Appl. Math. J. Chinese Univ.
2013, 28(4): 465-484

Wiener's lemma: localization and various approaches

SHIN Chang Eon[1] SUN Qi-yu[2]

In celebration of Professor Chen Jiangong 120th anniversary

Abstract. Matrices and integral operators with off-diagonal decay appear in numerous areas of mathematics including numerical analysis and harmonic analysis, and they also play important roles in engineering science including signal processing and communication engineering. Wiener's lemma states that the localization of matrices and integral operators are preserved under inversion. In this introductory note, we re-examine several approaches to Wiener's lemma for matrices. We also review briefly some recent advances on localization preservation of operations including nonlinear inversion, matrix factorization and optimization.

§1 Introduction

Let us start with recalling Lemma IIe in [49] by N. Wiener: "*If $f(x)$ is a function with an absolutely convergent Fourier series, which nowhere vanishes for real arguments, $1/f(x)$ has an absolutely convergent Fourier series.*" The above famous statement is now referred to as the classical Wiener's lemma.

Let $\mathcal{W}$ contain all periodic functions with absolutely convergent Fourier series. Then we can restate the classical Wiener's lemma as follows.

Theorem 1.1. *If $f \in \mathcal{W}$ and $f(t) \neq 0$ for all $t \in \mathbb{R}$, then $1/f \in \mathcal{W}$.*

Define

$$\|f\|_{\mathcal{W}} := \sum_{n\in\mathbb{Z}} |\hat{f}(n)|, \ \ f \in \mathcal{W},$$

if f has the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$. We may verify that $\mathcal{W}$ is a Banach algebra under the function multiplication. Here a Banach space $\mathcal{A}$ with norm $\|\cdot\|_{\mathcal{A}}$ is said to be a *Banach algebra* if it contains a unit element I, it has operation of multiplications possessing the usual algebraic properties, and $\|AB\|_{\mathcal{A}} \leq \|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}$ for all $A, B \in \mathcal{A}$.

Received: 2013-09-23.

MR Subject Classification: 46H10, 47B37, 15A09, 47A10, 46A45.

Keywords: Wiener's lemma, infinite matrix, stability, Wiener algebra, Beurling algebra, off-diagonal decay, inverse closedness.

Digital Object Identifier(DOI): 10.1007/s11766-013-3215-6.

The authors are partially supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2013R1A1A2005402), and National Science Foundation (DMS-1109063).

Given a Banach algebra $\mathcal{A}$, the family $\mathcal{W}(\mathcal{A})$ of all periodic functions $f(t) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$, $t \in \mathbb{R}$, with

$$\|f\|_{\mathcal{W}(\mathcal{A})} := \sum_{n\in\mathbb{Z}} \|\hat{f}(n)\|_{\mathcal{A}} < \infty$$

is also a Banach algebra, where the product of $f(t) \cdot g(t)$ is the function $f(t)g(t)$. The above Banach algebra $\mathcal{W}(\mathcal{A})$ is a noncommutative extension of the Wiener algebra $\mathcal{W}$. The following celebrated Bochner-Phillips theorem [10, Theorem 1] is a generalization of the classical Wiener's lemma from complex numbers to a Banach algebra $\mathcal{A}$.

Theorem 1.2. *Let $f \in \mathcal{W}(\mathcal{A})$. If $f(t)$ has left inverse in $\mathcal{A}$ for every $t \in \mathbb{R}$, then f has a left inverse in $\mathcal{W}(\mathcal{A})$.*

Let $\ell^q := \ell^q(\mathbb{Z}), 0 < q \le \infty$, be the space of all q-summable sequences on $\mathbb{Z}$ with its standard (quasi-)norm denoted by $\|\cdot\|_q$. We may associate a sequence $a := (a(n))_{n\in\mathbb{Z}} \in \ell^1$ with a matrix $A := \big(a(m-n)\big)_{m,n\in\mathbb{Z}}$ in $\mathcal{B}(\ell^q), 1 \le q \le \infty$, the Banach space of all bounded linear operators on ℓ^q under the standard operator norm. Denote the family of those matrices by

$$\tilde{\mathcal{W}} := \Big\{\big(a(m-n)\big)_{m,n\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} |a(k)| < \infty\Big\}. \tag{1.1}$$

Then the classical Wiener's lemma has the following equivalent matrix formulation: *If $A \in \tilde{\mathcal{W}}$ and A is invertible in $\mathcal{B}(\ell^2)$, then $A^{-1} \in \tilde{\mathcal{W}}$.*

Let

$$\mathcal{C}_1 := \Big\{\big(a(m,n)\big)_{m,n\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} \Big(\sup_{m-n=k} |a(m,n)|\Big) < \infty\Big\}. \tag{1.2}$$

The above family $\mathcal{C}_1$ of matrices is known as the *Baskakov-Gohberg-Sjöstrand class* [4, 17, 22, 36, 40]. Any matrix A in the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$ is a bounded linear operator on ℓ^q, where $1 \le q \le \infty$. Hence

$$\tilde{\mathcal{W}} \subset \mathcal{C}_1 \subset \mathcal{B}(\ell^q),\ 1 \le q \le \infty.$$

Applying Bochner-Phillips theorem, we have the following noncommutative extension of the classical Wiener's lemma to matrices in the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$.

Theorem 1.3. *Let A be a matrix in the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$. If A is invertible in $\mathcal{B}(\ell^2)$, then its inverse A^{-1} belongs to $\mathcal{C}_1$.*

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras with common identity I and assume that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$. We say that $\mathcal{A}$ is *inverse-closed* in $\mathcal{B}$ if $A \in \mathcal{A}$ and $A^{-1} \in \mathcal{B}$ implies $A^{-1} \in \mathcal{A}$. Inverse-closedness occurs under various names, such as spectral invariance, Wiener pair, local subalgebra, etc [16, 30, 46]. As the classical Wiener's lemma can be also stated as that the Wiener algebra $\tilde{\mathcal{W}}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$, we call the inverse-closed property for a Banach subalgebra as *Wiener's lemma for that subalgebra*.

Inverse-closedness (= Wiener's lemma) has been established for matrices and integral operators with various off-diagonal decay, see the survey papers [18, 27], samples of recent publications [6, 13, 14, 20, 21, 28, 33, 42, 43] and references therein. There are several ways to prove the inverse-closedness for matrices and integral operators, such as the Wiener's localization

[49], the Gelfand's technique [16], the Hulanicki's spectral method [24], the Brandenburg's trick [11], the Jaffard's boot-strap argument [25], and the Sjöstrand's commutator estimates [36]. In this introductory report, we present several techniques to establish localization preservation of various operations, such as inversion, factorization and optimization.

A Banach algebra $\mathcal{A}$ is said to be a **-algebra* if there is a continuous linear *involution* $*$ on $\mathcal{A}$ with the property that

$$(AB)^* = B^*A^* \ \text{ and } \ A^{**} = A \ \text{ for all } \ A, B \in \mathcal{A}.$$

A *-algebra $\mathcal{A}$ is called *symmetric* if $\sigma_{\mathcal{A}}(A^*A) \subset [0, \infty)$ for any $A \in \mathcal{A}$. The operator algebra $\mathcal{B}(\ell^2)$ is a symmetric *-algebra under the operator adjoint, while $\mathcal{B}(\ell^p)$ with $p \neq 2$ is not.

Define the *spectral set* $\sigma_{\mathcal{A}}(A)$ of A in a Banach algebra $\mathcal{A}$ with identity I by

$$\sigma_{\mathcal{A}}(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible in } \mathcal{A}\}$$

and the spectral radius $\rho_{\mathcal{A}}(A)$ of $A \in \mathcal{A}$ by

$$\rho_{\mathcal{A}}(A) := \max\{|\lambda| : \lambda \in \sigma_{\mathcal{A}}(A)\}.$$

For Banach algebras $\mathcal{A}$ and $\mathcal{B}$ with common identity I and the property that $\mathcal{A}$ is a subalgebra of $\mathcal{B}$,

$$\rho_{\mathcal{A}}(A) = \rho_{\mathcal{B}}(A) \quad \text{ for all } A \in \mathcal{A},$$

if $\mathcal{A}$ is inverse-closed in $\mathcal{B}$. The following famous Hulanicki's lemma shows that the converse holds for symmetric *-algebras [24].

Theorem 1.4. *Let $\mathcal{A} \subset \mathcal{B}$ be two *-algebras with common identity and involution. If $\mathcal{B}$ is a symmetric Banach algebra, then $\mathcal{A}$ is inverse-closed in $\mathcal{B}$ if and only if $\rho_{\mathcal{A}}(A) = \rho_{\mathcal{B}}(A)$ for all $A = A^* \in \mathcal{A}$.*

Given $0 < q \leq \infty$ and a weight $u = (u(n))_{n\in\mathbb{Z}}$ (a positive function on $\mathbb{Z}$), let

$$\mathcal{A}_{q,u} := \big\{A : \|A\|_{\mathcal{A}_{q,u}} < \infty\big\}, \tag{1.3}$$

where for $A := (a(m,n))_{m,n\in\mathbb{Z}}$,

$$\|A\|_{\mathcal{A}_{q,u}} := \max\Big\{\sup_{m\in\mathbb{Z}} \|(a(m,n)u(m-n))_{n\in\mathbb{Z}}\|_q, \sup_{n\in\mathbb{Z}} \|(a(m,n)u(m-n))_{m\in\mathbb{Z}}\|_q\Big\}.$$

The above family $\mathcal{A}_{q,u}, 0 < q \leq \infty$, of matrices is known as the *Gröchenig-Schur class* [22, 29, 34, 38, 40], while for $q = \infty$ it is also referred to as the *Jaffard class* [25]. For brevity, we write $\mathcal{A}_q$ instead of $\mathcal{A}_{q,u}$ when $u \equiv 1$ is the trivial weight.

