M}h(x) < h(x)$ for $x > 0$ and $\mathcal{M}h(0) = h(0)$. The results are obtained by (ii) of Theorem 3.2. □

3.3 General optimal problem

By the definitions of V_p, V_q and V, we can get the relationship $V(x) \geq \max(V_p(x), V_q(x))$. By Lemma 5.1 in Zhao et al. (2014), we give the following results.

Lemma 3.1. *For each initial capital* $x \geq 0$, *if the functions* $g(x)$ *and* $h(x)$ *satisfy the conditions of Theorem 3.1 and Theorem 3.2, respectively, we have*

(i) If $\mathcal{M}g(0) \le g(0)$, then $g(x) = V(x)$ and the optimal strategy $\pi^* = \pi_p^*$ is given by (14);
(ii) If $h(0) \ge -P$, then $h(x) = V(x)$ and the optimal strategy $\pi^* = \pi_q^*$ is given by (21);
(iii) In particular, if conditions in (i) and (ii) hold, then $g(x) = h(x) = V(x)$ and the optimal strategy $\pi^* = \pi_p^*$ or $\pi^* = \pi_q^*$.

§4 Hyper-exponential jumps

In this section, we construct a closed form solution of the value function when the income jumps follow a hyper-exponential distribution, i.e.,

$$p(y) = \sum_{i=1}^{m} w_i \alpha_i e^{-\alpha_i y}, \quad y \ge 0, \tag{23}$$

with $\sum_{i=1}^{m} w_i = 1$, $w_i > 0$ for $i = 1, \cdots, m$, and $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < \infty$. This distribution has the merit that many calculations can be performed explicitly. Moreover, they can be used to approximate 'completely monotone' probability distribution functions, including some long-tailed distributions such as the Pareto and Weibull.

It is easy to show that the characteristic equation

$$\frac{\sigma^2}{2}s^2 - cs - (\lambda + \delta + \gamma) + \lambda \sum_{i=1}^{m} w_i \frac{\alpha_i}{\alpha_i - s} = 0,$$

has exactly $m+2$ roots denoted by $r_0, r_1, \cdots, r_{m+1}$ satisfying the following condition

$$r_0 < 0 < r_1 < \alpha_1 < \cdots < r_m < \alpha_m < r_{m+1}. \tag{24}$$

These roots play an important role in constructing smooth solutions to optimal problems.

4.1 Solution to the optimal problem without capital injections

According to Corollary 3.1, we focus on constructing an increasing and concave function $g(x) \in \mathcal{C}^2((0, \infty))$ satisfying (13) for some $x_p \ge 0$. In the case of hyper-exponential jumps, it is reasonable to construct a candidate solution to the value function defined in (11) as follows

$$g(x) = \sum_{k=0}^{m+1} C_k(x_p) e^{r_k x} + Ax + B, \tag{25}$$

where $r_k, k = 0, \cdots, m+1$ are the roots of the characteristic equation. We need to determine the values of A, B and the point x_p, and the associated $m+2$ coefficients $C_k(x_p), k = 0, \cdots, m+1$. Substituting (25) into the equation $\mathcal{L}g(x) = 0$ and comparing the coefficients of constant term, x, $e^{r_k x}$ and $e^{\alpha_i x}$ yield that

$$A = \frac{\gamma}{\gamma + \delta}, \quad B = \frac{\gamma \mu}{(\gamma + \delta)^2},$$

$$\sum_{k=0}^{m+1} C_k(x_p) e^{r_k x_p} \frac{\alpha_i r_k}{\alpha_i - r_k} + \frac{\gamma}{\gamma + \delta} = 1, \quad i = 1, \cdots, m. \tag{26}$$

Combining with the boundary conditions $g(0)=-P$, $g'(x_p)=1$ and $g''(x_p)=0$, we have

$$\sum_{k=0}^{m+1} C_k(x_p)+\frac{\gamma\mu}{(\gamma+\delta)^2}=-P, \tag{27}$$

$$\sum_{k=0}^{m+1} r_k C_k(x_p)e^{r_k x_p}+\frac{\gamma}{\gamma+\delta}=1, \quad \sum_{k=0}^{m+1} r_k^2 C_k(x_p)e^{r_k x_p}=0. \tag{28}$$

