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On the maximal eccentric connectivity indices of graphs

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Abstract. For a connected simple graph G, the eccentricity $ec(v)$ of a vertex v in G is the distance from v to a vertex farthest from v, and $d(v)$ denotes the degree of a vertex v. The eccentric connectivity index of G, denoted by $\xi^c(G)$, is defined as $\sum_{v\in V(G)} d(v)ec(v)$. In this paper, we will determine the graphs with maximal eccentric connectivity index among the connected graphs with n vertices and m edges($n \leq m \leq n+4$), and propose a conjecture on the graphs with maximal eccentric connectivity index among the connected graphs with n vertices and m edges ($m \geq n+5$).

§1 Introduction and notation

For a connected simple graph G, we define the distance $d(u,v)$ between two vertices u and v as the length of the shortest path connecting u and v in G. The eccentricity $ec(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v. The diameter $Diam(G)$ of G is the maximum eccentricity among the vertices of G. The eccentric connectivity index of G [5], denoted by $\xi^c(G)$, is defined as

$$\xi^c(G) = \sum_{v\in V(G)} ec(v)d(v) = \sum_{uv\in E(G)} \omega_G(uv),$$

where $\omega_G(uv)$ denotes the weight of edge uv in G and equals $ec(u)+ec(v)$.

As a novel, distance-cum-adjacency topological descriptor, the eccentric connectivity index provides excellent correlations with regard to both physical and biological properties of chemical substances. The simplicity amalgamated with high correlating ability of this index can be easily exploited in QSPR/QSAR studies (see [2, 5]). Recently, many results on the eccentric connectivity index have been obtained for various classes of graphs, see, for example, [1, 3, 4, 6].

Let $d_{n,m} = \left\lfloor \frac{2n+1-\sqrt{17+8(m-n)}}{2} \right\rfloor$, and $E_{n,m}$ be the graph obtained from a path $P_{d_{n,m}+1} = v_0v_1\dots v_{d_{n,m}}$ by joining each vertex of $K_{n-d_{n,m}-1}$ to both $v_{d_{n,m}}$ and $v_{d_{n,m}-1}$, and by joining

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$m-n+1-\binom{n-d_{n,m}}{2}$ vertices of $K_{n-d_{n,m}-1}$ to $v_{d_{n,m}-2}$. Clearly, $E_{n,n-1}\cong P_n$. In this paper, we will show that for $8\le n\le m\le n+2$ and $14\le n+3\le m\le n+4$, $E_{n,m}$ is the unique graph with maximal eccentric connectivity index among the connected graphs with n vertices and m edges, respectively. Finally, we conjecture that $n+5\le m\le n(n-1)/2$ and $d_{n,m}\ge 3$, $E_{n,m}$ is the graph with maximal eccentric connectivity index among the connected graphs with n vertices and m edges.

§2 Main results

Let $T_{n,i}$ be the tree obtained from $P_{n-1}=v_0v_1\ldots v_{n-2}$ by attaching a pendent vertex v_{n-1} to v_i, where $1\le i\le\lfloor\frac{n-2}{2}\rfloor$. Let $T_a^{n,p}$ be the tree obtained by attaching a and $p-a$ pendent vertices respectively to the two end vertices of the path P_{n-p} for $1\le a\le\lfloor\frac{p}{2}\rfloor$, and let $T^{n,p}=\{T_a^{n,p}|1\le a\le\lfloor\frac{p}{2}\rfloor, 2\le p\le n-2\}$.

Lemma 2.1. *[6] Among the trees on n vertices with at least p pendent vertices, any tree $T\in T^{n,p}$ reaches the maximal value on the eccentric connectivity index. Moreover, P_n, $T_{n,1}$ and $T_{n,2}$ are respectively the only trees with the first three maximal eccentric connectivity indices.*

Theorem 2.1. *Let G be a graph with n vertices and n edges, that is, G is a unicyclic graph, where $n\ge 6$. Then $\xi^c(G)\le\xi^c(E_{n,n})$, with equality if and only if $G\cong E_{n,n}$.*

Proof. Let G be a unicyclic graph on n vertices. Then $Diam(G)\le n-2$ and $\omega(e)\le 2n-4$ for any edge $e\in E(G)$. If $G\ncong C_n$, then we can always find an edge e of G such that $G-e$ is a tree with at least three pendent vertices. By Lemma 2.1 we have $\xi^c(G)\le\xi^c(G-e)+\omega(e)\le\xi^c(T_{n,1})+2n-4=\xi^c(E_{n,n})$. Note that $\omega(e)=2n-4$ implies $G\cong E_{n,n}$, thus the equality holds if and only if $G\cong E_{n,n}$.