The Gröchenig-Schur class $\mathcal{A}_1$ is a Banach algebra containing the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$, i.e.,

$$\mathcal{C}_1 \subset \mathcal{A}_1,$$

but unlike the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$, it is not an inverse-closed Banach subalgebra of $\mathcal{B}(\ell^2)$ [48]. Applying the Hulanicki's spectral method and the Brandenburg's trick, we obtain that the Gröchenig-Schur class $\mathcal{A}_{q,w_\alpha}$ with polynomial weights $w_\alpha(t) := (1+|t|)^\alpha$ are inverse-closed Banach subalgebras of $\mathcal{B}(\ell^2)$, provided that $1 \leq q \leq \infty$ and $\alpha > 1 - 1/q$ [3, 22, 25, 38, 40].

Theorem 1.5. *Let $1 \le q \le \infty$ and $w_\alpha(t) := (1+|t|)^\alpha$ with $\alpha > 1 - 1/q$. Then the Gröchenig-Schur class $\mathcal{A}_{q,w_\alpha}$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$.*

Let $\mathcal{B}_1$ contain all matrices $A := (a(m,n))_{m,n\in\mathbb{Z}}$ such that

$$|a(m,n)| \le b(m-n) \quad \text{for all } m,n \in \mathbb{Z}$$

for some sequence $b := \{b(n)\}_{n\in\mathbb{Z}}$ being summable ($b \in \ell^1$), symmetric ($b(-n) = b(n)$ for all $n \in \mathbb{Z}$), and radically decreasing ($b(n) \le b(m)$ for all integers m, n with $|n| \ge |m|$) [42], i.e.,

$$\mathcal{B}_1 := \{A : \|A\|_{\mathcal{B}_1} < \infty\}, \tag{1.4}$$

where

$$\|A\|_{\mathcal{B}_1} := \sum_{k\in\mathbb{Z}} \Big(\sup_{|m-n|\ge|k|} |a(m,n)| \Big).$$

We call $\mathcal{B}_1$ the *Beurling class* since it is a noncommutative matrix extension of the Beurling algebra

$$A^*(\mathbb{T}) := \Big\{ \sum_{n\in\mathbb{Z}} a(n)e^{int} : \sum_{k\in\mathbb{Z}} \sup_{|n|\ge|k|} |a(n)| < \infty \Big\}$$

introduced by A. Beurling for establishing contraction properties of periodic functions [8]. The Beurling class $\mathcal{B}_1$ is a unital Banach algebra under matrix multiplication, and it is contained in the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$ (and hence also in the Gröchenig-Schur class $\mathcal{A}_1$),

$$\mathcal{B}_1 \subset \mathcal{C}_1 \subset \mathcal{A}_1.$$

Given $1 \le p < \infty$ and a weight $w := (w(n))_{n\in\mathbb{Z}}$, let

$$\ell^p_w := \Big\{ c : \|c\|_{p,w} ::= \Big(\sum_{n\in\mathbb{Z}} |c(n)|^p w(n) \Big)^{1/p} < \infty \Big\} \tag{1.5}$$

contain all weighted p-summable sequences $c := (c(n))_{n\in\mathbb{Z}}$ on $\mathbb{Z}$. We say that w is a *discrete Muckenhoupt A_p-weight* if

$$\sup_{m>n} (m-n)^{-p} \Big(\sum_{k=n}^{m} w(k) \Big) \Big(\sum_{k=n}^{m} (w(k))^{-1/(p-1)} \Big)^{p-1} < \infty \ \text{ if } 1 < p < \infty,$$

and

$$\sup_{m>n} \frac{(m-n)^{-1} \sum_{k=n}^{m} w(k)}{\inf_{n\le k\le m} w(k)} < \infty \ \text{ if } p = 1.$$

The polynomial weights $((1+|n|)^\alpha)_{n\in\mathbb{Z}}$ with $-1 < \alpha < p-1$ and $1 \le p < \infty$ are discrete Muckenhoupt A_p-weights. For $c := (c(n))_{n\in\mathbb{Z}}$, define the *discrete maximal function* Mc by

$$Mc(n) := \sup_{0\le N\in\mathbb{Z}} \frac{1}{2N+1} \sum_{m=n-N}^{n+N} |c(m)|, \ n \in \mathbb{Z}. \tag{1.6}$$

For any matrix A in the Beurling class $\mathcal{B}_1$ and any vector c, Ac is dominated by a multiple of the dominated function Mc,

$$|Ac(n)| \le 5\|A\|_{\mathcal{B}_1} Mc(n), \ n \in \mathbb{Z}.$$

Therefore any matrix A in the Beurling class $\mathcal{B}_1$ defines a bounded operator on the weighted sequence space ℓ^p_w,

$$\mathcal{B}_1 \subset \mathcal{B}(\ell^p_w),$$

where $1 \le p < \infty$ and w is a discrete Muckenhoupt A_p-weight. The reader may refer to [15, 37] for the theory of weighted inequalities and its ramifications.

For the Beurling algebra $A^*(\mathbb{T})$, it is shown that any function $f \in A^*(\mathbb{T})$ with $f(t) \ne 0$ for all $t \in \mathbb{R}$ has $1/f \in A^*(\mathbb{T})$ [7]. For the Beurling algebra $\mathcal{B}_1$, applying the Sjöstrand commutator estimates, we can prove Wiener's lemma for the Beurling subalgebra $\mathcal{B}_1$ of $\mathcal{B}(\ell^p_w)$ [42].

Theorem 1.6. *Let $1 \le p < \infty$ and w be a discrete Muckenhoupt A_p-weight. Then $\mathcal{B}_1$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^p_w)$.*

This note is organized as follows. In Section 2, we recall the Wiener's localization for Fourier series with slight modification and provide a proof of Wiener's lemma for the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$.

Given a Banach algebra $\mathcal{B}$, we say that its Banach subalgebra $\mathcal{A}$ is a *differential subalgebra of order* $\theta \in (0, 1]$ ([9, 38, 43]) if the norm $\|\cdot\|_{\mathcal{A}}$ on $\mathcal{A}$ is a differential norm of order θ, i.e., there exists a positive constant C such that

$$\|AB\|_{\mathcal{A}} \le C\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\Big(\Big(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}}\Big)^{\theta} + \Big(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}}\Big)^{\theta}\Big) \quad \text{for all } A, B \in \mathcal{A}. \tag{1.7}$$

The differential subalgebras have been widely used in operator theory and non-commutative geometry [9, 26, 32] and they could also be important in numerical analysis and optimization [29, 43, 44]. In Section 3, we first apply the Hulanicki's spectral method and the Brandenburg's trick to prove that a differential subalgebra of a symmetric *-algebra is inverse-closed (Theorem 3.1). In that section, we then show that the Gröchenig-Schur class associated with polynomial weights is a differential subalgebra of $\mathcal{B}(\ell^2)$ (Theorem 3.2), and hence an inverse-closed subalgebra in $\mathcal{B}(\ell^2)$.

The Sjöstrand's commutator estimates were introduced in [36] to provide an independent proof of the conclusion that the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$ is inverse-closed in $\mathcal{B}(\ell^2)$. The variations are used in showing stability of localized matrices and integral operators for different (un)weighted spaces, and inverse-closed subalgebras of a non-symmetric Banach algebra [2, 13, 33, 42, 47]. Included in Section 4 is Wiener's lemma for the Beurling class $\mathcal{B}_1$ in the non-symmetric algebra $\mathcal{B}(\ell^q), q \ne 2$, the unweighted version of Theorem 1.6.

Wiener's lemmas for matrices and integral operators can be informally interpreted as localization preservation under inversion. Such a localization preservation is of great importance in applied harmonic analysis, numerical analysis, optimization and many mathematical and engineering fields. In Section 5, as the supplement to survey papers [18, 27], we review briefly some recent advances on localization preservation under (non)linear operations such as matrix factorization and optimization.

All proofs in this note are based on the original arguments in the literature, mostly their simplified versions. For motivation, various applications, and historical remarks on Wiener's lemma for matrices, integral operators and pseudodifferential operators, we refer the reader to [18, 27, 42].

§2 Wiener's localization for Fourier series

In this section, we first recall the localization technique for Fourier series in [49], see Lemma 2.4. We then apply the Bochner-Phillips theorem to prove Wiener's lemma for the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$.

2.1 Wiener's localization for Fourier series

In this subsection, we follow Wiener's original arguments [49] with slight modification to prove Theorem 1.1. To do so, we need several lemmas. The first lemma shows that $\mathcal{W}$ is an algebra (cf. [49, Lemma IIa]).

