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Hyper-exponential jump-diffusion model under the barrier dividend strategy

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Abstract. In this paper, we consider a hyper-exponential jump-diffusion model with a constant dividend barrier. Explicit solutions for the Laplace transform of the ruin time, and the Gerber-Shiu function are obtained via martingale stopping.

§1 Introduction

In recent years, the ruin problem and the issue of dividend payment strategies have been received remarkable attention in the actuarial literature. Under a barrier strategy, when the surplus of an insurance company reaches a barrier level, premium income no longer goes into the surplus but is paid out as dividends to shareholders. Such a dividend-payment strategy was first discussed in De Finetti (1957) for a Bernoulli model. In this paper, we consider the classical problem of dividend payouts from a firm according to a dividend-barrier strategy, where the excess of the firm asset value above a threshold barrier will be automatically paid out to the shareholders.

Given a filtered complete probability space $\{\Omega, \Im, \{\Im_t\}, P\}$, all random variables and stochastic processes of this paper are assumed to be defined on it. Consider the surplus process of the insurance company modeled by:

$$X_t = u + ct + \sigma W_t + \sum_{i=1}^{N(t)} Z_i \hat{=} u + ct + \sigma W_t + S_t, \tag{1.1}$$

where $u > 0$ is the initial surplus, $c > 0$ is a constant, $\sigma > 0$ is a diffusion coefficient, $\{W_t; t \geq 0\}$ is a standard Brownian motion, $\{N(t); t \geq 0\}$ is a Poisson process with intensity $\lambda > 0$, $\{Z_i, i \geq 1\}$ is a sequence of independent and identically distributed random variables. Furthermore, it is assumed that $\{N(t); t \geq 0\}$, $\{Z_i, i \geq 1\}$ and $\{W_t; t \geq 0\}$ are mutually independent. If the

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jumps $\{Z_i, i \geq 1\}$ are negative, then the risk model (1.1) reduces to the classical risk process perturbed by diffusion introduced by Gerber (1970) in insurance mathematics. In this paper, we consider a modified version of (1.1), in which the jumps are assumed to be two-sided. As explained in Labbé, et al. (2012), if Z_i is negative, then it is interpreted as a claim, while it represents income from deceased annuitant if Z_i is positive. Many authors considered this kind of models, see for example, Zhang, et al. (2010) and Yuen and Yin (2011).

In recent years, studies of an insurance risk model have in general been focusing on analyzing the Gerber-Shiu expected discounted penalty function, which was first introduced by Gerber and Shiu (1998), and Gerber and Laudry (1998) to analyze the quantities, such as the time of ruin, the surplus immediately before ruin and the deficit at ruin, in a unified manner. It has proven to be a powerful analytical tool. The Gerber-Shiu function has been fully studied in the compound Poisson risk models, and the one-sided jump-diffusion models with and without dividend. Interestingly, Zhou (2005) made use of the fluctuation theory of Lévy processes to obtain a representation of the Gerber-Shiu function for the Cramér-Lundberg risk model with the barrier dividend strategy. But it is in general not easy to give a closed-form formula for the Gerber-Shiu function in a two-sided jump-diffusion model with the dividend strategy.

The double exponential jump-diffusion model is a special case of the jump-diffusion model with two-sided jumps which has been studied in finance by many authors due to its analytical tractability. By connecting the ruin problem of the ex-dividend risk process with the first passage problem of the Lévy process reflected at its running maximum, Bo, et al. (2012) gave the explicit expressions for the Laplace transform of the ruin time and the distribution of the deficit at ruin under the double exponential jump-diffusion model with a barrier dividend strategy. The hyper-exponential jump-diffusion model is a generalization of the double exponential jump-diffusion model. In fact, the hyper-exponential distribution is rich enough to approximate many other distributions, including any discrete distribution, the normal distribution, and various heavy-tailed distributions such as Gamma, Weibull and Pareto distributions. In this paper, motivated by Bo, et al. (2012), we consider the Gerber-Shiu function under the hyper-exponential jump-diffusion model with a barrier dividend strategy.