By solving equations (26) and (28), the coefficients $C_k(x_p)$ can be obtained

$$C_k(x_p)=-\frac{\delta}{r_k(\gamma+\delta)}e^{-r_k x_p}\prod_{i=1}^{m}\frac{r_k-\alpha_i}{\alpha_i}\prod_{j=0,j\neq k}^{m+1}\frac{r_j}{r_k-r_j}, \quad k=0,1,\cdots,m+1. \tag{29}$$

By (24), for all $x_p>0$, we have

$$\begin{aligned} &C_0(x_p)<0, \quad \lim_{x_p\to\infty} C_0(x_p)=-\infty;\\ &C_k(x_p)>0, \quad \lim_{x_p\to\infty} C_k(x_p)=0, \quad k=1,\cdots,m+1. \end{aligned} \tag{30}$$

The point x_p is determined by equation (27) which can be written as

$$-\sum_{k=0}^{m+1}\frac{\delta}{r_k(\gamma+\delta)}e^{-r_k x_p}\prod_{i=1}^{m}\frac{r_k-\alpha_i}{\alpha_i}\prod_{j=0,j\neq k}^{m+1}\frac{r_j}{r_k-r_j}+\frac{\gamma\mu}{(\gamma+\delta)^2}=-P. \tag{31}$$

Now, we show that the existence and uniqueness of the solution to equation (31). Let us define a function in x_p as

$$f(x_p)=\sum_{k=0}^{m+1} C_k(x_p)+\frac{\gamma\mu}{(\gamma+\delta)^2}+P, \tag{32}$$

then equation (31) is equivalent to $f(x_p)=0$. By (30), we obtain

$$\lim_{x_p\to\infty} f(x_p)=-\infty, \qquad f'(x_p)=-\sum_{k=0}^{m+1} r_k C_k(x_p)<0. \tag{33}$$

Hence, $f(x_p)$ is a decreasing function in x_p. From (16) and (29), we have

$$f(0)=\sum_{k=0}^{m+1} C_k(x_p)e^{r_k x_p}+\frac{\gamma\mu}{(\gamma+\delta)^2}+P=\frac{\mu}{\gamma+\delta}+P.$$

Due to the continuity of $f(x_p)$, we obtain that (31) has a unique solution $x_p^*>0$ if and only if $P>-\frac{\mu}{\gamma+\delta}$, which coincides with that in Remark 2.1.

Finally, we show that the function $g(x)$ satisfies all conditions of Corollary 3.1. By the above construction of $g(x)$, we only need to prove that $g(x)$ is increasing and concave for $x\geq 0$. Because of the same signs of r_k and $C_k(x_p^*)$, $k=0,\cdots,m+1$, we obtain

$$g'(x)=\sum_{k=0}^{m+1} r_k C_k(x_p^*)e^{r_k x}+\frac{\gamma}{\gamma+\delta}>0, \tag{34}$$

$$g'''(x)=\sum_{k=0}^{m+1} r_k^3 C_k(x_p^*)e^{r_k x}>0, \tag{35}$$

which together with $g''(x_p^*)=0$ yields the increase and concavity of $g(x)$.

Remark 4.1. *By (31) and (33), we have that x_p^* is an increasing function with respect to $P\in(-\frac{\mu}{\delta+\gamma},\infty)$. In other words, when the penalty for ruin is higher, instead of paying more* *dividend, the company had better reserve more money to guard against financial risks.*

4.2 Solution to the optimal problem without ruin

According to Corollary 3.2, we construct a candidate solution to the value function defined in (17) as follows, for some $x_q > 0$,

$$h(x) = \sum_{k=0}^{m+1} C_k(x_q)e^{r_k x} + \frac{\gamma}{\gamma+\delta}x + \frac{\gamma\mu}{(\gamma+\delta)^2}, \quad 0 \le x \le x_q. \tag{36}$$

The crucial levels x_q and η are determined by conditions (20), i.e.,

$$\sum_{k=0}^{m+1} r_k C_k(x_q)e^{r_k \eta} + \frac{\gamma}{\gamma+\delta} = \phi, \tag{37}$$

$$\sum_{k=0}^{m+1} C_k(x_q)e^{r_k \eta} - \sum_{k=0}^{m+1} C_k(x_q) = \eta(\phi - \frac{\gamma}{\gamma+\delta}) + K. \tag{38}$$