Lemma 2.1. *If f and g belong to $\mathcal{W}$, then their product fg belongs to $\mathcal{W}$ too. Moreover,*

$$\|fg\|_{\mathcal{W}} \le \|f\|_{\mathcal{W}}\|g\|_{\mathcal{W}}. \tag{2.1}$$

Proof. Clearly it suffices to prove (2.1). Take $f, g \in \mathcal{W}$, set $h = fg$, and let $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{int}$, $\sum_{n\in\mathbb{Z}} \hat{g}(n)e^{int}$ and $\sum_{n\in\mathbb{Z}} \hat{h}(n)e^{int}$ be the Fourier series of f, g and h respectively. One may verify that

$$\hat{h}(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\hat{g}(n-m), n \in \mathbb{Z}.$$

Thus

$$\begin{aligned}
\|fg\|_{\mathcal{W}} &= \sum_{n\in\mathbb{Z}} |\hat{h}(n)| \le \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\hat{f}(m)||\hat{g}(n-m)| \\
&= \sum_{m\in\mathbb{Z}} |\hat{f}(m)|\Big(\sum_{n\in\mathbb{Z}} |\hat{g}(n-m)|\Big) = \Big(\sum_{m\in\mathbb{Z}} |\hat{f}(m)|\Big)\Big(\sum_{n\in\mathbb{Z}} |\hat{g}(n)|\Big) \\
&= \|f\|_{\mathcal{W}}\|g\|_{\mathcal{W}}.
\end{aligned}$$

□

The second lemma, c.f. [49, Lemma IIb], says that any function coincident with functions in $\mathcal{W}$ locally belongs to $\mathcal{W}$.

Lemma 2.2. *Let f be a periodic function. If for any t_0 there exist $\epsilon_{t_0} > 0$ and a function $f_{t_0} \in \mathcal{W}$ such that $f(t) = f_{t_0}(t)$ for all $t \in (t_0 - \epsilon_{t_0}, t_0 + \epsilon_{t_0})$, then $f \in \mathcal{W}$.*

Proof. By the compactness of the set $[-\pi, \pi]$, there exists a finite periodic covering $\{(t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i}) + 2\pi\mathbb{Z}\}_{i=1}^N$ to the real line $\mathbb{R}$, where $N \ge 1$. Associated with the above finite periodic covering, we can find a periodic unit partition $\varphi_i, 1 \le i \le N$, such that $\varphi_i, 1 \le i \le N$, are smooth periodic functions supported in $(t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i}) + 2\pi\mathbb{Z}$ (hence $\varphi_i \in \mathcal{W}$) and $\sum_{i=1}^N \varphi_i(t) = 1$ for all $t \in \mathbb{R}$. Let $f_{t_i} \in \mathcal{W}$ be the periodic function coincident with f on $(t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i}) + 2\pi\mathbb{Z}, 1 \le i \le N$. Then $\varphi_i f_{t_i} \in \mathcal{W}$ for all $1 \le i \le N$ by Lemma 2.1, and

$$f = \sum_{i=1}^N \varphi_i f = \sum_{i=1}^N \varphi_i f_{t_i} \in \mathcal{W}.$$

□

The third lemma, c.f. [49, Lemma IIc], is related to the convergence of Neumann series in the algebra $\mathcal{W}$.

Lemma 2.3. *If the Fourier series $\sum_{n\in\mathbb{Z}} \hat f(n)e^{int}$ of a function $f \in \mathcal{W}$ satisfies $|\hat f(0)| > \sum_{n\neq 0} |\hat f(n)|$, then $1/f \in \mathcal{W}$.*

Proof. Let $g = f - \hat f(0)$. Then $g \in \mathcal{W}$ and $\|g\|_{\mathcal{W}} < |\hat f(0)|$ by our assumption. Hence

$$\begin{aligned}\Big\|\frac{1}{f}\Big\|_{\mathcal{W}} &= \frac{1}{|\hat f(0)|}\Big\|\sum_{n=0}^{\infty}\Big(-\frac{g}{\hat f(0)}\Big)^n\Big\|_{\mathcal{W}} \le \frac{1}{|\hat f(0)|}\sum_{n=0}^{\infty}\Big\|\Big(-\frac{g}{\hat f(0)}\Big)^n\Big\|_{\mathcal{W}}\\ &\le \frac{1}{|\hat f(0)|}\sum_{n=0}^{\infty}\Big(\frac{\|g\|_{\mathcal{W}}}{|\hat f(0)|}\Big)^n = \frac{1}{|\hat f(0)| - \|g\|_{\mathcal{W}}} < \infty,\end{aligned}$$

where the first equality follows from Neumann series

$$\hat f(0)/f = 1 + (-g/\hat f(0)) + (-g/\hat f(0))^2 + \cdots$$

and the second inequality holds by Lemma 2.1. □

The key lemma, c.f. [49, Lemma IId], introduces localization technique for functions in the algebra $\mathcal{W}$.

Lemma 2.4. *If $f \in \mathcal{W}$ satisfies $f(t_0) \neq 0$ for some $t_0 \in \mathbb{R}$, then there exist $\epsilon > 0$ and a function $g_\epsilon \in \mathcal{W}$ such that*

$$g_\epsilon(t) = f(t) \quad \text{for all } t \in (t_0 - \epsilon, t_0 + \epsilon), \tag{2.2}$$

and the Fourier series $\sum_{n=-\infty}^{\infty} \hat g_\epsilon(n)e^{int}$ of g_ϵ has the property that

$$|\hat g_\epsilon(0)| > \sum_{n\neq 0} |\hat g_\epsilon(n)|. \tag{2.3}$$

Proof. Without loss of generality, we assume that $t_0 = 0$. Let $\epsilon > 0$ be chosen later. Take an auxiliary periodic function $\varphi_\epsilon(t)$ whose restriction on $[-\pi, \pi]$ is $\varphi(t/\epsilon)$ for some smooth function $\varphi(t)$ such that $\varphi(t) = 1$ for $|t| \le 1$ and $\varphi(t) = 0$ for $|t| \ge 2$. One may verify that the Fourier series of the function φ_ϵ is given by $\epsilon \sum_{n\in\mathbb{Z}} \hat\varphi(\epsilon n)e^{int}$ and hence

$$\varphi_\epsilon \in \mathcal{W}, \tag{2.4}$$

where $\hat\varphi(\xi) = (2\pi)^{-1}\int_{\mathbb{R}} \varphi(t)e^{-it\xi}dt$ is the Fourier transform of φ on the real line.

Define

$$g_\epsilon := \varphi_\epsilon f + f(0)(1 - \varphi_\epsilon). \tag{2.5}$$

Then g_ϵ satisfies (2.2) and belongs to $\mathcal{W}$ by (2.4), the assumption $f \in \mathcal{W}$ and Lemma 2.1.

Now it remains to verify that the Fourier series $\sum_{n\in\mathbb{Z}} \hat g_\epsilon(n)e^{int}$ of g_ϵ satisfies (2.3). Let f have the Fourier series $\sum_{n\in\mathbb{Z}} \hat f(n)e^{int}$. By (2.5),

$$\begin{aligned}\hat g_\epsilon(n) &= f(0)\delta_{n0} + \epsilon\sum_{m\in\mathbb{Z}} \hat\varphi(\epsilon(n-m))\hat f(m) - f(0)\epsilon\hat\varphi(\epsilon n)\\ &= f(0)\delta_{n0} + \epsilon\sum_{m\in\mathbb{Z}} \big(\hat\varphi(\epsilon(n-m)) - \hat\varphi(\epsilon n)\big)\hat f(m),\ n \in \mathbb{Z},\end{aligned}$$

where δ_n stands for the Kronecker symbol. Therefore

$$\begin{aligned} |\hat{g}_\epsilon(0)| - \sum_{n\neq 0} |\hat{g}_\epsilon(n)| \quad &\geq \quad |f(0)| - \epsilon \sum_{n,m\in\mathbb{Z}} |\hat{\varphi}(\epsilon(n-m)) - \hat{\varphi}(\epsilon n)||\hat{f}(m)| \\ &=: \quad |f(0)| - \sum_{m\in\mathbb{Z}} a_\epsilon(m)|\hat{f}(m)|, \end{aligned} \tag{2.6}$$

where

$$a_\epsilon(m) = \epsilon \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\epsilon(n-m)) - \hat{\varphi}(\epsilon n)|, \; m \in \mathbb{Z}.$$

As φ is a compactly supported smooth function on the real line,

$$|\hat{\varphi}(\xi)| \leq C(1+|\xi|)^{-2} \text{ and } |(\hat{\varphi})'(\xi)| \leq C(1+|\xi|)^{-2}, \; \xi \in \mathbb{R},$$

where C is a positive constant. Therefore

$$0 \leq a_\epsilon(m) \leq 2\epsilon \sum_{n\in\mathbb{Z}} |\hat{\varphi}(\epsilon n)| \leq C_0 \epsilon \sum_{n\in\mathbb{Z}} (1+\epsilon|n|)^{-2} \leq C_0,$$

and

$$a_\epsilon(m) \leq C\epsilon \sum_{n\in\mathbb{Z}} \int_{-\epsilon|m|}^{\epsilon|m|} (1+|\epsilon n - t|)^{-2} dt \leq C_1 \epsilon |m|, \; m \in \mathbb{Z},$$

where C_0 and C_1 are positive constants independent of $\epsilon \in (0,1)$ and $m \in \mathbb{Z}$. Letting ϵ be chosen sufficiently small, the desired estimate (2.3) for the function g_ϵ follows from (2.6) by the dominated convergence theorem. □

We finish this subsection with the proof of the classical Wiener's lemma.