Assume the jump distribution has the common density function given by

$$f(x) = p\sum_{i=1}^{m} p_i\alpha_i e^{-\alpha_i x}1_{\{x\geq 0\}} + q\sum_{i=1}^{m} q_i\beta_i e^{\beta_i x}1_{\{x<0\}}, \tag{1.2}$$

where $0 < p, q < 1, p + q = 1, \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} q_i = 1$ with $0 < p_i, q_i < 1$ for $i = 1, \cdots, m$ and $\beta_m > \cdots > \beta_2 > \beta_1 > 0$, $\alpha_m > \alpha_{m-1} > \cdots > \alpha_1 > 0$. Let D_t be the aggregate dividends paid from 0 to t. Under the barrier dividend strategy, D_t can be expressed as follows,

$$D_t = \sup_{0\leq s\leq t}(X_s - b)^+, \tag{1.3}$$

where $b > 0$ is a constant. Since the risk model (1.1) is a cádlág process, it is separable and hence D_t is well-defined. Let

$$X_t^b = X_t - D_t \leq b$$

be the surplus process regulated by the dividend payment D_t. Furthermore, the dividend process D_t can be rewritten as

$$D_t = \int_{[0,t]} 1_{\{X_t^b = b\}} dD_t.$$

Define the ruin time as $\hat{\tau}_u = \inf\{t : X_t^b \le 0\}$, with $\hat{\tau}_u = +\infty$, if $X_t^b > 0$ for all t.

From now on, $\{P_x : x \in R\}$ denotes probabilities such that under P_x, $X_0 = x$ with probability one. $E_x[.]$ denotes the expectation operator associated to P_x.

Define the Gerber-Shiu function, introduced by Gerber and Landry (1998), as

$$\Psi(u) = E_u[e^{-\delta \hat{\tau}_u} \eta(|X^b_{\hat{\tau}_u}|) 1_{\{\hat{\tau}_u < \infty\}}], \tag{1.4}$$

where $\delta > 0$ is interpreted as the force of interest or the variable of a Laplace transform and $\eta(.)$ is a non-negative function defined on $[0, \infty)$. We remark that the penalty function $\eta(.)$ is not necessarily continuous at 0. The function $\Psi(u)$ embraces various quantities of ruin including the probability of ultimate ruin and the distribution of the deficit at ruin. For example, if we let $\eta(x) = 1$, then $\Psi(u) = E_u[e^{-\delta \hat{\tau}_u} 1_{\{\hat{\tau}_u < \infty\}}]$ is the Laplace transform of the ruin time. If we let $\eta(x) = 1_{\{x=0\}}$, then $\Psi(u) = E_u[e^{-\delta \hat{\tau}_u} 1_{\{\hat{\tau}_u < \infty, X^b_{\hat{\tau}_u} = 0\}}]$ is the Laplace transform of the ruin time due to oscillation. If we let $\eta(x) = 1_{\{x>0\}}$, then $\Psi(u) = E_u[e^{-\delta \hat{\tau}_u} 1_{\{\hat{\tau}_u < \infty, |X^b_{\hat{\tau}_u}| > 0\}}]$ is the Laplace transform of the ruin time due to a jump. Furthermore, in order to know the distribution of the deficit at ruin, we should consider $\eta(x) = 1_{\{x>l\}}, l > 0$. Then the Gerber-shiu function becomes $\Psi(u) = E_u[e^{-\delta \hat{\tau}_u} 1_{\{\hat{\tau}_u < \infty, |X^b_{\hat{\tau}_u}| > l\}}]$. These special cases have attracted a lot of attention.

In this paper, we shall obtain the explicit formula for $\Psi(u)$ and study several of its special case. The rest of the paper is organized as follows. In Section 2 we present some preliminary results. In Section 3 we derive the closed-form formula for the Gerber-Shiu function via martingale stopping. In Section 4, conclusions are given.

§2 Preliminary results

Define the Laplace exponent of $X_t - u$ as

$$g(s) = \frac{\ln E[e^{s(X_t - u)}]}{t} = \frac{1}{2}\sigma^2 s^2 + cs - \lambda + \lambda \sum_{i=1}^{m} \left(\frac{pp_i \alpha_i}{\alpha_i - s} + \lambda \frac{qq_i \beta_i}{\beta_i + s}\right). \tag{2.1}$$

In risk theory, the equation

$$g(s) = \delta, \quad \delta > 0 \tag{2.2}$$

is called the generalized Lundberg equation for the perturbed compound Poisson risk model with two-sided jumps. The roots of equation (2.2) play an important role in deriving the formula for the Gerber-Shiu function. Cai and Kou (2011) gave the following result.