Suppose that $G\cong C_n$, where $n\ge 6$. If n is even, then we have $\xi^c(C_n)=n^2<\frac{3(n-2)^2}{2}+4n-9=\xi^c(E_{n,n})$; if n is odd, then $\xi^c(C_n)=n^2-n<\frac{3(n-2)^2+1}{2}+4n-9=\xi^c(E_{n,n})$. The result now follows. □

Let $Q_{n,n}$ be the tree obtained from $P_{n-1}=v_0v_1\ldots v_{n-2}$ by joining an isolated vertex v_{n-1} to both v_{n-2} and v_{n-4}.

Lemma 2.2. *Let G be a unicyclic graph on n vertices, $n\ge 6$ and $G\ncong E_{n,n}$. Then $\xi^c(G)\le\xi^c(Q_{n,n})$, with equality if and only if $G\cong Q_{n,n}$.*

Proof. By applying an argument similar to the proof of Theorem 2.1, we can show that the result is true for the case $G\cong C_n$. Suppose that G is not a cycle on n vertices. Let $g(G)$ be the length of the unique cycle in G. If $g(G)\ge 5$, then $Diam(G)\le n-3$ and there exists an edge e in the unique cycle of G such that $G-e\ncong P_n, T_{n,1}, T_{n,2}$, and

$$\xi^c(G)\le\xi^c(G-e)+\omega(e)\le\xi^c(T_{n,2})+2n-6<\xi^c(T_{n,2})+2n-5=\xi^c(Q_{n,n}).$$

If $g(G)=4$ and $n\ge 7$, then $Diam(G)\le n-2$ and there is an edge e such that $G-e$ is a tree such that $G-e\ncong P_n, T_{n,1}$. Clearly, there exists no edge of weight $2n-4$ in G. Then

$$\xi^c(G)\le\xi^c(G-e)+\omega(e)\le\xi^c(T_{n,2})+2n-5=\xi^c(Q_{n,n}),$$

with equality only if $G - e \cong T_{n,2}$ and $\omega(e) = 2n - 5$. These imply that $G \cong Q_{n,n}$. By simple computation we can also obtain the result for $n = 6$.

If $g(G) = 3$, and $G \ncong E_{n,n}$, then there is an edge e such that $G - e \ncong P_n, T_{n,1}$. If $Diam(G) \leq n - 3$, similarly, the result follows. If $Diam(G) = n - 2$, then G is the graph obtained from $P_{n-1} = v_0 v_1 \dots v_{n-2}$ by joining an isolated vertex v_{n-1} to both v_i and v_{i+1}, where $1 \leq i \leq n - 4$. Then

$$\xi^c(Q_{n,n}) - \xi^c(G) = 4n - 12 - \omega(v_i v_{n-1}) - \omega(v_{i+1} v_{n-1}) \geq 4n - 12 - (2n - 7) - (2n - 6) > 0.$$

These complete the proof. □

Theorem 2.2. *Let G be a connected graph with n vertices and $n+1$ edges, where $n \geq 6$. Then $\xi^c(G) \leq \xi^c(E_{n,n+1})$, with equality if and only if $G \cong E_{n,n+1}$.*

Proof. Let G be a graph on n vertices with $n + 1$ edges and exactly two cycles. Then we know that $Diam(G) \leq n - 3$ and there exists an edge e in a cycle such that $G - e$ has two pendent vertices, thus $G - e$ is not isomorphic to $E_{n,n}$ and $Q_{n,n}$, and by Theorem 2.2, we have

$$\xi^c(G) \leq \xi^c(G - e) + \omega(e) < \xi^c(Q_{n,n}) + 2n - 6 = \xi^c(E_{n,n+1}).$$