Proof of Theorem 1.1. Let $f \in \mathcal{W}$ with $f(t) \neq 0$ for all $t \in \mathbb{R}$. Then $g = 1/f$ is a continuous function. For any $t_0 \in \mathbb{R}$, there exist $\epsilon > 0$ and a function $f_{t_0} \in \mathcal{W}$ by Lemma 2.4 such that

$$f(t) = f_{t_0}(t) \quad \text{for all } t \in (t_0 - \epsilon, t_0 + \epsilon)$$

and the Fourier series $\sum_{n\in\mathbb{Z}} \hat{f}_{t_0}(n) e^{int}$ of f_{t_0} satisfies

$$|\hat{f}_{t_0}(0)| > \sum_{n\neq 0} |\hat{f}_{t_0}(n)|.$$

Then the function $g_{t_0} := 1/f_{t_0}$ belongs to $\mathcal{W}$ by Lemma 2.3, and it satisfies

$$g(t) = g_{t_0}(t) \quad \text{for all } t \in (t_0 - \epsilon, t_0 + \epsilon).$$

Therefore $g = 1/f$ belongs to $\mathcal{W}$ by Lemma 2.2. □

2.2 Wiener's lemma for the Baskakov-Gohberg-Sjöstrand class

In this subsection, we prove Theorem 1.3.

Proof of Theorem 1.3. Take $A := (a(m,n))_{m,n\in\mathbb{Z}} \in \mathcal{C}_1$ with A being invertible in $\mathcal{B}(\ell^2)$. Let $e(t)$ be the diagonal matrix with diagonal entries $e^{imt}, m \in \mathbb{Z}$, and define $x(t) := e(t)A(e(t))^{-1} = e(t)Ae(-t)$. Then $x(t)$ is invertible in $\mathcal{B}(\ell^2)$ for all real t, and its inverse is given by

$$(x(t))^{-1} = e(t)A^{-1}(e(t))^{-1}. \tag{2.7}$$

Define $A_k = (a_k(m,n))_{m,n\in\mathbb{Z}}, k \in \mathbb{Z}$, by $a_k(m,n) = a(m,n)$ if $m-n=k$ and 0 otherwise. Then

$$\sum_{k\in\mathbb{Z}} \|A_k\|_{\mathcal{B}(\ell^2)} = \sum_{k\in\mathbb{Z}} \sup_{|m-n|=k} |a(m,n)| < \infty.$$

Observe that $x(t)$ has the Fourier series $\sum_{k\in\mathbb{Z}} A_k e^{ikt}$. Thus the Fourier series $\sum_{k\in\mathbb{Z}} B_k e^{ikt}$ of $(x(t))^{-1}$ satisfies

$$\sum_{k\in\mathbb{Z}} \|B_k\|_{\mathcal{B}(\ell^2)} < \infty$$

by Theorem 1.2 with $\mathcal{A}$ replaced by $\mathcal{B}(\ell^2)$. Observe that

$$\|B_k\|_{\mathcal{B}(\ell^2)} = \sup_{m-n=k} |b(m,n)|, \ k \in \mathbb{Z}$$

by (2.7), where $A^{-1} = (b(m,n))_{m,n\in\mathbb{Z}}$. This together with (2.7) proves that $A^{-1} \in \mathcal{C}_1$. □

§3 Spectral method for symmetric algebras

In this section, we show that a differential *-subalgebra of a symmetric *-algebra is inverse-closed, and then we apply it to establish Wiener's lemma for the Gröchenig-Schur class.

Replacing both A and B in (1.7) by $A^n, n \geq 1$, leads to

$$\|A^{2n}\|_{\mathcal{A}} \leq 2C\|A^n\|_{\mathcal{A}}^{2-\theta}\|A^n\|_{\mathcal{B}}^{\theta} \quad \text{for all } A \in \mathcal{A}.$$

Taking n-th roots and then letting $n \to \infty$ yields

$$\rho_{\mathcal{A}}(A) \leq \rho_{\mathcal{B}}(A) \quad \text{for all } A \in \mathcal{A}.$$

The above Brandenburg's trick together with the Hulanicki's lemma establishes Wiener's lemma for symmetric *-algebras [11, 19, 20, 38, 40, 42].

Theorem 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two *-algebras with common identity and involution. If $\mathcal{B}$ is a symmetric Banach algebra and $\mathcal{A}$ is a differential subalgebra of $\mathcal{B}$ of order $\theta \in (0,1]$, then $\mathcal{A}$ is inverse-closed in $\mathcal{B}$.*

Let $1 \leq q \leq \infty$ and set $q' = q/(q-1)$. Matrices in the Gröchenig-Schur class $\mathcal{A}_{q,u}$ are bounded linear operators on ℓ^2 if

$$\|u^{-1}\|_{q'} < \infty. \tag{3.1}$$

In fact,

$$\|A\|_{\mathcal{B}(\ell^2)} \le \|u^{-1}\|_{q'} \|A\|_{\mathcal{A}_{q,u}} \quad \text{for all } A \in \mathcal{A}_{q,u}.$$

The Gröchenig-Schur class $\mathcal{A}_{q,u}$ is a *-algebra under the matrix multiplication and the matrix conjugate if there exists another weight v, called the companion weight, such that

$$u(m+n) \le u(m)v(n) + v(m)u(n) \ \text{ for all } \ m, n \in \mathbb{Z} \tag{3.2}$$

and

$$\|vu^{-1}\|_{q'} < \infty. \tag{3.3}$$

For $q = 1$, the above requirements (3.2) and (3.3) are met if the weight u is *submultiplicative* [22, 40], i.e., there exists a positive constant C such that

$$u(m+n) \le Cu(m)u(n) \ \text{ for all } \ m, n \in \mathbb{Z}.$$

Denote by $\chi_{[-\tau,\tau]}$ the characteristic function on the interval $[-\tau, \tau]$. If the weight u and its companion weight v satisfy

$$\inf_{\tau>0} \|v\chi_{[-\tau,\tau]}\|_2 + t\|vu^{-1}(1-\chi_{[-\tau,\tau]})\|_{q'} \le Dt^{1-\theta} \text{ for all } t \ge 1, \tag{3.4}$$

where $D > 0$ and $\theta \in (0, 1)$, then $\mathcal{A}_{q,u}$ is a differential subalgebra of $\mathcal{B}(\ell^2)$ of order θ (and hence it is an inverse-closed subalgebra of $\mathcal{B}(\ell^2)$ by Theorem 3.1). We refer the reader to [13, 19, 20, 21, 25, 33, 38, 40, 41, 42] for various differential subalgebras of matrices and localized integral operators.

Theorem 3.2. *Let $1 \le q \le \infty$ and u be a weight satisfying (3.1)–(3.4) for some $D \in (0, \infty)$ and $\theta \in (0, 1)$. Then $\mathcal{A}_{q,u}$ is a differential algebra of $\mathcal{B}(\ell^2)$ of order θ.*

In literatures, we usually assume that weights have positive lower bounds [22, 40]. In this case, the assumptions (3.1) and (3.3) are satisfied if (3.4) holds. One may also verify that the requirement (3.4) is met if

$$\sup_{\tau>0} \big(\|v\chi_{[-\tau,\tau]}\|_2\big)^{1-\theta} \big(\|vu^{-1}(1-\chi_{[-\tau,\tau]})\|_{q'}\big)^{\theta} < \infty \tag{3.5}$$

and both weight u and its companion weight v are *slow-varying*, i.e.,

$$\sup_{|m-n|\le 1} \frac{v(m)}{v(n)} + \frac{u(m)}{u(n)} < \infty. \tag{3.6}$$

For the polynomial weight $w_\alpha(t) = (1+|t|)^\alpha$ with $\alpha > 1 - 1/q$, we may select $v_\alpha(t) = 2^\alpha$ as its companion weight, since

$$w_\alpha(m+n) \le \big(1 + 2\max(|m|, |n|)\big)^\alpha \le 2^\alpha w_\alpha(m) + 2^\alpha w_\alpha(n), \ m, n \in \mathbb{Z}.$$

Then the requirements (3.1)–(3.4) are met as both w_α and v_α have positive lower bounds, and satisfy (3.5) and (3.6), because

$$\begin{aligned} &\sup_{\tau>0} \big(\|v_\alpha\chi_{[-\tau,\tau]}\|_2\big)^{1-\theta} \big(\|v_\alpha w_\alpha^{-1}(1-\chi_{[-\tau,\tau]})\|_{q'}\big)^{\theta} \\ \le\ & 2^\alpha (2/(q'\alpha - 1))^{\theta/q'} \sup_{\tau>0} (2\tau+1)^{(1-\theta)/2} \tau^{-\theta(\alpha - 1/q')} < \infty \end{aligned}$$

for $\theta = 1/(1+2\alpha-2/q') \in (0,1)$. Then Theorem 1.5 follows from Theorem 3.2 with u replaced by polynomial weight w_α with $\alpha > 1-1/q$.

We finish this subsection with the proof of Theorem 3.2.