Lemma 2.1. *For $\delta > 0$, equation (2.2) has exactly $m+1$ roots $r_1, r_2, \cdots, r_{m+1}$, on the left-half complex plane, and exactly $m+1$ roots $r_{m+2}, \cdots, r_{2m+2}$, on the right-half complex plane, with $-\infty < r_1 < -\beta_m < r_2 < -\beta_{m-1} < \cdots < -\beta_1 < r_{m+1} < 0 < r_{m+2} < \alpha_1 < \cdots < \alpha_m < r_{2m+2} < +\infty$.*

Define $Y_t = M_t - X_t$ as the process X reflected at its running supremum M, where

$$M_t = \sup_{0 \le s \le t} X_s \vee 0.$$

Define the entrance time of Y into $[b,\infty)$ as

$$\tau_b = \inf\{t : Y_t \geq b\}, \tag{2.3}$$

with $\inf \emptyset = \infty$. By the spatial homogeneity of the surplus process X, Bo, et al. (2012) obtained the following results concerning the Laplace transform of the ruin time and the deficit at ruin.

Lemma 2.2. *For $\delta > 0$ and a nonnegative measurable function $\eta(.)$*

$$E_u[e^{-\delta\hat{\tau}_u}\eta(|X^b_{\hat{\tau}_u}|)1_{\{\hat{\tau}_u<\infty\}}] = E_{u-b}[e^{-\delta\tau_b}\eta(Y_{\tau_b}-b)], \quad 0<u<b. \tag{2.4}$$

In particular, the Laplace transform of the ruin time is given by

$$E_u[e^{-\delta\hat{\tau}_u}] = E_{u-b}[e^{-\delta\tau_b}]. \tag{2.5}$$

For $l \geq 0$, the deficit at ruin satisfies

$$E_u[e^{-\delta\hat{\tau}_u}1_{\{X^b_{\hat{\tau}_u}<-l\}}] = E_{u-b}[e^{-\delta\tau_b}1_{\{Y_{\tau_b}-b>l\}}]. \tag{2.6}$$

Therefore, it suffices to derive the expectation $E_{u-b}[e^{-\delta\tau_b}\eta(Y_{\tau_b}-b)]$. To this end, we first investigate the property of the overshoot $Y_{\tau_b}-b$. Note that, either a downward jump of X or the component $ct+\sigma W_t$ may lead Y to cross b. Define the events $A_0, A_1, A_2, \cdots, A_m$ by

$$A_0 = \{Y_{\tau_b} = b\},$$

$$A_i = \{Y \text{crosses } b \text{ by a downward jump of an } Exp(\beta_i) - \text{distributed random variable}\}.$$

Lemma 2.3. *For any $x > 0$, we have*

$$P(\tau_b \leq t, Y_{\tau_b}-b \geq x, A_i) = e^{-\beta_i x}P(\tau_b \leq t, Y_{\tau_b}-b>0, A_i). \tag{2.7}$$

In particular,

$$P(Y_{\tau_b}-b \geq x | Y_{\tau_b}-b>0, A_i) = e^{-\beta_i x}. \tag{2.8}$$

Furthermore,

$$\begin{aligned} & P(\tau_b \leq t, Y_{\tau_b}-b \geq x | Y_{\tau_b}-b>0, A_i) \\ = \; & P(\tau_b \leq t | Y_{\tau_b}-b>0, A_i)P(Y_{\tau_b}-b \geq x | Y_{\tau_b}-b>0, A_i). \end{aligned} \tag{2.9}$$

Proof. The proof is similar to that of Proposition 2.1 in Kou and Wang (2003). It suffices to show that (2.7), since (2.8) can be easily obtained by letting $t \to +\infty$ in (2.7). Since $x > 0$, the overshoot can only occur in the jump times of the Poisson process N. $T_i, i = 1, 2, \cdots$ denote the arrival times of the Poisson process N. Then

$$P(\tau_b \leq t, Y_{\tau_b}-b \geq x, A_i) = \sum_{n=1}^{\infty} P(\tau_b = T_n \leq t, Y_{\tau_b}-b \geq x, A_i) \hat{=} \sum_{n=1}^{\infty} P_n.$$

We can compute P_n as follows,

$$\begin{aligned} P_n &= P(\min_{0\leq s<T_n} Y_s - b < 0, Y_{T_n}-b \geq x, T_n \leq t, A_i) \\ &= E[P(Y_{T_n}-b \geq x, A_i | \Im_{T_n^-}, T_n)1_{\{\min_{0\leq s<T_n} Y_s-b<0, T_n\leq t\}}] \\ &= E[P(Y_{T_n^-} - Z_{T_n} - b \geq x, -Z_{T_n} \sim Exp(\beta_i) | \Im_{T_n^-}, T_n)1_{\{\min_{0\leq s<T_n} Y_s-b<0, T_n\leq t\}}] \\ &= e^{-\beta_i x}E[e^{-\beta_i(b-Y_{T_n^-})}1_{\{\min_{0\leq s<T_n} Y_s-b<0, T_n\leq t\}}] \\ &= e^{-\beta_i x}P(\tau_b = T_n \leq t, Y_{T_n}-b>0, A_i). \end{aligned}$$

