Appl. Math. J. Chinese Univ.
2013, 28(2): 187-203

Optimal investment for the defined-contribution pension with stochastic salary under a CEV model

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Abstract. In this paper, we study the optimal investment strategy of defined-contribution pension with the stochastic salary. The investor is allowed to invest in a risk-free asset and a risky asset whose price process follows a constant elasticity of variance model. The stochastic salary follows a stochastic differential equation, whose instantaneous volatility changes with the risky asset price all the time. The HJB equation associated with the optimal investment problem is established, and the explicit solution of the corresponding optimization problem for the CARA utility function is obtained by applying power transform and variable change technique. Finally, we present a numerical analysis.

§1 Introduction

There are two radically different methods to design a pension fund: the defined-benefit plan (hereinafter called DB) and the defined-contribution plan (hereinafter called DC). In DB, the benefits are fixed in advance by the sponsor and the contributions are adjusted in order to maintain the fund in balance, where the associated financial risks are assumed by the sponsor agent; in DC, the contributions are fixed and the benefits depend on the returns on the assets of the fund, where the associated financial risks are borne by the beneficiary. Historically, DB has been more popular. However, in recent years, owing to the demographic evolution and the development of the equity markets, DC plays a crucial role in the social pension systems.

The optimal investment for DC has been widely discussed in the literatures and focuses on the following aspects. (1) DC with guarantee. For example, Boulier, et al. (2001) dealt with the optimal management of DC with a protected benefit under the stochastic interest rates (the Vasicek model) and found that the guarantee depended on the level of the stochastic interest rates when the employee retired. Deelstra, et al. (2003) assumed that the stochastic

Received: 2012-07-16.
MR Subject Classification: 49L20, 91G10, 91G80.
Keywords: Defined contribution pension plan, Stochastic salary, constant elasticity of variance model, optimal investment.
Digital Object Identifier(DOI): 10.1007/s11766-013-3087-9.
Supported by the National Natural Science Foundation of Tianjin (07JCYBJC05200) and the Young Scholar Program of Tianjin University of Finance and Economics (TJYQ201201).

interest rates followed the affine dynamics (including the CIR model and the Vasicek model), described the contribution flow by a non-negative, progressive measurable and square-integrable process, and then studied optimal investment strategies for different examples of guarantees and contributions. (2) DC with stochastic salary. Besides Deelstra, et al. (2003), Battocchio and Menoncin (2004) took into account two background risks (the salary risk and the inflation) and analyzed the behavior of the optimal portfolio with respect to salary and inflation in detail. Cairns, et al. (2006) incorporated asset, salary (labor-income) and interest-rate risk, used the member's final salary as a numeraire, and discussed various properties and characteristics of the optimal asset-allocation strategies both with and without the presence of non-hedgeable salary risk. (3) DC with stochastic interest rates. Most studies mentioned are under stochastic interest. Besides, Gao (2008) studied the portfolio problem of DC with the stochastic interest rates (including the CIR model and the Vasicek model) by using the method of stochastic optimal control. (4) DC before and after retirement. For example, Devolder, et al. (2003) studied the optimal investment policies before and after retirement for different utility functions without considering interest risk and salary risk.

The studies mentioned above generally supposed the risky asset price following a geometric Brownian motion (hereinafter called GBM), which implies that the volatility of risky asset price is only a constant. The constant volatility is contrary to the empirical evidence of fluctuating historical volatility. However, the constant elasticity of variance (hereinafter called CEV) model which was originally proposed by Cox (1975) is a natural extension of GBM and can capture the implied volatility smile (the implied volatility skew) similar to the volatility smile curves observed in practice (e.g. Dennis and Mayhew, 2002). The CEV model is usually applied to calculating the theoretical price, sensitivities and implied volatility of options (e.g. Cox, 1996; Yuen, et al., 2001; Jones, 2003; Widdicks, et al., 2005; Hsu, et al., 2008). However, Xiao, et al. (2007) firstly studied the pension investment problem before and after retirement under the CEV model and derived the dual solution for the logarithm utility by Legendre transform and dual theory. Afterwards, under the CEV model, Gao (2009a,b) derived the explicit solutions for investors with CRRA and CARA utility functions in DC. In addition, Gu, et al. (2010) adopted the CEV model to describe the risky asset price dynamics in the optimal reinsurance and investment decision-making.

However, the researches mentioned above with a CEV model didn't consider the stochastic salary risk. The optimal asset allocation for a pension fund involves quite a long period, generally from 20 to 40 years, so it is crucial to take into account the salary risk. Our main objective in this paper is to find the optimal investment strategy for DC pension with the stochastic salary under a CEV model. The paper constructs a new model based on: (1) Battocchio and Menoncin (2004) and Cairns, et al. (2006) who proposed that the stochastic salary follows the GBM model; (2) Xiao, et al. (2007) and Gao (2009a,b) who studied the optimal investment strategies for DC pension under a CEV model. In our model, the pension investor's objective is to maximize his expected terminal utility by investing in a risk-free asset and a risky asset whose price follows a CEV model; meanwhile the salary is stochastic which follows a stochastic differential equation (SDE), the instantaneous volatility changes with the risky asset price all the time, and there is a multiple relationship between the risk sources of stock and stochastic salary. The most novel feature of our research is the application of the CEV model to DC pension with the stochastic salary model, which has not been reported in the existing literature. Applying the maximum principle, we derive a nonlinear second-order partial differential equation (PDE) for the value function. However, the PDE's coefficient variables are closely correlated, and it is difficult to characterize the solution structure. Therefore, we use power transform and variable change technique to transform it into a linear PDE and obtain the explicit solution for the CARA utility function.

The rest of the paper is organized as follows. In Section 2, we introduce the mathematical model including the financial market, the stochastic salary and the wealth process. In Section 3, we propose the optimization problem. In Section 4, we obtain the explicit solution for the CARA utility function by the maximum principle, power transform and variable change technique. In Section 5, we present a numerical analysis to demonstrate our results. Section 6 draws the conclusions.

§2 Mathematical model

In this section, we introduce the market structure and define the stochastic dynamics of assets' prices and the salary.

We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0, T]$, where $T > 0$ denotes the retirement time of a representative shareholder.