We will prove that there exists a unique pair (x_q^*, η) solution to the above equations with $0 < \eta < x_q^*$. For convenience, we first define a function

$$\psi(y) = \sum_{k=0}^{m+1} r_k C_k(y) + \frac{\gamma}{\gamma+\delta}, \quad y \ge 0. \tag{39}$$

Then, making the change of variable $y = x_q - \eta$, we can rewrite (37) as $\psi(y) = \phi$. Recalling the boundary conditions and (30), we obtain

$$\psi(0) = \sum_{k=0}^{m+1} r_k C_k(0) + \frac{\gamma}{\gamma+\delta} = h'(x_q) = 1 < \phi, \qquad \lim_{y\to\infty} \psi(y) = +\infty,$$

$$\psi'(0) = -\sum_{k=0}^{m+1} r_k^2 C_k(0) = -h''(x_q) = 0, \qquad \psi''(y) = \sum_{k=0}^{m+1} r_k^3 C_k(y) > 0.$$

Then $\psi(y)$ is continuous, increasing and convex. Furthermore, there exists a unique $x_1 > 0$ such that $\psi(x_1) = \phi$, i.e., $\eta = x_q - x_1$. We define another function with respect to x_q as follows

$$f_1(x_q) = \sum_{k=0}^{m+1} C_k(x_q)e^{r_k(x_q - x_1)} - \sum_{k=0}^{m+1} C_k(x_q) - (x_q - x_1)(\phi - \frac{\gamma}{\gamma+\delta}) - K, \; x_q \ge x_1. \tag{40}$$

Then equation (38) can be rewritten as $f_1(x_q) = 0$. Noting that

$$f_1(x_1) = -K < 0, \quad \lim_{x_q\to\infty} f_1(x_q) = +\infty, \quad f_1'(x_q) = \psi(x_q) - \phi \ge \psi(x_1) - \phi = 0,$$

we know that there exists a unique $x_q^* > x_1$ such that $f_1(x_q^*) = 0$, and so $\eta = x_q^* - x_1$ is also determined.

Finally, similar to (34) and (35), we can show that the function $h(x)$ is increasing and concave for $x \in (0, x_q^*]$. According to Corollary 3.2, the function $h(x)$ is the value function of optimal problem without ruin, and π_q^* in (21) is the optimal strategy.

Remark 4.2. *From (5.12) in Peng et al. (2012), we know* $\eta = \inf\{x > 0 : \int_0^x [\psi(y+x_1)-\phi]dy = K\}$. *Then the larger fixed cost* K *results in higher amount of capital injection, further higher dividend barrier* x_q^*. *Due to (40), larger proportional cost* ϕ *results in higher dividend level* x_q^*.

When K or ϕ increases, the transaction costs increase, the company had better increase the dividend barrier to guard against risks.

4.3 Solution to the general optimal problem

Lemma 4.1. *Assuming that $g(x)$ and $h(x)$ are respectively the value functions constructed in Subsection 4.1 and 4.2, we have*
(i) $\mathcal{M}g(0) \leq g(0)$ if and only if $x_p^* \leq x_q^*$;
(ii) $h(0) \geq -P$ if and only if $x_p^* \geq x_q^*$.

Proof. (i) Since $\mathcal{M}g(0) - g(0) = \max_{y\geq 0}\{g(y) - \phi y - K - g(0)\}$ and $g'(y) - \phi < 0$ for $x \geq x_p^*$, we have

$$0 \leq \widetilde{\eta} = \arg\max[g(y) - \phi y - K - g(0)] < x_p^*.$$

Because of the concavity of $g(x)$, we know $\widetilde{\eta} = 0$ if and only if $g'(0) \leq \phi$. Due to

$$g'(0) = \sum_{k=0}^{m+1} r_k C_k(x_p^*) + \frac{\gamma}{\gamma+\delta} = \psi(x_p^*) \leq \phi = \psi(x_1)$$