It follows that

$$P(\tau_b \le t, Y_{\tau_b} - b \ge x, A_i) = \sum_{n=1}^{\infty} e^{-\beta_i x} P(\tau_u = T_n \le t, Y_{T_n} - b > 0, A_i)$$
$$= e^{-\beta_i x} P(\tau_b \le t, Y_{\tau_b} - b > 0, A_i).$$

From (2.7) and (2.8),

$$P(\tau_b \le t, Y_{\tau_b} - b \ge x | Y_{\tau_b} - b > 0, A_i) = \frac{P(\tau_b \le t, Y_{\tau_b} - b \ge x, A_i)}{P(Y_{\tau_b} - b > 0, A_i)}$$
$$= \frac{e^{-\beta_i x} P(\tau_b \le t, Y_{\tau_b} - b > 0, A_i)}{P(Y_{\tau_b} - b > 0, A_i)}$$
$$= P(\tau_b \le t | Y_{\tau_b} - b > 0, A_i) P(Y_{\tau_b} - b \ge x | Y_{\tau_b} - b > 0, A_i).$$

The proof is completed. □

§3 First passage time of the reflected process

In this section, we consider the first passage time problem for Y. From Lemma 2.3 we have, conditional on $A_i, i = 1, \cdots, m$, τ_b and Y_{τ_b} are independent and furthermore the overshoot $Y_{\tau_b} - b$ is exponentially distributed with mean $1/\beta_i$. Consequently, for any $\delta > 0$ and $0 < y < b$,

$$E_{-y}[e^{-\delta \tau_b} \eta(Y_{\tau_b} - b)] = E_{-y}[e^{-\delta \tau_b} 1_{A_0}] \eta(0) + \sum_{i=1}^{m} E_{-y}[e^{-\delta \tau_b} 1_{A_i}] \int_0^{\infty} \eta(z) \beta_i e^{-\beta_i z} dz. \tag{3.1}$$

So we only need to calculate $E_{-y}[e^{-\delta \tau_b} 1_{A_i}], i = 0, 1, \cdots, m$. To this end, we first define an $(2m+2) \times (2m+2)$ matrix as follows

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{\beta_1}{\beta_1 + r_1} & \frac{\beta_1}{\beta_1 + r_2} & \cdots & \frac{\beta_1}{\beta_1 + r_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_m}{\beta_m + r_1} & \frac{\beta_m}{\beta_m + r_2} & \cdots & \frac{\beta_m}{\beta_m + r_{2m+2}} \\ r_1 e^{r_1 b} & r_2 e^{r_2 b} & \cdots & r_{2m+2} e^{r_{2m+2} b} \\ \frac{r_1 e^{r_1 b}}{\alpha_1 - r_1} & \frac{r_2 e^{r_2 b}}{\alpha_1 - r_2} & \cdots & \frac{r_{2m+2} e^{r_{2m+2} b}}{\alpha_1 - r_{2m+2}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{r_1 e^{r_1 b}}{\alpha_m - r_1} & \frac{r_2 e^{r_2 b}}{\alpha_m - r_2} & \cdots & \frac{r_{2m+2} e^{r_{2m+2} b}}{\alpha_m - r_{2m+2}} \end{pmatrix}. \tag{3.2}$$

Theorem 3.1. *Consider a nonnegative measurable function η such that for each $i = 1, 2, \cdots, m$, $\int_0^{+\infty} \eta(y) e^{-\beta_i y} dy < \infty$. Then for $\delta > 0$, we have*

$$E_{-y}[e^{-\delta \tau_b} \eta(Y_{\tau_b} - b)] = c_0 \eta(0) + \sum_{i=1}^{m} c_i \int_0^{\infty} \eta(z) \beta_i e^{-\beta_i z} dz, \quad 0 < y < b, \tag{3.3}$$

where $c_0, c_1, \cdots, c_m$ are determined by the following linear system

$$(c_0 \cdots c_m \quad d_0 \cdots d_m) A = (e^{r_1 (b-y)} \cdots e^{r_{m+1}(b-y)} \quad e^{r_{m+2}(b-y)} \cdots e^{r_{2m+2}(b-y)}), \tag{3.4}$$

with matrix A defined by (3.2) and $r_1, \cdots, r_{2m+2}$ are $2m+2$ distinct roots of the equation $g(x) = \delta$. If A is nonsingular, then $c_0, c_1, \cdots, c_m, d_0, d_1, \cdots, d_m$ uniquely solves (3.4).