Proof of Theorem 3.2. Let $A = (a(m,n))_{m,n\in\mathbb{Z}} \in \mathcal{A}_{q,u}$ and $B = (b(m,n))_{m,n\in\mathbb{Z}} \in \mathcal{A}_{q,u}$, and set $AB = C =: (c(m,n))_{m,n\in\mathbb{Z}}$. Then

$$c(m,n) = \sum_{k\in\mathbb{Z}} a(m,k)b(k,n),\ \ m,n\in\mathbb{Z}.$$

Applying Hölder inequality and using (3.2) yield

$$\|C\|_{\mathcal{A}_{q,u}} \le \|A\|_{\mathcal{A}_{q,u}}\|B\|_{\mathcal{A}_{1,v}} + \|A\|_{\mathcal{A}_{1,v}}\|B\|_{\mathcal{A}_{q,u}}, \tag{3.7}$$

where v is the companion weight in (3.2). Observe that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} |a(m,n)|v(m-n)\\
\le\ & \inf_{\tau>0}\Big\{\Big(\sum_{|m-n|\le\tau}|a(m,n)|^2\Big)^{1/2}\Big(\sum_{|m-n|\le\tau}|v(m-n)|^2\Big)^{1/2}\\
&+\Big(\sum_{|m-n|>\tau}|a(m,n)u(m,n)|^q\Big)^{1/q}\Big(\sum_{|m-n|>\tau}\Big|\frac{v(m-n)}{u(m-n)}\Big|^{q'}\Big)^{1/q'}\Big\}\\
\le\ & \inf_{\tau>0}\big\{\|A\|_{\mathcal{B}(\ell^2)}\|v\chi_{[-\tau,\tau]}\|_2 + \|A\|_{\mathcal{A}_{q,u}}\|vu^{-1}(1-\chi_{[-\tau,\tau]})\|_{q'}\big\}\\
\le\ & D\|A\|^{1-\theta}_{\mathcal{A}_{q,u}}\|A\|^{\theta}_{\mathcal{B}(\ell^2)} \quad \text{for all } m\in\mathbb{Z},
\end{aligned}\tag{3.8}$$

where the last inequality follows from (3.4) and the second one holds as

$$\sup_{m\in\mathbb{Z}}\Big(\sum_{n\in\mathbb{Z}}|a(m,n)|^2\Big)^{1/2} \le \|A\|_{\mathcal{B}(\ell^2)}.$$

Similarly,

$$\sup_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}|a(m,n)|v(m-n) \le D\|A\|^{\theta}_{\mathcal{B}(\ell^2)}\|A\|^{1-\theta}_{\mathcal{A}_{q,u}}. \tag{3.9}$$

Combining (3.8) and (3.9) leads to

$$\|A\|_{\mathcal{A}_{1,v}} \le D\|A\|^{1-\theta}_{\mathcal{A}_{q,u}}\|A\|^{\theta}_{\mathcal{B}(\ell^2)}. \tag{3.10}$$

Applying the same argument gives

$$\|B\|_{\mathcal{A}_{1,v}} \le D\|B\|^{1-\theta}_{\mathcal{A}_{q,u}}\|B\|^{\theta}_{\mathcal{B}(\ell^2)}. \tag{3.11}$$

Combining (3.7), (3.10) and (3.11) proves that $\mathcal{A}_{q,u}$ is a differential subalgebra of $\mathcal{B}(\ell^2)$ of order $\theta\in(0,1)$. □

§4 Commutators for infinite matrices

In this section, we recall commutator estimates for infinite matrices [36, 42], and then we apply them to establish Wiener's lemma for the Beurling class $\mathcal{B}_1$, particularly the following unweighted version of Theorem 1.6.

Theorem 4.1. *The Beurling class $\mathcal{B}_1$ is an inverse-closed subalgebra of $\mathcal{B}(\ell^p), 1 \le p < \infty$.*

Let h be a Lipschitz function such that $h(t) = 1$ if $|t| \le 1$, $h(t) = 0$ if $|t| \ge 2$ and $0 \le h(t) \le 1$ for all $t \in \mathbb{R}$. For instance, the trapezoidal-shaped membership function $h(t) := \min(\max(2 - |t|, 0), 1)$ is such an example. Given $1 \le p \le \infty$ and $A \in \mathcal{B}(\ell^p)$, define localization operators Ψ_i^N and commutators $[\Psi_i^N, A], i \in \mathbb{Z}$, by

$$\Psi_i^N c := \big(h(n/N - i)c(n)\big)_{n\in\mathbb{Z}}$$

and

$$[\Psi_i^N, A]c = \Psi_i^N Ac - A\Psi_i^N c \quad \text{for } c := (c(n))_{n\in\mathbb{Z}} \in \ell^p,$$

where N is a sufficiently large integer.

For $1 \le p < \infty$ and $A \in \mathcal{B}(\ell^p)$, we say that a matrix A has *ℓ^p-stability* if

$$0 < \inf_{\|d\|_p=1} \|Ad\|_p \le \sup_{\|d\|_p=1} \|Ad\|_p < \infty.$$

The ℓ^p-stability is one of the basic assumptions for matrices arising in the study of time-frequency analysis and nonuniform sampling etc (see [1, 12, 23, 39, 43, 45] and the references therein). Any matrix $A \in \mathcal{B}(\ell^p)$ with a left inverse in $\mathcal{B}(\ell^p)$ will have ℓ^p-stability. For matrices having ℓ^p-stability, we have the following localization property.

Lemma 4.2. *Let $1 \le p < \infty$. If $A \in \mathcal{B}_1$ has ℓ^p-stability, then there exist sequences $(V_N(i))_{i\in\mathbb{Z}}$, $N \ge 1$, such that*

$$\lim_{N\to\infty} \sum_{n\in\mathbb{Z}} \Big(\sup_{|i|\ge|n|} V_N(i)\Big) = 0, \tag{4.1}$$

and

$$\big(\inf_{\|d\|_p=1} \|Ad\|_p\big)\|\Psi_i^N c\|_p \le \|\Psi_i^N Ac\|_p + \sum_{j\in\mathbb{Z}} V_N(i-j)\|\Psi_j^N c\|_p \tag{4.2}$$

for all $c \in \ell^p, i \in \mathbb{Z}$ and $N \ge 1$.

Proof. We follow the arguments in [42]. Without loss of generality, we assume that $\inf_{\|d\|_p=1} \|Ad\|_p = 1$. Define the linear operator Φ_N on $\ell^p, 1 \le p < \infty$, by

$$\Phi_N c := \big(H(n/N)c(n)\big)_{n\in\mathbb{Z}} \text{ for } c := (c(n))_{n\in\mathbb{Z}} \in \ell^p,$$

where $H(t) = \big(\sum_{i\in\mathbb{Z}^d}(h(t-i))^2\big)^{-1}, t \in \mathbb{R}$. Then

$$\|\Phi_N c\|_p \le \|c\|_p \ \text{ for all } c := (c(n))_{n\in\mathbb{Z}} \in \ell^p, \tag{4.3}$$

as Φ_N is a diagonal matrix with diagonal entries bounded by one.

For all $i \in \mathbb{Z}$ and $c := (c(n))_{n\in\mathbb{Z}} \in \ell^p$, it follows from the ℓ^p-stability of the matrix A that

$$\begin{aligned}\|\Psi_i^N c\|_p &\le \|A\Psi_i^N c\|_p \le \|\Psi_i^N Ac\|_p + \|(\Psi_i^N A - A\Psi_i^N)c\|_p \\ &\le \|\Psi_i^N Ac\|_p + \sum_{j\in\mathbb{Z}^d} \|(\Psi_i^N A - A\Psi_i^N)\Psi_j^N \Phi_N \Psi_j^N c\|_p. \end{aligned} \tag{4.4}$$

For commutators $[\Psi_i^N, A] := \Psi_i^N A - A\Psi_i^N, i \in \mathbb{Z}$, we have that

$$\begin{aligned}\|[\Psi_i^N, A]\Psi_j^N c\|_p &= \Big\{\sum_{m\in\mathbb{Z}}\Big|\sum_{n\in\mathbb{Z}}\big(h(m/N-i)-h(n/N-i)\big)a(m,n)h(n/N-j)c(n)\Big|^p\Big\}^{1/p} \\ &\le \Big(\sum_{k\in\mathbb{Z}}\min(\|h'\|_\infty|k|/N,1)\big(\sup_{m-n=k}|a(m,n)|\big)\Big)\|c\|_p\end{aligned}$$

if $|i-j| \le 8$, and

$$\begin{aligned}\|[\Psi_i^N, A]\Psi_j^N c\|_p &= \Big(\sum_{m\in\mathbb{Z}}\Big|\sum_{n\in\mathbb{Z}} h(m/N-i)a(m,n)h(n/N-j)c(n)\Big|^p\Big)^{1/p} \\ &\le \Big(\sum_{m\in\mathbb{Z}}\Big|\sum_{(|i-j|-4)N<|m-n|<(|i-j|+4)N}|a(m,n)||c(n)|\Big|^p\Big)^{1/p} \\ &\le \Big(\sum_{(|i-j|-4)N<|k|<(|i-j|+4)N}\big(\sup_{m-n=k}|a(m,n)|\big)\Big)\|c\|_p\end{aligned}$$

if $|i-j| > 8$. The above two estimates for commutators $[\Psi_i^N, A], i \in \mathbb{Z}$, together with (4.3) and (4.4) prove (4.2) with

$$V_N(i) = \begin{cases} \sum_{k\in\mathbb{Z}}\min(\|h'\|_\infty|k|/N,1)\big(\sup_{m-n=k}|a(m,n)|\big) & \text{if } |i| \le 8, \\ \sum_{(|i|-4)N<|k|<(|i|+4)N}\big(\sup_{m-n=k}|a(m,n)|\big) & \text{if } |i| > 8. \end{cases}$$

Now it remains to prove (4.1). Recall that $\|h'\| \ge 1$, we then obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sup_{|i|\ge|k|} V_N(i) &\le 17\sum_{k\in\mathbb{Z}}\min(\|h'\|_\infty|k|/N,1)\big(\sup_{|m-n|\ge|k|}|a(m,n)|\big) \\ &\quad + \sum_{|i|\ge 8}\sum_{(|i|-4)N<|k|<(|i|+4)N}\big(\sup_{|m-n|\ge|k|}|a(m,n)|\big) \\ &\le 17\sum_{k\in\mathbb{Z}}\min(\|h'\|_\infty|k|/N,1)\big(\sup_{|m-n|\ge|k|}|a(m,n)|\big) \\ &\quad +8\sum_{|k|\ge 4N}\big(\sup_{|m-n|\ge|k|}|a(m,n)|\big) \\ &\to 0 \quad \text{as } N \to +\infty\end{aligned}$$

by the dominated convergence theorem and the assumption that $A \in \mathcal{B}_1$. □

The localization technique in Lemma 4.2 could be used to establish equivalence of stability in different sequence spaces [2, 6, 13, 33, 35]. In the following, we apply it to establish Wiener's lemma for the Beurling class $\mathcal{B}_1$ [42].