2.1. The financial market

We suppose that the market structure consists of two financial assets, a risk-free asset and a single risky asset. Let $B(t)$ be the price of the risk-free asset (bond or bank account) at time t, which evolves according to the following equation:

$$dB(t) = r_0 B(t)dt,\ B(0) = B_0,\ r_0 > 0, \tag{1}$$

where B_0 stands for an initial price of the risk-free asset, and r_0 is a constant rate of interest. We assume that the price of the risky asset is a continuous time stochastic process. We apply the CEV model instead of GBM in the price of risky asset. Let $S(t)$ be the price of the risky asset (stock) at time t, which follows the CEV model:

$$dS(t) = r_1 S(t)dt + \sigma_1 S(t)^{\beta+1} dW(t),\ S(0) = S_0, \tag{2}$$

where S_0 stands for an initial price of the risky asset, and r_1 and σ_1 are constant parameters. r_1 is an expected instantaneous rate of return of the risky asset and satisfies the general condition $r_1 > r_0$. $\sigma_1 S(t)^\beta$ is the instantaneous volatility, β is the elasticity parameter and satisfies the general condition $\beta < 0$. $\{W(t) : t \geq 0\}$ is a standard Brownian motion defined on a complete probability space (Ω, F, P), where Ω is the real space, P is the probability measure. The filtration $F = \{F_t\}$ is a right continuous filtration of sigma-algebras on this space and denotes the information structure generated by the Brownian motion.

Remark 1. In Eq.(2), if the elasticity parameter $\beta = 0$, then the price process of the risky asset reduces to a GBM; if $\beta = -1$, it is the Ornstein-Uhlenbeck process; if $\beta = -1/2$, it is the model firstly presented by Cox (1975) as an alternative diffusion process for valuation of options; if $\beta < 0$, the instantaneous volatility $\sigma_1 S(t)^\beta$ increases as the stock price decreases, and can generate a distribution with a fatter left tail; if $\beta > 0$, the situation is reversed and unrealistic.

2.2. The stochastic salary

Let $L(t)$ be the salary at time t, which follows the stochastic differential equation (SDE):

$$dL(t) = r_2 L(t)dt + \sigma_2(t)L(t)dW(t),\ L(0) = L_0, \tag{3}$$

where L_0 stands for an initial salary, r_2 is an expected instantaneous growth rate of the salary and is a real constant. $\sigma_2(t)$ is the instantaneous volatility. Similar to the work of Battocchio and Menoncin (2004), there is the multiple relationship between stock and salary with respect to the instantaneous volatility, i.e., $\sigma_2(t) = \eta\sigma_1 S(t)^\beta$ and η is volatility scale factor. As mentioned above, $\{W(t) : t \geq 0\}$ is a standard Brownian motion defined on a complete probability space (Ω, F, P).

Remark 2. In Eq.(3), notice that the salary is influenced by the financial market (i.e., stock). That is to say, $\sigma_2(t)$ is a hedgeable volatility whose risk source belongs to the set of the financial market risk source. This assumption is similar to Deelstra, et al. (2003), whose risks are both just generated by the financial market (i.e., stock), but is different from Battocchio and Menoncin (2004), and Cairns, et al. (2006), who also assumed that the salary was affected by non-hedgeable risk source (i.e., non-financial market). Additionally, they all assumed that the salary's instantaneous volatility was constant, but in Eq.(3), the stochastic salary's instantaneous volatility changes by the risky asset price all the time, and there is a multiple relationship.

2.3. The wealth process

We consider that the contributions are continuously paid into the pension fund at the rate of $kL(t)$. Let $X(t)$ be the wealth of pension fund at time $t \in [0, T]$, π_t and $1 - \pi_t$ be the proportion of the pension fund invested in the stock and the bank, respectively. Accordingly, let $\Lambda(t)$ be the optimal investment amount invested in the stock.

The dynamics of the pension fund is given by

$$dX(t) = (1 - \pi_t)\, X(t)\frac{dB(t)}{B(t)} + \pi_t X(t)\frac{dS(t)}{S(t)} + kL(t)dt,\ X(0) = X_0, \tag{4}$$

where X_0 stands for an initial wealth.

According to Eqs.(1)-(3), the evolution of pension wealth can be rewritten as

$$dX(t) = [(1 - \pi_t)\, X(t)r_0 + \pi_t X(t)r_1 + kL(t)]\, dt + \pi_t X(t)\sigma_1 S(t)^\beta dW(t). \tag{5}$$

Remark 3. The contribution term $kL(t)dt$ in Eq.(4) is the same as Cairns, et al. (2006), but Battocchio and Menoncin (2004) adopted $kdL(t)$. Their most significant difference is whether the contribution is proportional to the salary or the change of salary. In addition, Ma (2011) pointed out that the assumption of Battocchio and Menoncin (2004) was incorrect and the correct expression should be $kL(t)dt$, because it implies that the change of salary $dL(t)$ (rather than salary L itself) contributes to the growth of pension wealth.

Please note that $dL(t)$ does not mean that a constant wage implies no contribution. $L(t)$ is the total labor income, i.e., the sum of all the wages earned between the initial date and time t. Thus, its difference $dL(t)$ coincides with the increment in the total labor income or, in other words, with wage.

In fact, the main question that must be answered is: at time t, does the pension fund know the contribution that it receives between t and $t+dt$? If the answer is yes, then $kL(t)dt$ is more suitable; if no, then $kdL(t)$ is more suitable. So, the two assumptions are correct. However, this paper chooses the former.

§3 The optimization program

The plan member will retire at time T and is risk averse, so the utility function $U(x)$ is typically increasing and concave ($U''(x) < 0$). In this section, we are interested in maximizing the utility of the pension investor's terminal wealth. For a strategy π_t, we define the value function attained by the pension investor from state x at time t as

$$H_{\pi_t}(t,s,l,x) = E_{\pi_t}\left[U(X(T))|S(t)=s, L(t)=l, X(t)=x\right]. \tag{6}$$

Our objective is to find the optimal value function $H(t,s,l,x) = \sup H_{\pi_t}(t,s,l,x)$ and the optimal strategy π_t^* such that $H_{\pi_t^*}(t,s,l,x) = H(t,s,l,x)$.

The Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is

$$\begin{aligned}&H_t + r_1 s H_s + r_2 l H_l + \tfrac{1}{2}\sigma_1^2 s^{2\beta+2} H_{ss} + \tfrac{1}{2}\sigma_2^2 l^2 H_{ll} + \sigma_1\sigma_2 s^{\beta+1} l H_{sl} +\\ &\max_{\{\pi_t\}}\left\{\left[(1-\pi_t)\, x r_0 + \pi_t x r_1 + kl\right] H_x + \pi_t x \sigma_1^2 s^{2\beta+1} H_{sx}\right.\\ &\left.+\pi_t x \sigma_1\sigma_2 l s^{\beta} H_{lx} + \tfrac{1}{2}\pi_t^2 x^2 \sigma_1^2 s^{2\beta} H_{xx}\right\} = 0,\end{aligned} \tag{7}$$

with $H(T,s,l,x) = U(x)$, where $H_t, H_s, H_l, H_x, H_{ss}, H_{ll}, H_{sx}, H_{lx}$ and H_{xx} denote partial derivatives of first and second orders with respect to time, stock price, salary and wealth.

The first order maximizing condition for the optimal strategy π_t^* is

$$\pi_t^* = -\left\{\frac{r_1 - r_0}{x\sigma_1^2 s^{2\beta}}\frac{H_x}{H_{xx}} + \frac{s}{x}\frac{H_{sx}}{H_{xx}} + \frac{\sigma_2 l}{x\sigma_1 s^{\beta}}\frac{H_{lx}}{H_{xx}}\right\}. \tag{8}$$

Putting this into Eq.(7), and noting that $\sigma_2(t) = \eta\sigma_1 S(t)^{\beta}$, we obtain a partial differential equation (PDE) for the value function H:

$$\begin{aligned}&H_t + r_1 s H_s + r_2 l H_l + \tfrac{1}{2}\sigma_1^2 s^{2\beta+2} H_{ss} + \tfrac{1}{2}\eta^2\sigma_1^2 s^{2\beta} l^2 H_{ll} + \eta\sigma_1^2 s^{2\beta+1} l H_{sl}\\ &+ (x r_0 + kl)\, H_x - \tfrac{1}{2}\tfrac{(r_1-r_0)^2}{\sigma_1^2 s^{2\beta}}\tfrac{H_x^2}{H_{xx}} - s\,(r_1 - r_0)\,\tfrac{H_x H_{sx}}{H_{xx}} - \eta l\,(r_1 - r_0)\,\tfrac{H_x H_{lx}}{H_{xx}}\\ &-\tfrac{1}{2}\sigma_1^2 s^{2\beta+2}\tfrac{H_{sx}^2}{H_{xx}} - \eta l \sigma_1^2 s^{2\beta+1}\tfrac{H_{sx}H_{lx}}{H_{xx}} - \tfrac{1}{2}\eta^2\sigma_1^2 s^{2\beta} l^2 \tfrac{H_{lx}^2}{H_{xx}} = 0,\end{aligned} \tag{9}$$

with $H(T,l,s,x) = U(x)$.

Here, we notice that the stochastic control problem described in the previous section has been transformed into a PDE. The problem now is to solve Eq.(9) for the value function and replace it in Eq.(8) in order to obtain the optimal strategy.

The following section provides the explicit solution for the exponential utility function.

§4 Explicit solution for the exponential utility function

Assume that the pension investor takes an exponential utility function,

$$U(x) = -\frac{1}{q}e^{-qx},\ (q > 0). \tag{10}$$

The absolute risk aversion of a decision maker with utility described in Eq.(10) is constant, thus the exponential untility is a CARA utility.

We conjecture a solution to Eq.(10) with the following form:

$$H(t,s,l,x) = -\frac{1}{q}\exp\left\{-q\left[c(t)x + f(t,s) + g(t,l)\right]\right\} \tag{11}$$

with the boundary conditions given by $c(T) = 1,\ f(T,s) = g(T,l) = 0$.

Then,

$$\begin{aligned}
&H_t = -q\left[c_t x + f_t + g_t\right]H,\ H_s = -qf_sH,\ H_l = -qg_lH,\\
&H_x = -qc(t)H, H_{xx} = q^2c(t)^2H,\ H_{sx} = q^2c(t)f_sH,\\
&H_{lx} = q^2c(t)g_lH,\ \ H_{ss} = \left(q^2f_s^2 - qf_{ss}\right)H,\\
&H_{ll} = \left(q^2g_l^2 - qg_{ll}\right)H,\ H_{sl} = q^2f_sg_lH.
\end{aligned}$$

Introducing these derivatives in Eq.(9), we derive

$$\begin{aligned}
&\left(c_t + r_0c\right)x + klc + f_t + g_t + r_0sf_s + \left(r_2 - \eta\left(r_1 - r_0\right)\right)lg_l\\
&+\tfrac{1}{2}\sigma_1^2s^{2\beta+2}f_{ss} + \tfrac{1}{2}\eta^2\sigma_1^2s^{2\beta}l^2g_{ll} + \tfrac{1}{2}\tfrac{(r_1-r_0)^2}{q\sigma_1^2s^{2\beta}} = 0.
\end{aligned} \tag{12}$$

Again we can split the above equation into two equations

$$c_t + r_0c = 0, \tag{13}$$

$$\begin{aligned}
&klc + f_t + g_t + r_0sf_s + \left(r_2 - \eta\left(r_1 - r_0\right)\right)lg_l +\\
&\tfrac{1}{2}\sigma_1^2s^{2\beta+2}f_{ss} + \tfrac{1}{2}\eta^2\sigma_1^2s^{2\beta}l^2g_{ll} + \tfrac{1}{2}\tfrac{(r_1-r_0)^2}{q\sigma_1^2s^{2\beta}} = 0.
\end{aligned} \tag{14}$$

Taking into account the boundary condition $c(T) = 1$, the solution to Eq.(13) is

$$c(t) = e^{r_0(T-t)}. \tag{15}$$

Note that Eq.(14) is a nonlinear second-order PDE, it is difficult to find an explicit solution. Therefore, we use power transformation and variable change technique proposed by Cox (1996) to transform the nonlinear equation into a linear one.

Let

$$f(t,s) = h(t,y)\ ,\ y = s^{-2\beta}. \tag{16}$$

Then,

$$\begin{aligned}
&f_t = h_t, f_s = -2\beta s^{-2\beta-1}h_y,\\
&f_{ss} = 2\beta\left(2\beta+1\right)s^{-2\beta-2}h_y + 4\beta^2s^{-4\beta-2}h_{yy}.
\end{aligned}$$

Putting these derivatives in Eq.(14), we derive

$$\begin{aligned}
&h_t + g_t - 2r_0\beta yh_y + \beta\left(2\beta+1\right)\sigma_1^2h_y + \left(r_2 - \eta\left(r_1 - r_0\right)\right)lg_l\\
&+2\beta^2\sigma_1^2yh_{yy} + \tfrac{1}{2}\eta^2\sigma_1^2y^{-1}l^2g_{ll} + klc(t) + \tfrac{1}{2}\tfrac{(r_1-r_0)^2}{q\sigma_1^2}y = 0.
\end{aligned} \tag{17}$$

We try to find a solution of the above equation with the following structure

$$h(t,y) + g(t,l) = I(t) + J(t)y + K(t)l \tag{18}$$

with boundary conditions $I(T) = J(T) = K(T) = 0$.