In particular, we have for $v < \beta_1$ and $l \geq 0$,

$$E_{-y}[e^{-\delta\tau_b}] = \sum_{i=0}^{m} c_i, \tag{3.5}$$

$$E_{-y}[e^{-\delta\tau_b} 1_{\{Y_{\tau_b} > b+l\}}] = \sum_{i=1}^{m} c_i e^{-\beta_i l}, \tag{3.6}$$

$$E_{-y}[e^{-\delta\tau_b + v(Y_{\tau_b} - b)}] = c_0 + \sum_{i=1}^{m} \frac{c_i \beta_i}{\beta_i - v}. \tag{3.7}$$

Proof. Consider a function $f(t, Y_t) = e^{-at+\gamma(Y_t - b)}$ for any $a > 0$ and γ with $\mathbf{R}(\gamma) = 0$. Applying Itó's formula to $f(t, Y_t)$, we obtain

$$\begin{aligned} e^{-at+\gamma(Y_t-b)} &= e^{-\gamma b}(e^{\gamma y} - \int_0^t a e^{-as+\gamma Y_s} ds + \int_0^t \gamma e^{-as+\gamma Y_s} d(M_s^c - X_s^c) \\ &\quad + \frac{1}{2}\int_0^t \gamma^2 \sigma^2 e^{-as+\gamma Y_s} ds + \sum_{0 \leq s \leq t, \Delta Y_s \neq 0} (e^{-as+\gamma Y_s} - e^{-as+\gamma Y_{s-}})). \end{aligned} \tag{3.8}$$

Note that

$$M_t^c = \int_0^t 1_{\{Y_s = 0\}} dM_s^c,$$

where M_t^c is the continuous part of M. Then (3.8) becomes

$$\begin{aligned} e^{-at+\gamma(Y_t-b)} &= e^{-\gamma b}(e^{\gamma y} + \int_0^t (\frac{1}{2}\gamma^2\sigma^2 - c\gamma - a) e^{-as+\gamma Y_s} ds + \int_0^t \gamma e^{-as} 1_{\{Y_s=0\}} dM_s^c \\ &\quad - \int_0^t \gamma e^{-as+\gamma Y_s} \sigma dW_s + \sum_{0 \leq s \leq t, \Delta Y_s \neq 0} (e^{-as+\gamma Y_s} - e^{-as+\gamma Y_{s-}})). \end{aligned} \tag{3.9}$$

The last term of (3.9) can be rewritten as

$$\begin{aligned} &\sum_{0 \leq s \leq t, \Delta Y_s \neq 0} (e^{-as+\gamma Y_s} - e^{-as+\gamma Y_{s-}}) \\ = &\sum_{0 \leq s \leq t} (1_{\{\Delta M_s = 0, \Delta Y_s \neq 0\}} + 1_{\{\Delta M_s > 0, \Delta Y_s \neq 0\}})(e^{-as+\gamma Y_s} - e^{-as+\gamma Y_{s-}}). \end{aligned}$$

It is obvious that if $\Delta M_s > 0$, then $M_s = X_s$. That is to say, if $\Delta M_s > 0$, then $Y_s = 0$. Therefore,

$$\begin{aligned} &\sum_{0 \leq s \leq t, \Delta Y_s \neq 0} (e^{-as+\gamma Y_s} - e^{-as+\gamma Y_{s-}}) \\ = &\sum_{0 \leq s \leq t} 1_{\{\Delta Y_s \neq 0, \Delta M_s = 0\}} (e^{-as+\gamma(Y_{s-} - \Delta X_s)} - e^{-as+\gamma Y_{s-}}) \\ &+ \sum_{0 \leq s \leq t} 1_{\{\Delta Y_s \neq 0, \Delta M_s > 0\}} (e^{-as} - e^{-as+\gamma Y_{s-}}) \\ = &\sum_{0 \leq s \leq t, \Delta Y_s \neq 0} (e^{-as+\gamma(Y_{s-} - \Delta X_s)} - e^{-as+\gamma Y_{s-}}) \\ &+ \sum_{0 \leq s \leq t, \Delta Y_s \neq 0} 1_{\{\Delta M_s > 0\}} (e^{-as} - e^{-as-\gamma \Delta M_s})), \end{aligned} \tag{3.10}$$

where the last equality holds because $Y_s = Y_{s-} + \Delta M_s - \Delta X_s = 0$ when $\Delta M_s > 0$.