and the increase of ψ where ψ is given by (39), we have $\mathcal{M}g(0) - g(0) = -K < 0$ if and only if $x_p^* \leq x_1 \leq x_q^*$. If $0 < \widetilde{\eta} < x_p^*$, $g'(\widetilde{\eta}) = \phi$ i.e., $\psi(x_p^* - \widetilde{\eta}) = \phi$, then $x_p^* - \widetilde{\eta} = x_1$. By the definition of f_1 in (40) and the increase of f_1, we have $\mathcal{M}g(0) - g(0) = f_1(x_p^*) \leq 0 = f_1(x_q^*)$ if and only if $x_p^* \leq x_q^*$.
(ii) Recalling the definition of f in (32), we have $h(0) = \sum_{k=0}^{m+1} C_k(x_q^*) + \frac{\gamma\mu}{(\gamma+\delta)^2} = f(x_q^*) - P$. Because of the decrease of f and $f(x_p^*) = 0$, we obtain $h(0) = f(x_q^*) - P \geq f(x_p^*) - P$ if and only if $x_p^* \geq x_q^*$. □

Similar to Theorem 5.1 in Zhao et al. (2014), by Lemma 3.1 and Lemma 4.1, we obtain the following results. For the general control problem , we have
(i) If $x_p^* \leq x_q^*$, the value function $V(x) = V_p(x)$ and the optimal control strategy $\pi^* = \pi_p^*$ is given by (14);
(ii) If $x_p^* > x_q^*$, the value function $V(x) = V_q(x)$ and the optimal control strategy $\pi^* = \pi_q^*$ is given by (21).
In other words, the value function $V(x) = \max\{V_p(x), V_q(x)\}$ and the optimal dividend barrier $x^* = \min\{x_p^*, x_q^*\}$.

§5 Numerical illustrations

In this section, a series of numerical examples are provided to illustrate the results and to show the impacts of the penalty P, the transaction costs ϕ and K, and the killing rate γ on the optimal control problem. Based on the numerical results, some interesting economic insights are given. In the following examples, we assume $c = 1$, $\delta = 0.05$, $\lambda = 1.5$, $\sigma = 2$, and

$$p(y) = \frac{1}{2} \times \frac{2}{3} \times e^{-\frac{2}{3}y} + \frac{1}{2} \times 2 \times e^{-2y}, \quad y \geq 0.$$

• **Influences of the penalty P**

We first consider the impact of the penalty P on the optimal strategy. Let $\gamma = 0.05$, $\phi = 1.1$ and $K = 0.25$, then $\frac{\mu}{\delta+\gamma} = 5$. Due to Theorem 2.2, we have $x^* = x_p^* = 0$ as $P \in (-\infty, -5]$. From (37) and (38), we otain $x_q^* = 5.3447$. We give the values of x_p^* for the different values of the penalty P in Table 1. Table 1 shows that x_p^* increases with the increase of P, and that the optimal strategy π^* switches from π_p^* to π_q^* with the increase of the penalty. In other words, it is more advisable for the company to inject capital to avoid ruin for large penalty. Here, the maximum penalty P that the company can afford is -1.7451. That is, when $P > -1.7451$, we have $x^* = x_q^* = 5.3447$ and $V(x) = V_q(x)$, or else $x^* = x_p^*$ and $V(x) = V_p(x)$.

Table 1: The influences of P on x_p^* and x^*

P $\uparrow$	$(-\infty, -5]$	-4	-3	-2	**-1.7451**	-1.5	-1	0	0.5
x_p^* $\uparrow$	0	1.9459	3.6458	5.0357	**5.3447**	5.6264	6.1586	7.0784	7.4791
x_q^* $\equiv$	5.3447	5.3447	5.3447	5.3447	**5.3447**	5.3447	5.3447	5.3447	5.3447
x^* $\uparrow$	x_p^*	x_p^*	x_p^*	x_p^*	$\mathbf{x_p^* = x_q^*}$	x_q^*	x_q^*	x_q^*	x_q^*

• Influences of the transaction costs ϕ and K

Here, we set $\gamma = 0.05$ and $P = -1.5$. By Table 1, the level of the optimal dividend barrier $x_p^* = 5.6264$. From Table 2, the dividend barriers x_q^* and x^* increase with the increase of ϕ or K, while the amount of capital injection η increases when ϕ decreases or K increases. There is a reasonable economic explanation for this phenomenon. Larger ϕ or K means higher costs of capital injection, the company had better reserve more money instead of paying more dividends in order to reduce or avoid capital injection, which calls for the higher dividend barrier. While ϕ decreases or K increases, it is advisable to increase the amount of capital injection η so as to cut down costs of capital injection. The optimal strategy π^* switches from π_q^* to π_p^* with the increase of ϕ or K. Furthermore, when $K = 0.25$ and $\phi \geq 1.1319$ or when $\phi = 1.1$ and $K \geq 0.3260$, the optimal strategy $\pi^* \equiv \pi_p^*$, i.e., the company prefer declaring ruin to injecting capital whenever it is on the edge of ruin for large enough transaction costs.