Putting Eq.(18) into Eq.(17), we get

$$\begin{aligned}&I_t+\beta\left(2\beta+1\right)\sigma_1^2J(t)+y\left\{J_t-2r_0\beta J(t)+\tfrac{1}{2}\tfrac{(r_1-r_0)^2}{q\sigma_1^2}\right\}+\\&l\left\{K_t+\left(r_2-\eta\left(r_1-r_0\right)\right)K(t)+kc(t)\right\}=0.\end{aligned}\tag{19}$$

By matching coefficients, we derive

$$I_t+\beta\left(2\beta+1\right)\sigma_1^2J(t)=0,\tag{20}$$

$$J_t-2r_0\beta J(t)+\frac{1}{2}\frac{\left(r_1-r_0\right)^2}{q\sigma_1^2}=0,\tag{21}$$

$$K_t+\left(r_2-\eta\left(r_1-r_0\right)\right)K(t)+kc(t)=0.\tag{22}$$

Taking the boundary conditions $I(T)=J(T)=K(T)=0$ into account, the solutions to Eqs.(20)-(22) are

$$J(t)=\frac{\gamma}{2r_0\beta}\left(1-e^{-2r_0\beta(T-t)}\right),\ \gamma=\frac{1}{2}\frac{\left(r_1-r_0\right)^2}{q\sigma_1^2},\tag{23}$$

$$I(t)=\theta\left(T-t-\frac{1-e^{-2r_0\beta(T-t)}}{2r_0\beta}\right),\quad \theta=\frac{(2\beta+1)(r_1-r_0)^2}{4r_0q}.\tag{24}$$

$$K(t)=\frac{k}{r_0+\lambda}\left(e^{-\lambda(T-t)}-e^{r_0(T-t)}\right),\ \lambda=-\left(r_2-\eta\left(r_1-r_0\right)\right).\tag{25}$$

From the above calculations, we obtain the optimal strategy under the exponential utility function.

Proposition 1. *The optimal strategy invested in the stock is given by*

$$\pi_t^*=x^{-1}\left[M(t,\sigma_t)N(t)+k\eta lP(t)\right],\tag{26}$$

where

$$\begin{aligned}&M(t,\sigma_t)=\tfrac{r_1-r_0}{q\sigma_t^2}e^{-r_0(T-t)},\ N(t)=1+\tfrac{r_1-r_0}{2r_0}\left(1-e^{-2r_0\beta(T-t)}\right),\\&\sigma_t=\sigma_1s^\beta,P(t)=\tfrac{1}{r_0+\lambda}\left(1-e^{-(r_0+\lambda)(T-t)}\right),\quad\lambda=-\left(r_2-\eta\left(r_1-r_0\right)\right).\end{aligned}$$

Proof. From the Eqs.(8), (11), (15)-(16), (18), (23), (25) and $\sigma_2(t)=\eta\sigma_1S(t)^\beta$, we obtain

$$\begin{aligned}\pi_t^*&=-\left\{\tfrac{r_1-r_0}{x\sigma_1^2s^{2\beta}}\tfrac{H_x}{H_{xx}}+\tfrac{s}{x}\tfrac{H_{sx}}{H_{xx}}+\tfrac{\sigma_2l}{x\sigma_1s^\beta}\tfrac{H_{lx}}{H_{xx}}\right\}\\&=\tfrac{r_1-r_0}{x\sigma_1^2s^{2\beta}}\tfrac{1}{qc(t)}-\tfrac{s}{x}\tfrac{f_s}{c(t)}-\tfrac{\eta l}{x}\tfrac{g_l}{c(t)}\\&=\left\{\tfrac{r_1-r_0}{xq\sigma_1^2s^{2\beta}}+\tfrac{2\beta s^{-2\beta}h_y}{x}-\tfrac{\eta lg_l}{x}\right\}e^{-r_0(T-t)}\\&=\left\{\tfrac{r_1-r_0}{xq\sigma_1^2s^{2\beta}}+\tfrac{2\beta s^{-2\beta}J(t)}{x}-\tfrac{\eta lK(t)}{x}\right\}e^{-r_0(T-t)}\\&=\tfrac{r_1-r_0}{xq\sigma_1^2s^{2\beta}}\left\{1+\tfrac{r_1-r_0}{2r_0}\left(1-e^{-2r_0\beta(T-t)}\right)\right\}e^{-r_0(T-t)}\\&\quad+\tfrac{k\eta l}{x(r_0+\lambda)}\left(1-e^{-(r_0+\lambda)(T-t)}\right)=x^{-1}\left[M(t,\sigma_t)N(t)+k\eta lP(t)\right],\end{aligned}$$

where

$$\begin{aligned}&M(t,\sigma_t)=\tfrac{r_1-r_0}{q\sigma_t^2}e^{-r_0(T-t)},N(t)=1+\tfrac{r_1-r_0}{2r_0}\left(1-e^{-2r_0\beta(T-t)}\right),\\&\sigma_t=\sigma_1s^\beta,P(t)=\tfrac{1}{r_0+\lambda}\left(1-e^{-(r_0+\lambda)(T-t)}\right),\quad\lambda=-\left(r_2-\eta\left(r_1-r_0\right)\right).\end{aligned}$$

□

Remark 4. If we use the amount of wealth to denote the optimal strategy, we can conclude the optimal strategies as holding the amount $x^{-1}\left[M(t,\sigma_t)N(t)+k\eta lP(t)\right]$ of current wealth in the stock, and putting the rest in the bank.

The optimal proportion invested in risky assets for the pension investor with the exponential utility function is divided into four parts.

The first part $M(t,\sigma_t)$ reflects the effect of the pension investor's risk appetite, instantaneous volatility and difference of return between riskless assets and risky assets. Comparing to results of Devolder, et al. (2003) and Gao (2009a,b), we define it as a "moving Merton strategy" for DC pension.