Combining (3.9) with (3.10), we have that

$$\begin{aligned} Z_t &\hat{=} e^{-\gamma b}(e^{-at+\gamma Y_t} - e^{\gamma y} - \int_0^t (g(-\gamma)-a)e^{-as+\gamma Y_s}ds \\ &\quad - \sum_{0\le s\le t,\Delta Y_s\neq 0,\Delta M_s>0} e^{-as}(1-e^{-\gamma\Delta M_s}) - \int_0^t \gamma e^{-as}1_{\{Y_s=0\}}dM_s^c) \\ &= -e^{-\gamma b}(\int_0^t \gamma e^{-as+\gamma Y_s}\sigma dW_s - \sum_{0\le s\le t,\Delta Y_s\neq 0}(e^{-as+\gamma(Y_{s^-}-\Delta X_s)} - e^{-as+\gamma Y_{s^-}}) \\ &\quad +\lambda\int_0^t\int_{-\infty}^{\infty} e^{-as+\gamma Y_{s^-}}(e^{-\gamma z}-1)f(z)dzds) \end{aligned}$$

is a zero mean martingale. Then applying Doob's optional stopping theorem we have $E[Z_{\tau_b}]=0$. An application of (3.1) yields

$$\begin{aligned} &c_0 + \sum_{i=1}^m \frac{c_i\beta_i}{\beta_i-\gamma} - e^{\gamma(y-b)} + (g(-\gamma)-a)E_{-y}[\int_0^{\tau_b} e^{-as+\gamma(Y_s-b)}ds] \\ &-E_{-y}[\int_0^{\tau_b}\gamma e^{-as-\gamma b}dM_s^c] - E_{-y}[\sum_{0\le s\le t} 1_{\{\Delta M_s>0\}}e^{-as-\gamma b}(1-e^{-\gamma\Delta M_s})] = 0. \quad (3.11) \end{aligned}$$

Since M_t jumps upward only driven by a positive jump of X, define the events $H_1,\cdots,H_m$ by

$$H_i = \{M \text{ crosses the supremum by a jump of an } Exp(\alpha_i) - \text{distributed random variable}\}.$$

Note that due to the memoryless property of the exponential distribution, given H_j, the overshoot that M crosses the supremum is exponentially distributed with mean $1/\alpha_j$. Therefore, the last term of (3.11) can be rewritten as

$$E_{-y}[\sum_{0\le s\le t} 1_{\{\Delta M_s>0\}}e^{-as-\gamma b}(1-e^{-\gamma\Delta M_s})] = \sum_{i=1}^m d_i\frac{\gamma}{\alpha_i+\gamma}e^{-\gamma b}, \quad (3.12)$$

where $d_i = E_{-y}[\sum_{0\le s\le t} e^{-as}1_{\{\Delta M_s>0,H_i\}}], i=1,\cdots,m$.

Write $d_0 = E_{-y}[\int_0^{\tau_b} e^{-as}dM_s^c]$. Then substituting (3.12) into (3.11), we have

$$\begin{aligned} &c_0 + \sum_{i=1}^m \frac{c_i\beta_i}{\beta_i-\gamma} - e^{\gamma(y-b)} - \gamma d_0 e^{-\gamma b} - \sum_{i=1}^m \frac{d_i\gamma}{\alpha_i+\gamma}e^{-\gamma b} \\ &+(g(-\gamma)-a)E_{-y}[\int_0^{\tau_b} e^{-as+\gamma Y_s}ds] = 0. \end{aligned}$$

By analytic continuation, the above equation can be extended to hold for $\gamma\in\mathbf{C}\backslash\{-\alpha_1,\cdots,-\alpha_m\}$. If we replace γ by $-r_i, i=1,\cdots,2m+2$, then we can obtain

$$c_0 + \sum_{i=1}^m \frac{c_i\beta_i}{\beta_i+r_j} + d_0 r_j e^{r_j b} + \sum_{i=1}^m \frac{d_i r_j}{\alpha_i - r_j}e^{r_j b} = e^{r_j(b-y)}, j=1,\cdots,r_{2m+2}.$$

Eqs. (3.5)-(3.7) can be easily obtained by taking particular forms of $\eta(.)$. □

From Theorem 3.1 and Lemma 2.1, we can directly obtain the solution for the Gerber-Shiu function for a general nonnegative measurable function η. We also present some special cases which are very useful in deriving the distributions of the ruin time and the deficit at ruin under the Hyper-exponential jump-diffusion model with a constant dividend barrier.