Proof of Theorem 1.6. Take $A \in \mathcal{B}_1$ that has an inverse $A^{-1} \in \mathcal{B}(\ell^p)$. Without loss of generality, we assume that $\|A^{-1}\|_{\mathcal{B}(\ell^p)} = 1$, otherwise replacing A by $(\|A^{-1}\|_{\mathcal{B}(\ell^p)})A$. Write $A^{-1} := (c(m,n))_{m,n\in\mathbb{Z}}$, set $c_m := (c(n,m))_{n\in\mathbb{Z}}$, and define $c_m^{l_0} := (c_{l_0}(n,m))_{n\in\mathbb{Z}}$ for $l_0 \geq 1$ and $m \in \mathbb{Z}$, where $c_{l_0}(n,m) := c(n,m)$ if $|m-n| \leq l_0$ and 0 otherwise. Then $c_m^{l_0}$ has finite support and

$$\lim_{l_0\to\infty} \|c_m^{l_0} - c_m\|_p = 0. \tag{4.5}$$

By Lemma 4.2, there exist a positive integer N and a sequence $(V_N(i))_{i\in\mathbb{Z}}$ with

$$r_0 := \sum_{k\in\mathbb{Z}} \big(\sup_{|i|\geq|k|} V_N(i)\big) < \frac{1}{5} \tag{4.6}$$

such that

$$\|\Psi_i^N c\|_p \leq \|\Psi_i^N Ac\|_p + \sum_{j\in\mathbb{Z}} V_N(i-j)\|\Psi_j^N c\|_p, \ i \in \mathbb{Z}, \tag{4.7}$$

where $c \in \ell^p$.

Define sequences $V_N^l := (V_N^l(i))_{i\in\mathbb{Z}}, l \geq 1$, by

$$\begin{cases} V_N^l(i) := V_N(i) & \text{if } l = 1 \text{ and } i \in \mathbb{Z}, \\ V_N^l(i) := \sum_{j\in\mathbb{Z}} V_N(i-j)V_N^{l-1}(j) & \text{if } l \geq 2 \text{ and } i \in \mathbb{Z}, \end{cases}$$

and set

$$\epsilon_N^l := \sum_{k\in\mathbb{Z}} \sup_{|i|\geq|k|} |V_N^l(i)|.$$

Inductively for $l \geq 2$,

$$\epsilon_N^l \quad \leq \quad \epsilon_N^{l-1} \sum_{k\in\mathbb{Z}} \sup_{|i|\geq|k|/2} |V_N(i)| + \epsilon_N^1 \sum_{k\in\mathbb{Z}} \sup_{|n|\geq|k|/2} |V_N^{l-1}(i)| \leq 5\epsilon_N^1\epsilon_N^{l-1}.$$

This together with (4.1) implies that $\epsilon_N^l, l \geq 1$, has exponential decay,

$$\epsilon_N^l \leq (5r_0)^l \quad \text{for all } l \geq 1. \tag{4.8}$$

For any $i \in \mathbb{Z}$, we get from replacing c in (4.7) by $c_m^{l_0}$ and then applying it repeatedly that

$$\begin{aligned} \|\Psi_i^N c_m^{l_0}\|_p \quad &\leq \quad \|\Psi_i^N Ac_m^{l_0}\|_p + \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} V_N^l(i-j)\|\Psi_j^N Ac_m^{l_0}\|_p + \sum_{j\in\mathbb{Z}} V_N^{L+1}(i-j)\|\Psi_j^N c_m^{l_0}\|_p \\ &\to \quad \sum_{j\in\mathbb{Z}} W_N(i-j)\|\Psi_j^N Ac_m^{l_0}\|_p \end{aligned} \tag{4.9}$$

as $L \to \infty$, where

$$W_N(k) = \begin{cases} 1 + \sum_{l=1}^{\infty} V_N^l(0) & \text{if } k = 0 \\ \sum_{l=1}^{\infty} V_N^l(k) & \text{if } 0 \neq k \in \mathbb{Z}. \end{cases}$$

By (4.6) and (4.8),

$$\sum_{k\in\mathbb{Z}} \sup_{|n|\geq|k|} W_N(n) \leq 1 + (1-5r_0)^{-1} < \infty. \tag{4.10}$$

Taking limit $l_0 \to \infty$ in (4.9), and then applying (4.5) and (4.10), we get

$$\|\Psi_i^N c_m\|_p \le \sum_{j\in\mathbb{Z}} W_N(i-j)\|\Psi_j^N A c_m\|_p \tag{4.11}$$

for any $i, m \in \mathbb{Z}$. Given any $n \in \mathbb{Z}$, let $i(n)$ be the unique integer in $\mathbb{Z}$ with $i(n)N \le n < (i(n)+1)N$. Then it follows from (4.11) that

$$\begin{aligned} |c(n,m)| &= |c_m(n)| \le \|\Psi_{i(n)}^N c\|_q \le \sum_{j\in\mathbb{Z}} W_N(i(n)-j)\|\Psi_j^N A c_m\|_1 \\ &\le \sum_{j\in\mathbb{Z}} W_N(i(n)-j) \sum_{k\in\mathbb{Z}} h(k/N-j)|Ac_m(k)| \\ &\le \sum_{l=-3}^{3} W_N(i(n-m)+l), \end{aligned} \tag{4.12}$$

where the last inequality holds by the definition of the function h and the sequence c_m. Hence the conclusion $A^{-1} \in \mathcal{B}$ follows from (4.10) and (4.12). □

§5 Localization preservation under inversion, factorization and optimization

In the section, we review some recent advances on norm-controlled inversion; inverse-closed q-Banach subalgebras; localization of matrix factorization, nonlinear mapping and nonlinear optimization; and convergence preservation for localized iterative algorithms.

5.1 Norm-controlled inversion

We say a Banach subalgebra $\mathcal{A}$ of a unital Banach algebra $\mathcal{B}$ admits *norm-controlled inversion* if there exists a continuous function h from $[0,\infty)\times[0,\infty)$ to $[0,\infty)$ such that

$$\|A^{-1}\|_{\mathcal{A}} \le h(\|A\|_{\mathcal{A}}, \|A^{-1}\|_{\mathcal{B}}) \tag{5.1}$$

for all $A \in \mathcal{A}$ being invertible in $\mathcal{B}$. Clearly Wiener's lemma holds for a Banach subalgebra $\mathcal{A}$ admitting norm-controlled inversion, but the converse is not true. The classical Banach algebra $\mathcal{W}$ (and also the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$) is inverse-closed but it does not have norm-controlled inversion in $\mathcal{B}(\ell^2)$ [31].

A differential *-algebra $\mathcal{A}$ of a symmetric $*$-algebra $\mathcal{B}$ with common identity and involution would have norm-control inversion [19, 20, 43], while for the Jaffard class $\mathcal{J}_\alpha := \mathcal{A}_{\infty,w_\alpha}$ with polynomial weight $w_\alpha(t) = (1+|t|)^\alpha$, a polynomial could be used as the function h in (5.1), see [5, 21].

Theorem 5.1. *Let $\alpha > 1$. If $A \in \mathcal{J}_\alpha$ is invertible in $\mathcal{B}(\ell^2)$, then $A^{-1} \in \mathcal{J}_\alpha$ and*

$$\|A^{-1}\|_{\mathcal{J}_\alpha} \le C_r \|A\|_{\mathcal{J}_\alpha}^{2r+2+2/(r-1)} \|A^{-1}\|_{\mathcal{B}(\ell^2)}^{2r+3+2/(r-1)}$$

for some absolute constant C_r depending on $r > 1$ only.