The second part $N(t)$ represents a supplementary term resulted from the changes of the volatility under a CEV model. In other words, it reflects the pension investor's decision to hedge the volatility risk. Based on the work of Gao (2009a,b), we define it as a "correction factor" for a "moving Merton strategy".

The third part $k\eta l$ reflects how the plan member's salary affect the optimal investment strategy. So we call it a "stochastic salary strategy".

The fourth part $P(t)$ represents an adjustment term for the plan member's salary, which is affected by the time t. Thus we call it an "adjustment factor" for a "stochastic salary strategy".

Remark 5. In Eq.(3), if $\eta=0$, the salary isn't stochastic, then the contribution isn't stochastic. As a result, the optimal strategy for $\eta=0$ is $\pi_t^*=x^{-1}M(t,\sigma_t)N(t)$, which is the same as the result of Gao (2009a,b). If we further assume that $\beta=0$, the optimal strategy is just the result of Devolder, et al. (2003). Compared with their works, Proposition 1 can be considered as an extension of Devolder, et al. (2003) and Gao (2009a,b).

In addition, under the exponential utility function, if assuming that the salary isn't stochastic, the contribution and wealth have no influence on the optimal strategy (Gao, 2009a,b). However, we confirm that the stochastic salary is an indispensable influence factor for the optimal strategy. Meanwhile, the optimal strategy isn't affected by the wealth. This can be explained by the risk tolerance, namely $U'(x)/U''(x)=-q^{-1}$, which is only a constant. This indicates that for an exponential utility, due to the independence of a risk tolerance coefficient on a wealth, the optimal strategy is independent of a wealth.

Corollary 1. *The correction factor $N(t)$ is a monotone increasing function with respect to the time and such that $1+\frac{r_1-r_0}{2r_0}\left(1-e^{-2r_0\beta(T-t)}\right)\le N(t)\le 1$.*

Proof. Note that $N(t)=1+\frac{r_1-r_0}{2r_0}\left(1-e^{-2r_0\beta(T-t)}\right), r_1>r_0>0,\ \beta<0$, and then,

$$N'(t)=-\beta\left(r_1-r_0\right)e^{-2r_0\beta(T-t)}>0.$$

This implies that $N(t)$ is a monotone increasing function with respect to time t. Taking into account the boundary conditions of $N(t)$ at time $t=0$ and $t=T$, we have

$$1+\frac{r_1-r_0}{2r_0}\left(1-e^{-2r_0\beta(T-t)}\right)\le N(t)\le 1.$$

□

Remark 6. Corollary 1 shows that the behavior of the correction factor depends only on time t. Without considering the other terms, the change of $N(t)$ implies that the pension investor will invest a small proportion of wealth in the stock at the beginning of the investment horizon, and steadily increase the proportion as time passes. In addition, the correction factor is less than or equal to one, meaning that the investor will reduce the proportion of wealth in stock suggested by $M(t,\sigma_t)$. Here, we should remember even though the change of the correction factor can be determined, we can't judge the trends of the optimal strategy just by the correction factor.

Corollary 2. *When $N(t) < 0$, the optimal strategy π_t^* is a monotone increasing function with respect to the elasticity coefficient β.*

Proof. Note that $e^{-r_0(T-t)} > 0,\ e^{-2r_0\beta(T-t)} > 0, r_1 > r_0 > 0,\ x > 0,\ q > 0,\ \beta < 0$, and then,

$$\begin{aligned}
&M(t,\sigma_t) > 0,\ N(t) < 0,\\
&\tfrac{dM(t,\sigma_t)}{d\beta} = -\tfrac{2(r_1-r_0)}{q\sigma_1^2}e^{-r_0(T-t)}s^{-2\beta-1} < 0,\\
&\tfrac{dN(t)}{d\beta} = e^{-2r_0\beta(T-t)}\left(r_1-r_0\right)(T-t) > 0,\\
&\tfrac{d\pi_t^*}{d\beta} = x^{-1}\left(\tfrac{dM(t,\sigma_t)}{d\beta}N(t) + \tfrac{dN(t)}{d\beta}M(t,\sigma_t)\right) > 0.
\end{aligned}$$

This shows that π_t^* is a monotone increasing function with respect to parameter β for $N(t) < 0$. Thus $x^{-1}k\eta lP(t) \le \pi_t^* \le x^{-1}\left[M(t,\sigma_t) + k\eta lP(t)\right]$. □

Remark 7. Corollary 2 indicates that whether the optimal strategy π_t^* has a monotone trend depends on the value of $N(t)$. When $N(t) < 0$, the pension investor will invest a higher proportion of wealth in stock as the parameter β goes on. However, if $0 \le N(t) \le 1$, we can't judge the trend of the optimal strategy with respect to the elasticity coefficient β. Thus, before making investment decisions, we can firstly find the range of $N(t)$ in order to estimate if the optimal strategy has a monotone trend.

Corollary 3. *The optimal strategy π_t^* is a piecewise monotone function with respect to the absolute risk aversion coefficient q, which depends on the correction factor of $N(t)$.*

Proof. Note that $e^{-r_0(T-t)} > 0,\ N(t) \le 1,\ r_1 > r_0 > 0,\ x > 0$ and $\beta < 0$, we have,

$$\frac{d\pi_t^*}{dq} = x^{-1}\frac{dM(t,\sigma_t)}{dq}N(t) = -\frac{r_1-r_0}{xq^2\sigma_1^2 s^{2\beta}}e^{-r_0(T-t)}N(t)\begin{cases} < 0, & 0 < N(t) \le 1,\\ \ge 0, & N(t) \le 0.\end{cases}$$

□

Remark 8. Corollary 3 reflects that how the optimal strategy π_t^* is affected by the absolute risk aversion coefficient q. Obviously, there are two kinds of opposite monotone trends, which depend on the sign of $N(t)$. When $0 < N(t) \le 1$, the optimal strategy (π_t^*) is a monotone decreasing function with respect to the absolute risk aversion coefficient q, which means the pension investor will be more reluctant to avoid risk as investing a lower proportion of wealth in stock as q increases. On the contrary, when $N(t) \le 0$, the pension investor is likely to invest a higher proportion of wealth in risky assets, i.e., the optimal strategy (π_t^*) is monotonously increasing as q increases.

Remark 9. From Proposition 1, whether the optimal strategy π_t^* has monotone trends with respect to the stochastic salary is difficult to find, because the mathematical structure is more complex and some parameters' values can't be determined.