Corollary 3.1. *Consider a nonnegative measurable function* $\eta(.)$ *such that for* $i = 1, \cdots, m$, $\int_0^{+\infty} \eta(y)e^{-\beta_i y}dy < \infty$. *For* $\delta > 0$, *we have*

$$E_u[e^{-\delta\hat{\tau}_u}\eta(-X^b_{\hat{\tau}_u})] = \overline{c}_0\eta(0) + \sum_{i=1}^m \overline{c}_i \int_0^\infty \eta(z)\beta_i e^{-\beta_i z}dz, \quad 0 < u < b, \tag{3.13}$$

where $\overline{c}_0, \overline{c}_1, \cdots, \overline{c}_m$ *are determined by the following linear system*

$$(\overline{c}_0 \cdots \overline{c}_m \quad \overline{d}_0 \cdots \overline{d}_m)A = (e^{r_1 u} \cdots e^{r_{m+1}u} \quad e^{r_{m+2}u} \cdots e^{r_{2m+2}u}), \tag{3.14}$$

and $r_1, \cdots, r_{2m+2}$ *are* $2m + 2$ *distinct roots of the equation* $g(x) = \delta$. *If* A *is nonsingular, then* $\overline{c}_0, \overline{c}_1, \cdots, \overline{c}_m, \overline{d}_0, \overline{d}_1, \cdots, \overline{d}_m$ *uniquely solve (3.14).*

In particular, we have for $v < \beta_1$ *and* $l \geq 0$,

$$E_u[e^{-\delta\hat{\tau}_u}] = \sum_{i=0}^m \overline{c}_i, \quad 0 < u < b, \tag{3.15}$$

$$E_u[e^{-\delta\hat{\tau}_u}1_{\{X^b_{\hat{\tau}_u} < -l\}}] = \sum_{i=1}^m \overline{c}_i e^{-\beta_i l}, \quad 0 < u < b, \tag{3.16}$$

$$E_u[e^{-\delta\hat{\tau}_u - vX^b_{\hat{\tau}_u}}] = \overline{c}_0 + \sum_{i=1}^m \frac{\overline{c}_i\beta_i}{\beta_i - v}, \quad 0 < u < b. \tag{3.17}$$

Remark. Based on equations (3.15)-(3.16), one can obtain the marginal distributions of the ruin time and the deficit at ruin: $P(\hat{\tau}_u \leq t)$, $P(X^b_{\hat{\tau}_u} < -l)$ and $P(X^b_{\hat{\tau}_u} = 0)$. The joint distribution of the ruin time and the deficit at ruin can be obtained by inverting the joint Laplace transform (3.17).

If the matrix A given by (3.2) is nonsingular, then solving equation (3.14) yields

$$(\overline{c}_0 \cdots \overline{c}_m \quad \overline{d}_0 \cdots \overline{d}_m) = (e^{r_1 u} \cdots e^{r_{m+1}u} \quad e^{r_{m+2}u} \cdots e^{r_{2m+2}u})A^{-1}.$$

Therefore, substituting the solutions for $\overline{c}_i, i = 0, 1, \cdots, m$ into equations (3.15)-(3.17), we have the following theorems concerning the Laplace transforms of $\hat{\tau}_u$ and $X^b_{\hat{\tau}_u}$.

Theorem 3.2. *If the matrix* A *given by* (3.2) *is nonsingular, then for* $\delta > 0$, *we have the following Laplace transforms concerning* $\hat{\tau}_u$ *and* $X^b_{\hat{\tau}_u}$:

$$E_u[e^{-\delta\hat{\tau}_u}] = \sum_{i=1}^{2m+2} k_i e^{r_i u}, \quad 0 < u < b, \tag{3.18}$$

$$E_u[e^{-\delta\hat{\tau}_u}1_{\{X^b_{\hat{\tau}_u} < -l\}}] = \sum_{i=1}^{2m+2} m_i e^{r_i u}, \quad 0 < u < b, l \geq 0, \tag{3.19}$$

$$E_u[e^{-\delta\hat{\tau}_u}1_{\{X^b_{\hat{\tau}_u} = 0\}}] = \sum_{i=1}^{2m+2} n_i e^{r_i u}, \quad 0 < u < b, \tag{3.20}$$