Table 2: The influences of ϕ and K on η, x_q^* and x^*

	$K = 0.25$					$\phi = 1.1$			
$\phi \uparrow$	1.05	1.1	**1.1319**	1.2	$K \uparrow$	0.25	0.3	**0.3260**	0.35
η $\downarrow$	2.6710	2.3208	**2.1744**	1.9518	$\uparrow$	2.3208	2.5112	**2.6025**	2.6825
x_q^* $\uparrow$	4.8193	5.3447	**5.6264**	6.1393	$\uparrow$	5.3447	5.5351	**5.6264**	5.7064
x^* $\uparrow$	x_q^*	x_q^*	$\mathbf{x_q^* = x_p^*}$	x_p^*	$\uparrow$	x_q^*	x_q^*	$\mathbf{x_q^* = x_p^*}$	x_p^*

• Influences of the killing rate γ

Finally, we discuss the effects of γ on the dividend barriers x_p^*, x_q^* and x^*, and on the value function. Let $\phi = 1.1$, $K = 0.25$. By Theorem 2.2, if $P \leq -10$, the value function $V(x) = x - P$

and the optimal strategy is *take-the-money-and-run* for all $\gamma \geq 0$. As $\gamma \to \infty$, if $P \leq 0$ the value function $V(x) \to x - P$, or else $V(x) \to x$. Now assuming $P = 0.1$, we plot Figure 1 as follows. The left figure in Figure 1 shows that as γ increases, x_p^*, x_q^* and η decrease, and that the optimal strategy π^* switches from π_q^* to π_p^*. When γ increases, it is earlier to kill the surplus process, and so it calls for the lower levels of dividend barriers for paying more dividends before stopping the business. When $\gamma = 0, 0.05, 0.4$, we obtain the levels of optimal dividend barriers $x^* = x_q^* = 5.4143$, $x^* = x_q^* = 5.3447$ and $x^* = x_q^* = 4.9532$. The right figure in Figure 1 shows that as expected the value function decreases with increasing γ due to expected earlier killing of the surplus process.

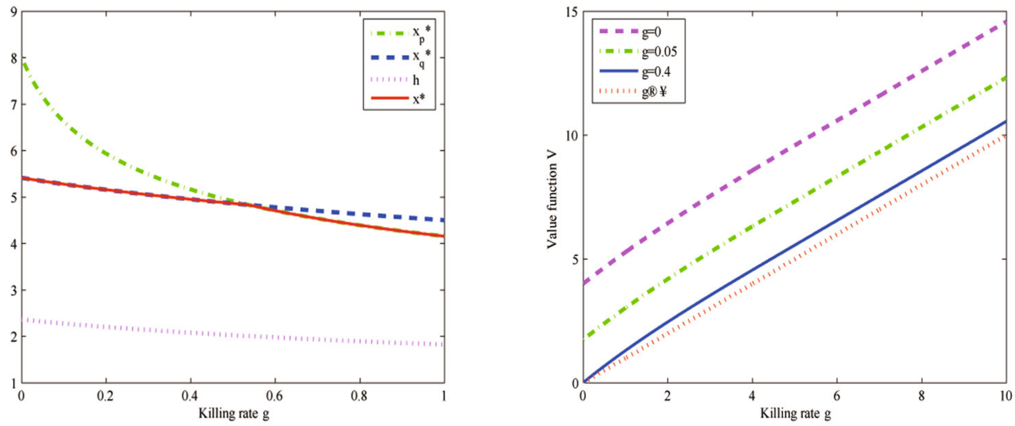

Figure 1: LEFT: The influences of γ on the optimal dividend barriers. RIGHT: The value function $V(x)$ as γ changes.

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