§5 Numerical analysis

This section provides some numerical analysis to examine the properties of the correction factor and the optimal investment strategy. Firstly, according to Proposition 1, we calculate the optimal investment amount's expectation $E\Lambda^*(t)$. Secondly, based on previous research results, we determine the basic parameters' values. Thirdly, we analyze the sensitivities of the correction factor and the optimal investment amount's expectation with respect to the elasticity coefficient and risk aversion coefficients etc.

5.1 Optimal investment amount's expectation

Taking expectations on both sides of (2) and (3), we obtain the following linear ordinary differential equations

$$dES(t) = r_1 ES(t)dt,\ S(0) = S_0,\ dEL(t) = r_2 EL(t)dt,\ L(0) = L_0.$$

Solving them, we obtain $ES(t) = S_0 e^{r_1 t}$ and $EL(t) = L_0 e^{r_2 t}$.

We denote the optimal investment amount at time t by $\Lambda^*(t)$. Therefore, according to Proposition 1, $\Lambda^*(t) = \pi_t^* X(t) = M(t, \sigma_t)N(t) + k\eta L(t)P(t)$. Taking expectation on it, we obtain, $E\Lambda^*(t) = N(t)EM(t, \sigma_t) + k\eta P(t)EL(t)$. Therefore,

$$E\Lambda^*(t) = N(t)\frac{(r_1 - r_0)(2\beta + 1)\left(1 - e^{-2\beta r_1 t}\right)}{2qr_1 e^{r_0(T-t)}} + N(t)\frac{r_1 - r_0}{q\sigma_1^2 S_0^{2\beta}} e^{-r_0(T-t) - 2\beta r_1 t} + k\eta P(t) L_0 e^{r_2 t},$$

where $N(t) = 1 + \frac{r_1 - r_0}{2r_0}\left(1 - e^{-2r_0\beta(T-t)}\right)$, $P(t) = \frac{1}{r_0 + \lambda}\left(1 - e^{-(r_0+\lambda)(T-t)}\right)$, $\lambda = \eta\,(r_1 - r_0) - r_2$.

5.2 Values of model parameters

Based on previous research results, we determine the basic parameters. Yuen, et al. (2001) estimated the CEV model parameters for Hong Kong stock option market, and those parameters' values are given by $r_0 = 0.03$, $r_1 = 0.12$, $\sigma_1 = 16.16$, $S_0 = 67$, $\beta = -1$. Following the assumption of Gao (2009a), we take the values of the investment time horizon $T = 20$, the contribution rate $k = 0.12$, and the risk aversion coefficient $q = 0.05$. According to the work of Battocchio and Menoncin (2004), the parameters' values of stochastic salary are given by $r_2 = 0.05$, $L_0 = 100$. In addition, the instantaneous volatility of salary is obviously lower than the one of stock, so we consider the value of the multiple factor $\eta = 0.01$. It is worthwhile to note that unless otherwise stated, the parameters' values are given as described above.

5.3 Sensitivity analysis

Fig.1 illustrates the dynamic behavior of the correction factor $N(t)$. From Fig.1, we find that the correction factors in the cases of $\beta = -1$ and $\beta = -2$ are both increasing with respect to time t and the values are both less than or equal to one. This implies that the correction factor advises the investor to invest a smaller proportion of wealth in stock at the initial time, and steadily increase the proportion as time passes. Moreover, the investor should reduce the proportion of wealth in stock suggested by the "moving Merton strategy". So Corollary 1 is validated exactly.

Additionally, we notice the correction factors have the same trends in the cases of $\beta = -1$ and $\beta = -2$, but the correction factors have one obvious difference. That is to say, the correction factor $\beta = -1$ is always higher than that of $\beta = -2$, but the gap is gradually shrinking. Furthermore, before about 12 years, the correction factors are both negative, which tells the investor to take short positions in stock. However, once the correction factors increase to positive values, the investor should start taking long positions in stock.

Fig.1. Evolution of the correction factors in the case of b=−1 and b=−2

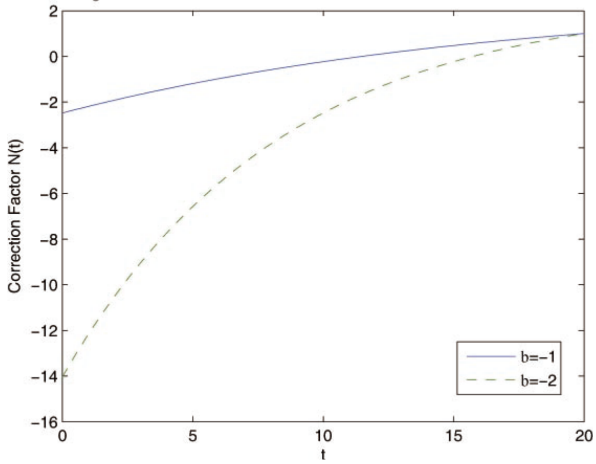
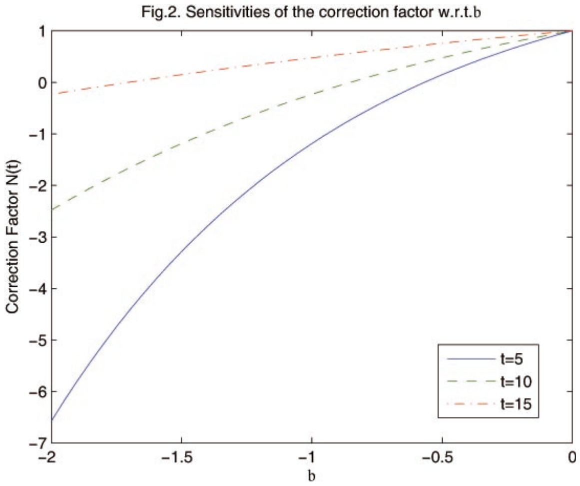


Fig.2 shows the effect of the elasticity coefficient β on the correction factors. From Fig. 2, we find that there is a positive relationship between the correction factor and β. That is to say, as β increases, so does the correction factor. This indicates that the correction factor advises the investor to take long positions in stock as β increases. Moreover, as β deceases to a certain point, the correction factor will be negative, which tells the investor to take short positions in stock.

Besides, Fig.2 also illustrates the trends of the correction factors at different time t ($t = 5, 10, 15$) with respect to β. Like Fig.1, it also further indicates that there is a positive relationship between the correction factor and t.