where $r_1, \cdots, r_{2m+2}$ *are* $2m + 2$ *distinct roots of the equation* $g(x) = \delta$, $\mathbf{k} = (k_1, \cdots, k_{2m+2})'$, $\mathbf{m} = (m_1, \cdots, m_{2m+2})'$, *and* $\mathbf{n} = (n_1, \cdots, n_{2m+2})'$ *are vectors uniquely determined by the following linear systems*

$$A\mathbf{k} = I, \ \ A\mathbf{m} = J_1, \quad A\mathbf{n} = J_2,$$

with $I = (1, 1, \cdots, 1, 0, 0 \cdots, 0)' \in \mathbf{R^{2m+2}}$, $J_1 = (0, e^{-\beta_1 l}, \cdots, e^{-\beta_m l}, 0, \cdots, 0)' \in \mathbf{R^{2m+2}}$ *and* $J_2 = (1, 0, \cdots, 0)' \in \mathbf{R^{2m+2}}$, *respectively.*

Remark. If $m = 1$, then (3.18) is the same as (3.23) in Bo, et al. (2012).

Remark. If $b = +\infty$, then from Cai and Kou (2011) we can obtain the Laplace transform of the ruin time given by

$$E[e^{-\delta\hat{\tau}_u}] = \sum_{i=1}^{m+1} \tilde{c}_i e^{r_i u}, u > 0,$$

where $r_1, r_2, \cdots, r_{m+1}$ are $m+1$ negative roots of the equation $g(s) = \delta$, $\tilde{c}_1, \cdots, \tilde{c}_{m+1}$ are determined by

$$(\tilde{c}_1, \cdots, \tilde{c}_{m+1})B = (1, \cdots, 1),$$

with the nonsingular matrix B given by

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{\beta_1}{\beta_1+r_1} & \frac{\beta_1}{\beta_1+r_2} & \cdots & \frac{\beta_1}{\beta_1+r_{m+1}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\beta_m}{\beta_m+r_1} & \frac{\beta_m}{\beta_m+r_2} & \cdots & \frac{\beta_m}{\beta_m+r_{m+1}} \end{pmatrix}$$

The proof of the non-singularity of B can be found in Cai and Kou (2011). But the techniques they adopted can not be used to justify if the column vectors or row vectors of the matrix A are linearly independent. We will consider the problem in the future's research.

The following theorem gives the joint Laplace transform of $\hat{\tau}_u$ and $X^b_{\hat{\tau}_u}$.

Theorem 3.3. *If the matrix A given by (3.2) is nonsingular, then for $v < \beta_1$ and $l \geq 0$,*

$$E_u[e^{-\delta\hat{\tau}_u - vX^b_{\hat{\tau}_u}}] = \sum_{i=1}^{2m+2} w_i e^{r_i u}, \quad 0 < u < b, \tag{3.21}$$

where $r_1, \cdots, r_{2m+2}$ are $2m+2$ roots of the equation $g(x) = \delta$, and $\mathbf{w} = (w_1, w_2, \cdots, w_{2m+2})'$ is a vector uniquely determined by

$$A\mathbf{w} = J_3,$$

with $J_3 = (1, \frac{\beta_1}{\beta_1-v}, \cdots, \frac{\beta_m}{\beta_m-v}, 0, \cdots, 0)' \in \mathbf{R}^{\mathbf{2m+2}}$.

Remark. If the parameters p, q, p_i and q_i in (1.2) are allowed to be negative, then the risk model (1.1) is called a mixed-exponential jump-diffusion process. Note that, under the mixed-exponential jump-diffusion process, the roots of the equation $g(x) = \delta$ will not necessarily be distinct. Furthermore, the events which lead the process Y_t to cross b can not be divided into $A_0, A_1, \cdots, A_m$. Therefore, the techniques used in this paper for studying the Gerber-Shiu function are not feasible under the mixed-exponential jump-diffusion process with a dividend barrier.

§4 Conclusions

This article considers the Gerber-Shiu expected discounted penalty function under a constant dividend barrier for a two-sided jump-diffusion model whose jump sizes have the hyper-exponential distribution, which is rich enough to approximate many other distributions, including some heavy-tail distributions. Using martingale stopping, we obtain the explicit formulas for the Laplace transform of the ruin time and the expected discounted deficit under the hyper-exponential jump-diffusion model with a constant dividend barrier.

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