Let G have three cycles, then there is an edge e such that $G - e \ncong E_{n,n}$. If $Diam(G) = n - 2$, then G is obtained from $P_{n-1} = v_0 v_1 \dots v_{n-2}$ by joining an isolated vertex v_{n-1} with v_{i-1}, v_i and v_{i+1}, where $1 \leq i \leq \lfloor \frac{n-2}{2} \rfloor$. If $i = 1$, then $G \cong E_{n,n+1}$. Let $i \geq 2$. It is easy to prove that $\xi^c(G) < \xi^c(E_{n,n+1})$. If $Diam(G) \leq n - 3$, by Theorem 2.2 we have

$$\xi^c(G) \leq \xi^c(G - e) + \omega(e) \leq \xi^c(Q_{n,n}) + 2n - 6 = \xi^c(E_{n,n+1}).$$

If $G - e \cong Q_{n,n}$, then $Diam(G) < Diam(G - e)$. Thus $\xi^c(G) < \xi^c(G - e) + \omega(e)$ and $\xi^c(G) < \xi^c(E_{n,n+1})$. If $G - e \ncong Q_{n,n}$, then by Lemma 2.2 we have that $\xi^c(G - e) < \xi^c(Q_{n,n})$. Thus $\xi^c(G) < \xi^c(E_{n,n+1})$. □

Theorem 2.3. *Let G be a connected graph with n vertices and $n+2$ edges, where $n \geq 8$. Then $\xi^c(G) \leq \xi^c(E_{n,n+2})$, with equality if and only if $G \cong E_{n,n+2}$.*

Proof. If G has a spanning tree T with at least four pendent vertices, and let e_1, e_2, e_3 be the edges not in T, that is, $T + e_1 + e_2 + e_3 = G$. Note that $Diam(G) \leq n - 3$, by Lemma 2.1 we have

$$\xi^c(G) \leq \xi^c(T) + \omega(e_1) + \omega(e_2) + \omega(e_3) \leq \xi^c(T_1^{n,4}) + 3(2n - 6) = \xi^c(E_{n,n+2}).$$

It is easy to prove that the above equality holds if and only if $G \cong E_{n,n+2}$.

Suppose that all spanning trees of G have at most three pendent vertices. If $G - e_1$ has two edge-disjoint cycles for any edge e_1 of a cycle in G, then we can always find edges e_2, e_3 such that $G - e_1 - e_2 - e_3$ is a spanning trees with at least four pendent vertices, this is a contradiction. Then for any edge e_1 in a cycle of G, $G - e_1$ must have exactly three cycles, and

G has exactly a non-trivial block (the block has at least three edges) that is showed in Fig.1.

H_1 H_2

Fig.1. Graphs H_1 and H_2.

Clearly, $H_2 - e_1 - e_2 - e_3$ have four pendent vertices, then $G - e_1 - e_2 - e_3$ must contain at least four pendent vertices and the non-trivial block of G is not isomorphic to H_2. If $G \ncong H_1$, then G can be obtained from H_1 by planting many trees to it. Suppose that there is a tree planting to a vertex but not u_1 of P^4 in Fig.1, we choose three edges e_1, e_2, e_3 respectively at the path P^1, P^2, P^3 in Fig.1. Then $G - e_1 - e_2 - e_3$ have four pendent vertices, that is a contradiction. Thus it must be $G \cong H_1$ and $Diam(G) \leq \lfloor \frac{n}{2} \rfloor$. For $n \geq 8$, we have $\xi^c(G) \leq 2(n+2)\lfloor \frac{n}{2} \rfloor < \left\lfloor \frac{3(n-3)^2+1}{2} \right\rfloor + 10n - 32 = \xi^c(E_{n,n+2})$. □

Theorem 2.4. *Let G be a connected graph with n vertices and m edges, where $n \geq 11$ and $m = n+3$. Then $\xi^c(G) \leq \xi^c(E_{n,m})$, with equality if and only if $G \cong E_{n,m}$.*