5.2 Wiener's lemma for q-Banach algebras

Let $0 < q \le 1$. We say that a complex vector space $\mathbf{A}$ is a *q-Banach space* if it is complete under the metric $d(x,y) := \|x-y\|_{\mathbf{A}}^q$, where the q-norm $\|\cdot\|_{\mathbf{A}} : \mathbf{A} \to \mathbb{R}$ satisfies (i) $\|x\|_{\mathbf{A}} \ge 0$ for all $x \in \mathbf{A}$, and $\|x\|_{\mathbf{A}} = 0$ if and only if $x = 0$; (ii) $\|\alpha x\|_{\mathbf{A}} = |\alpha|\|x\|_{\mathbf{A}}$ for all $\alpha \in \mathbb{C}$ and $x \in \mathbf{A}$; and (iii) $\|x+y\|_{\mathbf{A}}^q \le \|x\|_{\mathbf{A}}^q + \|y\|_{\mathbf{A}}^q$ for all $x, y \in \mathbf{A}$. We say that a q-Banach space $\mathcal{A}$ with q-norm $\|\cdot\|_{\mathcal{A}}$ is a *q-Banach algebra* if it contains a unit element I, it has operation of multiplications possessing the usual algebraic properties, and $\|AB\|_{\mathcal{A}} \le \|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}$ for all $A, B \in \mathcal{A}$.

Theorem 5.2. [29] *Let $0 < q \le 1, \mathcal{A}$ be a q-Banach algebra and $\mathcal{B}$ be a C^*-algebra. If $\mathcal{A}$ and $\mathcal{B}$ be two *-algebras with common identity and involution, and $\mathcal{A}$ is a differential subalgebra of $\mathcal{B}$ of order $\theta \in (0,1]$, then $\mathcal{A}$ is inverse-closed in $\mathcal{B}$.*

For $0 < q \le 1$, the Gröchenig-Schur class $\mathcal{A}_{q,u}$ is a q-Banach subalgebra of $\mathcal{B}(\ell^2)$ and satisfies

$$\|AB\|_{\mathcal{A}_{q,u}}^q \le D\|A\|_{\mathcal{A}_{q,u}}^q \|B\|_{\mathcal{A}_{q,u}}^q \Big(\Big(\frac{\|A\|_{\mathcal{B}(\ell^2)}}{\|A\|_{\mathcal{A}_{q,u}}}\Big)^{q\theta} + \Big(\frac{\|B\|_{\mathcal{B}(\ell^2)}}{\|B\|_{\mathcal{A}_{q,u}}}\Big)^{q\theta}\Big)$$

(hence it is inverse-closed in $\mathcal{B}(\ell^2)$ by Theorem 5.2) if the weight u is bounded below and there exists a companion weight v satisfying (3.2), $\|vu^{-1}\|_\infty < \infty$ and

$$\inf_{\tau>0} \|v\chi_{[-\tau,\tau]}\|_{2/(2-q)} + t\|vu^{-1}(1-\chi_{[-\tau,\tau]})\|_\infty \le Dt^{(1-\theta)}, t \ge 1,$$

for some positive constant $D > 0$ and $\theta \in (0,1)$. For instance, polynomial weights $w_\alpha := ((1+|n|)^\alpha)_{n\in\mathbb{Z}}$ with $\alpha > 0$ and subexponential weights $e_{\tau,\delta} = (\exp(\tau|n|^\delta))_{n\in\mathbb{Z}}$ with $\tau \in (0,\infty)$ and $\delta \in (0,1)$ satisfy the above requirements.

For a matrix $A = (a(m,n))_{m,n\in\mathbb{Z}}$, denote by $\|A\|_{\mathcal{S}_0}$ the maximum number of nonzero entries in all rows and columns of the matrix A, i.e.,

$$\|A\|_{\mathcal{S}_0} = \max\Big\{\sup_{m\in\mathbb{Z}} \big\|(a(m,n))_{n\in\mathbb{Z}}\big\|_{\ell^0}, \sup_{n\in\mathbb{Z}} \big\|(a(m,n))_{m\in\mathbb{Z}}\big\|_{\ell^0}\Big\},$$

where the ℓ^0 quasi-norm $\|c\|_{\ell^0}$ of a vector $c = (c(n))_{n\in\mathbb{Z}}$ is the cardinality of the set of its nonzero entries. The q-norm measure $\|A\|_{\mathcal{A}_{q,u}}$ of a matrix A could be considered as a relaxation of its sparsity measure $\|A\|_{\mathcal{S}_0}$, since

$$\lim_{q\to 0} \|A\|_{\mathcal{A}_{q,u}}^q = \|A\|_{\mathcal{S}_0}$$

for a matrix A with $\|A\|_{\infty,u} < \infty$. Thus matrices in the Gröchenig-Schur class $\mathcal{A}_{q,u}$ with $0 < q \le 1$ are numerically sparse and have certain off-diagonal decay.

5.3 Localized matrix factorizations

Localized factorizations started with Wiener's work on spectral factorization [49] and they have been shown to greatly reduce computational complexity in lots of numerical algorithms. Significant progress was made recently in [28], where localized matrices give rise to LU- Cholesky, QR- and polar factorizations, whose factors inherit the same localization.

Theorem 5.3. *Let A be an invertible matrix in the Baskakov-Gohberg-Sjöstrand class $\mathcal{C}_1$. Then* *(i) If $A = QR$ for some unitary matrix Q and upper triangular matrix R, then $Q, R \in \mathcal{C}_1$.*

(ii) If $A = LU$ for some lower triangular matrix L with its diagonal entries being all equal to one and some upper triangular matrix U, then $L, U \in \mathcal{C}_1$.

5.4 Wiener's lemma for strictly monotonic functions

Let $\mathbf{H}$ be a Hilbert space and f be a function on $\mathbf{H}$. We say that f is *differentiable on* $\mathbf{H}$ if it is differentiable at every $x \in \mathbf{H}$, that is, there exists a linear operator, denoted by $f'(x)$, in $\mathcal{B}(\mathbf{H})$ such that

$$\lim_{y \to 0} \frac{\|f(x+y) - f(x) - f'(x)y\|}{\|y\|} = 0;$$

and that f is *strictly monotonic* ([50]) if there exist positive constants m_0 and M_0 such that

$$m_0\|x - x'\|^2 \le \langle x - x', f(x) - f(x')\rangle \le M_0\|x - x'\|^2 \quad \text{for all } x, x' \in \mathbf{H}.$$

In [43], it is shown that a strictly monotonic function on a Hilbert space $\mathbf{H}$ with its derivative being continuous and bounded in an inverse-closed subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathbf{H})$ is invertible and its inverse has the same localization property.

Theorem 5.4. *Let $\mathbf{H}$ be a Hilbert space and $\mathcal{A}$ be a Banach subalgebra of $\mathcal{B}(\mathbf{H})$ that admits norm control in $\mathcal{B}(\mathbf{H})$. If f is a strictly monotonic function on $\mathbf{H}$ such that*

$$\sup_{x \in \mathbf{H}} \|f'(x)\|_{\mathcal{A}} < \infty \quad \text{and} \quad \lim_{y \to x} \|f'(y) - f'(x)\|_{\mathcal{A}} = 0 \quad \text{for all } x \in \mathbf{H},$$

then f is invertible and the derivative g' of its inverse $g = f^{-1}$ is bounded and continuous in $\mathcal{A}$.

5.5 Contraction and optimization

Given a Banach space $\mathbf{B}$, a function $f : \mathbf{B} \to \mathbf{B}$ is said to be a *contraction* on $\mathbf{B}$ if there exists $0 \le r_0 < 1$ such that

$$\|f(x) - f(y)\|_{\mathbf{B}} \le r_0\|x - y\|_{\mathbf{B}} \ \text{ for all } x, y \in \mathbf{B}.$$

The classical Banach fixed point theorem states that a contraction f on a Banach space $\mathbf{B}$ has a unique fixed-point x^* (i.e., $f(x^*) = x^*$), and the fixed point x^* is the limit of $x_n, n \ge 0$, in the iterative algorithm

$$x_{n+1} = f(x_n), n \ge 0, \tag{5.2}$$

with an arbitrary initial element $x_0 \in \mathbf{B}$.

In [43, 44], we developed a fixed point problem for a function f on a Banach space, whose restriction on its dense Hilbert subspace is a contraction.

Theorem 5.5. *Let $\mathbf{B}$ be a Banach space, $\mathbf{H}$ be a Hilbert space dense in $\mathbf{B}$, and let $\mathcal{A}$ be a Banach subalgebra of both $\mathcal{B}(\mathbf{B})$ and $\mathcal{B}(\mathbf{H})$ that is a differential subalgebra of $\mathcal{B}(\mathbf{H})$ of order*

$\theta \in (0,1]$. *If* $f : \mathbf{B} \to \mathbf{B}$ *is differentiable in* $\mathbf{B}$ *with its derivative being continuous and bounded on* $\mathcal{A}$*, and there exists a positive constant* $r_0 \in [0,1)$ *such that*

$$\|f'(x)\|_{\mathcal{B}(\mathbf{H})} \le r_0 \quad \text{for all } x \in \mathbf{B},$$

then the sequence $x_n, n \ge 0$*, in the iterative algorithm* (5.2) *with arbitrary initial* $x_0 \in \mathbf{B}$ *converges exponentially to the unique fixed point* x^* *of the function* f *on* $\mathbf{B}$*.*

The proof of Theorem 5.5 depends on the following observation for a differential subalgebra $\mathcal{A}$ of Banach algebra $\mathcal{B}$ of order $\theta \in (0,1]$: for any positive constants C_0 and r_0 there exists a positive constant C for any $r_1 > r_0$ such that

$$\sup_{A_1,\ldots,A_n \in \mathcal{A}(C_0,r_0)} \|A_1 A_2 \cdots A_n\|_{\mathcal{A}} \le C r_1^n, \ n \ge 1, \tag{5.3}$$

where $\mathcal{A}(C_0, r_0)$ contains all $A \in \mathcal{A}$ with $\|A\|_{\mathcal{A}} \le C_0$ and $\|\mathcal{A}\|_{\mathcal{B}} \le r_0$.