Fig.2. Sensitivities of the correction factor w.r.t.b



From Fig.3, we find that before about β=-1.6 the positive relationship between the optimal investment amount's expectation and the elasticity coefficient β is extremely significant, which indicates that as the coefficient β increases the investor will increase the optimal expected amount in stock. When β exceeds -1.6, the optimal expected amount $E\Lambda^*(t)$ nearly equals to zero.

The above situation can be explained by Corollary 2. According to Corollary 2 and Remark 6, when $N(t) < 0$, the pension investor will invest a higher proportion of wealth in stock as the parameter β rises, but if $0 \leq N(t) \leq 1$, we can't judge the trend of the optimal strategy with respect to the elasticity coefficient β. From Fig.2, we find the correction factor $N(t)$ is less than zero before about $\beta = -1.6$. Accordingly, the optimal proportion in stock (π_t^*) increases with β rising. Therefore, the trend of the optimal expected amount in stock $E\Lambda^*(t)$ is monotonously increasing before about $\beta = -1.6$ (see Fig.3).

Fig.3. Sensitivities of optimal investment amount's expectation w.r.t.b

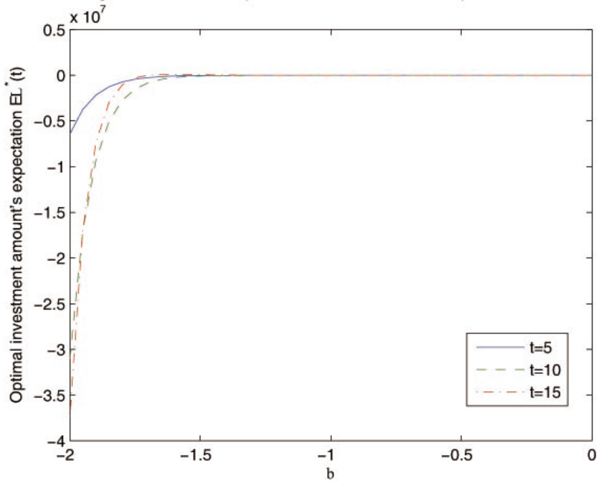


Fig.4 shows the relationship between the optimal investment amount's expectation $E\Lambda^*(t)$ and the absolute risk aversion coefficient q. We find the monotone trends of $E\Lambda^*(t)$ depend on the correction factors $N(t)$. As Fig.4 shows, when $N(t) > 0$, $E\Lambda^*(t)$ is a monotone decreasing function with respect to q. On the contrary, $E\Lambda^*(t)$ is a monotone increasing function with respect to q.

Corollary 3 and Remark 8 further analyze the essential of the above phenomenon by mathematic derivation. That is to say, the optimal strategy π_t^* is a piecewise monotone function with respect to q, which depends on the sign of $N(t)$. When $0 < N(t) \leq 1$, π_t^* is a monotone decreasing function with respect to q. On the contrary, when $N(t) \leq 0$, π_t^* increases as q rising. Obviously, the optimal investment amount's expectation $E\Lambda^*(t)$ also shows the same trend.

Fig.4. Sensitivities of optimal investment amount's expectation w.r.t. q

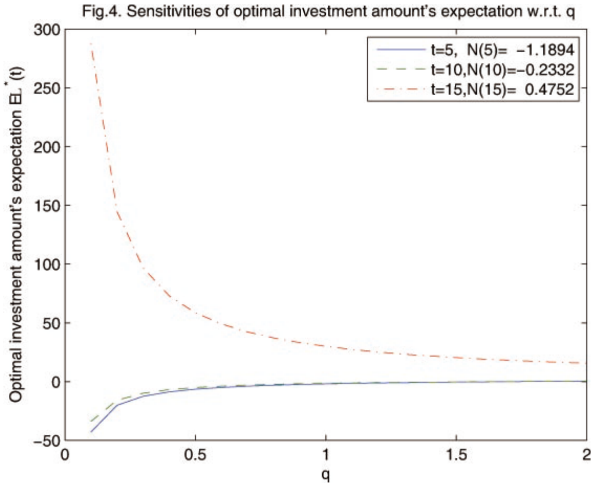
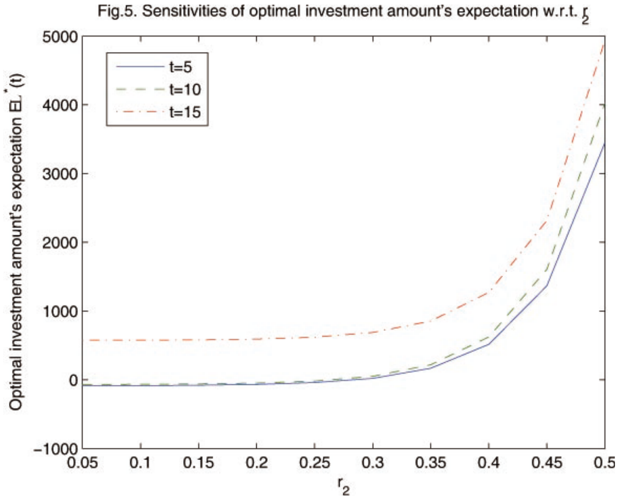


Fig.5 illustrates the impact of the expected return of the stochastic salary r_2 on the optimal investment amount's expectation $E\Lambda^*(t)$. We notice that there is a positive relationship between $E\Lambda^*(t)$ and r_2. This implies that the optimal investment expected amount increases when the expected return of the stochastic salary increases. Moreover, before about $r_2 = 0.3$ the change of $E\Lambda^*(t)$ is relatively small, whereas it sharply rises after about $r_2 = 0.3$. It follows that the investor will allocate more assets to stock when the expected return of the stochastic salary increases and the trend becomes particularly strong after a certain point (about $r_2 = 0.3$).

This is easily understood. Initially, when the expected return of the stochastic salary is low, the investor tends to take short positions in stock in order to avoid risk. However, after it continues to increase a certain point, the higher the expected return of the stochastic salary is, the more the investor is likely to take risks. And thus the investor can increase the amount invested in the stock.

Fig.5. Sensitivities of optimal investment amount's expectation w.r.t. r_2



η is the multiple relationship between the stock and salary with respect to the instantaneous volatility. From Fig.6, we find that the functional curves show that $E\Lambda^*(t)$ increases with respect to η. In fact, it reflects that the multiple factor η significantly affects the optimal investment expected amount $E\Lambda^*(t)$. Hence, we can say that how to allocate the assets to the stock is dependent on the value of the multiple factor η.