Proof. If there is an edge e such that $\omega(e) \leq 2n - 12$ and $G - e$ is connected, then we have

$$\xi^c(G) \leq \xi^c(G-e) + 2n - 12 \leq \xi^c(E_{n,n+2}) + 2n - 12 = \xi^c(E_{n,n+3}).$$

If $G - e \cong E_{n,n+2}$, then G is obtained by adding an edge to $E_{n,n+2}$, and the first inequality above is strict. If $G - e \ncong E_{n,n+2}$, then by Theorem 2.3 $\xi^c(G-e) < \xi^c(E_{n,n+2})$ and $\xi^c(G) < \xi^c(E_{n,n+3})$. Thus the conclusion is correct for $Diam(G) \leq n - 6$.

Let $Diam(G) = n - 5$. If $n \leq 13$, then $\xi^c(G) \leq 2(n-5)m < \left\lfloor \frac{3(n-3)^2+1}{2} \right\rfloor + 12n - 44 = \xi^c(E_{n,n+3})$.

Let $n \geq 14$. Suppose that $v_0 v_1 \ldots v_{n-7} v_{n-6} v_{n-5}$ is a diametral path of G, and v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1} are the vertices not in the given diametral path. We need to consider two cases:(a) there exists an edge $v_i v_{i+1}$ $(1 \leq i \leq n-7)$ in a non-trivial block of G(the block has at least three edges) and (b) each edge $v_i v_{i+1}$ for $1 \leq i \leq n-7$ is a cut edge of G.

Case (a): there exists an edge $v_i v_{i+1}$ $(1 \leq i \leq n-7)$ in a non-trivial block of G. Let $k \in \{n-4, n-3, n-2, n-1\}$.

If the path $P_{v_i v_k}$ of length $d(v_i, v_k)$ passes a cut vertex in the given diametral path, then $d(v_i, v_k) \leq \max\{d(v_i, v_0), d(v_i, v_{n-5})\}$.

If it does not pass any cut vertex in the given diametral path, then there exists i such that $d(v_i, v_k) \leq 10/2 \leq \frac{14-3}{2} \leq \max\{d(v_i, v_0), d(v_i, v_{n-5})\}$.

Thus there is an edge $v_i v_{i+1}$ whose weight is not greater than $2n - 12$. From the first part of the theorem, the conclusion is correct.

Case (b): each edge $v_i v_{i+1}$ for $1 \leq i \leq n-7$ is a cut edge of G. In this case, for any vertex u, $ec(u) = \max\{d(u, v_0), d(u, v_{n-5})\}$, by comparing the sum of the weights of edges in G with these of $E_{n,m}$, we obtain the result.

Let $Diam(G) = n-4$, and $v_0v_1\ldots v_{n-4}$ be a diametral path. Since $n \geq 11$, we can show that $ec(v_i) = \max\{d(v_i, v_0), d(v_i, v_{n-4})\}$ for $0 \leq i \leq n-4$ similarly. Then

$$\xi^c(G) \leq \xi^c(P_{n-3}) + (m-n+4)(2n-8) = \left\lfloor \frac{3(n-4)^2+1}{2} \right\rfloor + (m-n+4)(2n-8) < \xi^c(E_{n,m}).$$

Let $Diam(G) = n-3$. Then there are exactly two vertices not in a given diametral path, each of which is adjacent to at least two vertices of the path, and the sum of the weights of edges not in the given diametral path of G is not greater than that of $E_{n,m}$. Thus we complete the proof. □

Since $\xi^c(E_{n,n+4}) - \xi^c(E_{n,n+3}) = 2n-12$ and $d_{n,n+4} = d_{n,n+3} = n-3$, we can obtain the following theorem similarly.

Theorem 2.5. *Let G be a connected graph with n vertices and m edges, where $n \geq 11$ and $m = n+4$. Then $\xi^c(G) \leq \xi^c(E_{n,m})$, with equality if and only if $G \cong E_{n,m}$.*

Finally, we propose

Conjecture 2.1. *Let $d_{n,m} \geq 3$. Then $E_{n,m}$ is the unique graph with maximal eccentric connectivity index among all connected graphs with n vertices and m edges.*

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