We say that a bounded linear operator A on ℓ^2 is *exponentially stable* if there exist positive constants E and α such that

$$\|e^{-At}\|_{\mathcal{B}(\ell^2)} \le E e^{-\alpha t}, \ t \ge 0.$$

We can use the observation (5.3) to solve algebraic Lyapunov equations and Riccati equations in a q-Banach algebra [29].

Theorem 5.6. *Let* $0 < q \le 1$ *and* $\mathcal{A}$ *be a q-Banach algebra. Assume that* $\mathcal{A}$ *is a differential subalgebra of* $\mathcal{B}(\ell^2)$ *of order* $\theta \in (0,1]$*,* $Q \in \mathcal{A}$ *is strictly positive on* ℓ^2 *and* A *is exponentially stable on* ℓ^2*. Then the unique strictly positive solution of the Lyapunov equation*

$$AP + PA^* + Q = 0$$

belongs to $\mathcal{A}$*.*

References

[1] A. Aldroubi and K. Gröchenig. *Nonuniform sampling and reconstruction in shift-invariant space*, SIAM Rev, 2001, 43: 585-620.

[2] A. Aldroubi, A. Baskakov and I. Krishtal. *Slanted matrices, Banach frames, and sampling*, J Funct Anal, 2008, 255: 1667-1691.

[3] B. A. Barnes. *The spectrum of integral operators on Lesbesgue spaces*, J Operator Theory, 1987, 18: 115-132.

[4] A. G. Baskakov. *Wiener's theorem and asymptotic estimates for elements of inverse matrices*, Funktsional Anal i Prilozhen, 1990, 24: 64-65; translation in Funct Anal Appl, 1990, 24: 222-224.

[5] A. G. Baskakov. *Estimates for the elements of matrices of inverse operators, and harmonic analysis*, Sibirsk Mat Zh, 1997, 38: 14-28.

[6] A. G. Baskakov and I. L. Krishtal. *Memory estimation of inverse operators*, Arxiv: 1103.2748.

[7] E. S. Belinskii, E. R. Liflyand, and R. M. Trigub. *The Banach algebra A^* and its properties*, J Fourier Anal Appl, 1997, 3: 103-129.

[8] A. Beurling. *On the spectral synthesis of bounded functions*, Acta Math, 1949, 81: 225-238.

[9] B. Blackadar and J. Cuntz. *Differential Banach algebra norms and smooth subalgebras of C^*-algebras*, J Operator Theory, 1991, 26: 255-282.

[10] S. Bochner and R. S. Phillips. *Absolutely convergent Fourier expansions for non-commutative normed rings*, Ann Math, 1942, 43: 409-418.

[11] L. Brandenburg. *On idenitifying the maximal ideals in Banach Lagebras*, J Math Anal Appl, 1975, 50: 489-510.

[12] O. Christensen and T. Strohmer. *The finite section method and problems in frame theory*, J Approx Theory, 2005, 133: 221-237.

[13] Q. Fang, C. E. Shin and Q. Sun. *Wiener's lemma for singular integral operators of Bessel potential type*, Monatsh Math, accepted.

[14] B. Farrell and T. Strohmer. *Inverse-closedness of a Banach algebra of integral operators on the Heisenberg group*, J Operator Theory, 2010, 64: 189-205.

[15] J. Garcia-Cuerva and J. L. Rubio de Francia. *Weighted Norm Inequalities and Related Topics*, Elsevier Science, New York, 1985.

[16] I. M. Gelfand, D. A. Raikov, and G. E. Silov. *Commutative Normed Rings*, Chelsea, New York, 1964.

[17] I. Gohberg, M. A. Kaashoek, and H. J. Woerdeman. *The band method for positive and strictly contractive extension problems: an alternative version and new applications*, Integral Equations Operator Theory, 1989, 12: 343-382.

[18] K. Gröchenig. *Wiener's lemma: theme and variations, an introduction to spectral invariance and its applications*, In: *Four Short Courses on Harmonic Analysis: Wavelets, Frames, Time-Frequency Methods, and Applications to Signal and Image Analysis*, editor by P. Massopust and B. Forster, Birkhauser, Boston, 2010.

[19] K. Gröchenig and A. Klotz. *Noncommutative approximation: inverse-closed subalgebras and off-diagonal decay of matrices*, Constr Approx, 2010, 32: 429-466.

[20] K. Gröchenig and A. Klotz. *Norm-controlled inversion in smooth Banach algebra I*, J London Math Soc, 2013, DOI: 10.1112/jlms/jdt004.

[21] K. Gröchenig and A. Klotz. *Norm-controlled inversion in smooth Banach algebra II*, Math Nachr, to appear.

[22] K. Gröchenig and M. Leinert. *Symmetry and inverse-closedness of matrix algebras and functional calculus for infinite matrices*, Trans Amer Math Soc, 2006, 358: 2695-2711.

[23] K. Gröchenig, Z. Rzeszotnik, and T. Strohmer. *Convergence analysis of the finite section method and Banach algebras of matrices*, Integral Equations Operator Theory, 2010, 67: 183-202.

[24] A. Hulanicki. *On the spectrum of convolution operators on groups with polynomial growth*, Invent Math, 1972, 17: 135-142.

[25] S. Jaffard. *Properiétés des matrices bien localisées prés de leur diagonale et quelques applications*, Ann Inst H Poincaré, 1990, 7: 461-476.

[26] E. Kissin and V. S. Shulman. *Differential properties of some dense subalgebras of C^*-algebras*, Proc Edinburgh Math Soc, 1994, 37: 399-422.

[27] I. Krishtal. *Wiener's lemma: pictures at exhibition*, Rev Un Mat Argentina, 2011, 52: 61-79.

[28] I. Krishtal, T. Strohmer and T. Wertz. *Localization of matrix factorizations*, Arxiv: 1305.1618.

[29] N. Motee and Q. Sun. *Sparsity measures for spatially decaying systems*, in preparation.

[30] M. A. Naimark. *Normed Algebras*, Wolters-Noordhoff Publishing Groningen, 1972.

[31] N. Nikolski. *In search of the invisible spectrum*, Ann Inst Fourier (Grenoble), 1999, 49: 1925-1998.

[32] M. A. Rieffel. *Leibniz seminorms for "matrix algebras converge to the sphere"*, In: *Quanta of Maths*, Clay Math Proc vol 11, Amer Math Soc, 2010, pages 543-578.

[33] K. S. Rim, C. E. Shin and Q. Sun. *Stability of localized integral operators on weighted L^p spaces*, Numer Funct Anal Optim, 2012, 33: 1166-1193.

[34] I. Schur. *Bemerkungen zur theorie der beschrankten bilinearformen mit unendlich vielen veranderlichen*, J Reine Angew Math, 1911, 140: 1-28.

[35] C. E. Shin and Q. Sun. *Stability of localized operators*, J Funct Anal, 2009, 256: 2417-2439.

[36] J. Sjöstrand. *Wiener type algebra of pseudodifferential operators*, Cent Math, Ecole Polytechnique, Palaiseau France, Seminaire 1994, 1995, December 1994.

[37] E. M. Stein. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[38] Q. Sun. *Wiener's lemma for infinite matrices with polynomial off-diagonal decay*, C R Acad Sci Paris Ser I, 2005, 340: 567-570.

[39] Q. Sun. *Non-uniform sampling and reconstruction for signals with finite rate of innovations*, SIAM J Math Anal, 2006, 38: 1389-1422.

[40] Q. Sun. *Wiener's lemma for infinite matrices*, Trans Amer Math Soc, 2007, 359: 3099-3123.

[41] Q. Sun. *Wiener's lemma for localized integral operators*, Appl Comput Harmon Anal, 2008, 25: 148-167.

[42] Q. Sun. *Wiener's lemma for infinite matrices II*, Constr Approx, 2011, 34: 209-235.

[43] Q. Sun. *Localized nonlinear functional equations and two sampling problems in signal processing*, Adv Comput Math, 2014, DOI: 10.1007/s10444-013-9314-3.

[44] Q. Sun and W.-S. Tang. *Nonlinear frames and sparse reconstructions*, submitted.

[45] Q. Sun and J. Xian. *Rate of innovation for (non)-periodic signals and optimal lower stability bound for filtering*, J Fourier Anal Appl, accepted.

[46] M. Takesaki. *Theory of Operator Algebra I*, Springer-Verlag, 1979.

[47] R. Tessera. *Left inverses of matrices with polynomial decay*, J Funct Anal, 2010, 259: 2793-2813.

[48] R. Tessera. *The Schur algebra is not spectral in $\mathcal{B}(\ell^2)$*, Monatsh Math, 2010, 164: 115-118.

[49] N. Wiener. *Tauberian theorem*, Ann Math, 1932, 33: 1-100.

[50] E. Zeidler and P. R. Wadsack. *Nonlinear Functional Analysis and its Applications*, Vol 1, Springer-Verlag, 1998.

[1] Department of Mathematics, Sogang University, Seoul, 121-742, Korea.
Email: shinc@sogang.ac.kr

[2] Department of Mathematics, University of Central Florida, Orlando, FL 32816, USA.
Email: qiyu.sun@ucf.edu