In addition, given the instantaneous volatility of the stock, the higher η is, the higher the instantaneous volatility of the salary is. At that time, the investor is likely to allocate more assets to the stock to earn a big profit.

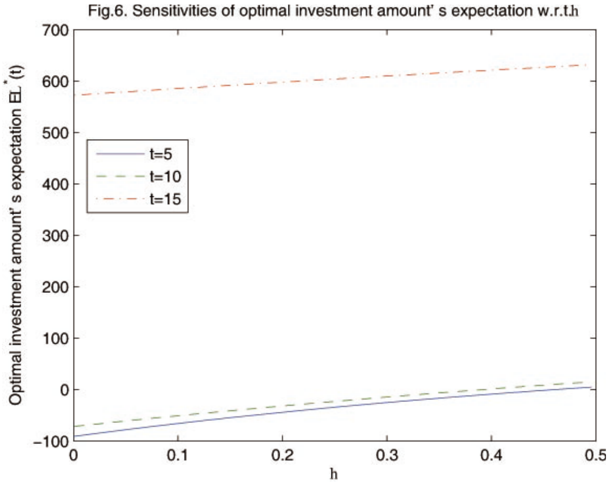


Fig.6. Sensitivities of optimal investment amount' s expectation w.r.t.h

5.4 Dynamic behavior of the optimal strategy

Fig.7 illustrates the evolution of the stock price over time under the CEV model. From Fig.7, the fluctuation of the stock price is very fierce.

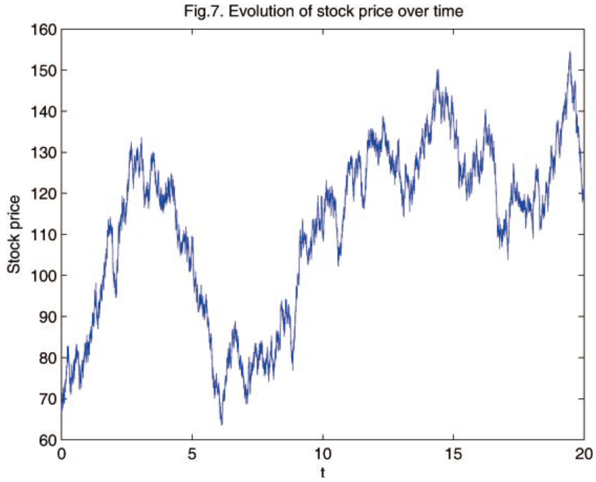
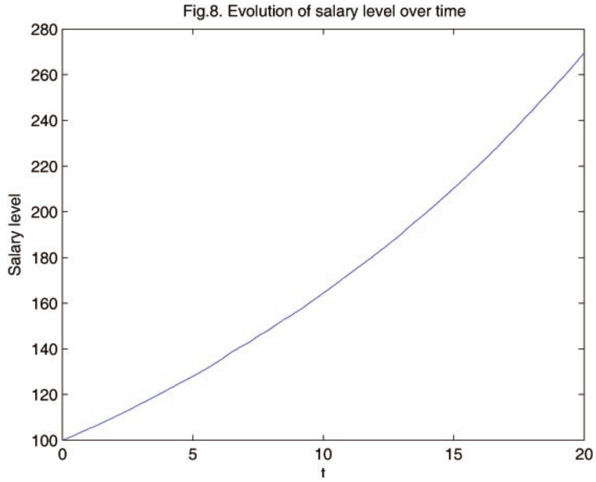
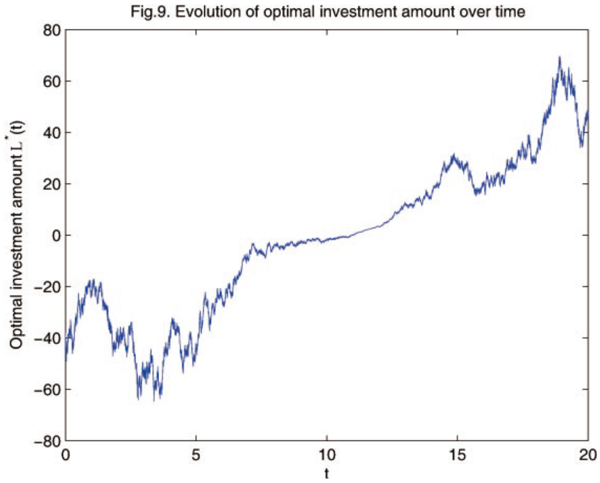


Fig.7. Evolution of stock price over time

Fig.8 shows the evolution of the salary level over time. From Fig.8, we find that the salary level is increasing with respect to time t and the value is greater than the initial value, 100.



The optimal investment amount of the stock is depicted in Fig.9. As shown in Fig.9, the overall tendency of the optimal amount invested in the stock $\Lambda^*(t)$ is increasing with respect to time t.



§6 Conclusions

In this paper, we study an investment problem for a defined contribution pension plan with the stochastic salary. Especially, we have focused on the CEV model to describe the dynamic movements of the risky asset price, which is a generalization of the GBM. The stochastic salary follows a stochastic differential equation (SDE), the instantaneous volatility changes with the risky asset price all the time, and there is a multiple relationship between the risk sources of stock and stochastic salary. The investment objective is to maximize the exponential utility of terminal wealth. The explicit solution has been obtained by power transform and variable change technique. The optimal investment strategy contains a "moving Merton strategy", a "correction factor", a "stochastic salary strategy" and an "adjustment factor". By restricting some coefficients to the special values, the model or the optimal investment strategy can all reduce to the predecessors' achievements, so we generalize the existing works of Devolder, et al. (2003) and Gao (2009a,b). At last, we have provided a numerical analysis to investigate the dynamic behaviors of the correlation factor and the optimal strategy.

Since the GBM is a special form of the CEV model, the further researches on the stochastic optimal control of DC mainly spread our work under the CEV model: (i) introducing different stochastic interest rate models under the research framework; (ii) increasing the number of the risky asset which follows the CEV model etc. It is noteworthy that the optimization problem with the CEV model is very difficult. Nevertheless, the above methodology can't be applied to the extended framework. It will result in a more sophisticated nonlinear partial differential equation, and it is difficult to gain the explicit solution at present. Thus we must make a breakthrough on the solving technology.

Acknowledgement. The authors would like to thank the editor and the anonymous reviewers for careful reading and helpful comments on the earlier version of the paper, which led to a considerable improvement of the presentation of the work.

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