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Maximum two-dimensional $(u \times v, 4, 1, 3)$-OOCs

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Abstract. Let $\Phi(u \times v, k, \lambda_a, \lambda_c)$ be the largest possible number of codewords among all two-dimensional $(u \times v, k, \lambda_a, \lambda_c)$ optical orthogonal codes. A 2-D $(u \times v, k, \lambda_a, \lambda_c)$-OOC with $\Phi(u \times v, k, \lambda_a, \lambda_c)$ codewords is said to be maximum. In this paper, the number of codewords of a maximum 2-D $(u \times v, 4, 1, 3)$-OOC has been determined.

§1 Introduction

Let u, v, k, λ_a and λ_c be positive integers. $\Omega(u \times v, k)$ denotes the set of all k-subsets of $I_u \times Z_v$, where $I_u = \{0, 1, \ldots, u-1\}$ and Z_v is the residue group of integers modulo v. A *two-dimensional* $(u \times v, k, \lambda_a, \lambda_c)$ *optical orthogonal code* (briefly 2-D $(u \times v, k, \lambda_a, \lambda_c)$-OOC), $\mathcal{C}$, can be viewed as a family of k-subsets (codewords) of $\Omega(u \times v, k)$ satisfying the following two properties:

1) The auto-correlation property: $|X \cap (X + \tau)| \leq \lambda_a$ for any $X \in \mathcal{C}$ and every $\tau \in Z_v \setminus \{0\}$;

2) The cross-correlation property: $|X \cap (Y + \tau)| \leq \lambda_c$ for any $X, Y \in \mathcal{C}$ with $X \neq Y$ and every $\tau \in Z_v$,

where $X + \tau = \{(x, i + \tau) : (x, i) \in X\}$ and all the additive operations are performed in Z_v.

For given positive integers u, v, k, λ_a and λ_c, let $\Phi(u \times v, k, \lambda_a, \lambda_c)$ denote the largest possible number of codewords among all 2-D $(u \times v, k, \lambda_a, \lambda_c)$-OOCs. A 2-D $(u \times v, k, \lambda_a, \lambda_c)$-OOC with $\Phi(u \times v, k, \lambda_a, \lambda_c)$ codewords is said to be *maximum* (or *optimal*). A 2-D $(1 \times v, k, \lambda_a, \lambda_c)$-OOC is usually called one-dimensional $(v, k, \lambda_a, \lambda_c)$-OOC (or 1-D $(v, k, \lambda_a, \lambda_c)$-OOC). When $\lambda_a = \lambda_c = \lambda$, the notions of $(u \times v, k, \lambda)$-OOC and $\Phi(u \times v, k, \lambda)$ are employed for abbreviation.

The study of optical orthogonal codes has been motivated by applications in an optical code-division multiple access (OCDMA) system. For more details, the interested reader may refer to [18, 20, 21]. For a long time, the research on optical orthogonal codes has been concentrated

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on 1-D OOCs, such as [1–3,5,7–13,15,17,19,24,25]. Furthermore, there are several research on maximum 2-D OOCs either, the reader may refer to [4,6,14,22,23].

For any $g \in Z_v$ and $X \in \Omega(u \times v, k)$, define $X + g = \{(x, i+g) : (x, i) \in X\}$. Then Z_v acts on $\Omega(u \times v, k)$. The set $O(X) = \{X + g : g \in Z_v\}$ is called the *orbit* containing X. The orbit with v elements is said to be *full*, otherwise *short*. It is clear that $\Omega(u \times v, k)$ can be partitioned into some orbits under the action of Z_v.

Let $X \in \Omega(u \times v, k)$. For any $x \in I_u$, we define the list of (x, x) pure differences of X by $\Delta_{xx}(X) = \{j - i : (x, i), (x, j) \in X, i \neq j\}$, as a multiset, where the minus operation is performed in Z_v. In addition, let $\lambda(X)$ denote the maximum multiplicity of the differences in the multiset $\cup_{x \in I_u} \Delta_{xx}(X)$. Then we have the following formula

$$\lambda(X) = \max\{|X \cap (X + \tau)| : \tau \in Z_v \setminus \{0\}\}. \tag{1}$$

Formula (1) is proved as follows. If $\lambda(X) = t$, let τ be a difference with multiplicity t in the multiset $\cup_{x \in I_u} \Delta_{xx}(X)$. Then there are exactly t pairs (x_s, i_s) and (x_s, j_s) from X such that $\tau = j_s - i_s$, where $1 \leq s \leq t$. Note that (x_s, j_s), $1 \leq s \leq t$, are pairwise distinct and $\{(x_s, j_s) : 1 \leq s \leq t\} \subseteq X \cap (X + \tau)$. Hence, we have $\lambda(X) \leq \max\{|X \cap (X + \tau)| : \tau \in Z_v \setminus \{0\}\}$. On the other hand, suppose $\max\{|X \cap (X + \tau)| : \tau \in Z_v \setminus \{0\}\} = t$, then there exist some $\tau \in Z_v \setminus \{0\}$ such that $|X \cap (X + \tau)| = t$. For each $(x, i) \in X \cap (X + \tau)$, we have $(x, i - \tau), (x, i) \in X$, this pair leads a pure difference τ. Then the multiplicity of τ in the multiset $\cup_{x \in I_u} \Delta_{xx}(X)$ is at least t, and hence $t \leq \lambda(X)$. This completes the proof of the formula (1).

By the definition of 2-D OOC and the formula (1), it is not difficult to observe that a 2-D $(u \times v, k, \lambda, k-1)$-OOC is a set $\mathcal{C}$ of distinct representatives of full orbits in $\Omega(u \times v, k)$ under the action of Z_v satisfying that $\lambda = \max\{\lambda(X) : X \in \mathcal{C}\}$, and $\lambda \leq k - 1$. Furthermore, $\Phi(u \times v, k, \lambda, k-1)$ is the number of all full orbits $O(X)$ in $\Omega(u \times v, k)$ under the action of Z_v satisfying that $\lambda(X) \leq \lambda$. Then we have the following result.

Lemma 1.1. *Let u, v, k and α be positive integers and $1 \leq \alpha \leq k - 2$. Then*

$$\Phi(u \times v, k, \alpha, k-1) = \Phi(u \times v, k, \alpha + 1, k-1) - |\Theta(\alpha)|,$$

where $\Theta(\alpha)$ is the set of all full orbits $O(X)$ in $\Omega(u \times v, k)$ under the action of Z_v with $\lambda(X) = \alpha + 1$.

In what follows, $\Phi(1 \times v, k, \lambda_a, \lambda_c)$ is briefly denoted by $\Phi(v, k, \lambda_a, \lambda_c)$. When $I_u = \{x\}$, applying Lemma 1.1, we have

$$\Phi(v, k, \alpha, k-1) = \Phi(v, k, \alpha + 1, k-1) - |\Theta_x(\alpha)|,$$

where $\Theta_x(\alpha)$ is the set of all full orbits $O(X)$ in $\Omega(1 \times v, k)$ under the action of Z_v with $X \in \Omega(1 \times v, k)$ and $\lambda(X) = \alpha + 1$. Then we have following conclusion.

Corollary 1.1. *Let u, v, k and α be positive integers and $1 \leq \alpha \leq k - 2$. Then*

$$|\theta_\alpha| = u[\Phi(v, k, \alpha + 1, k-1) - \Phi(v, k, \alpha, k-1)],$$

where θ_α is the set of all full orbit $O(X)$ in $\Omega(u \times v, k)$ under the action of Z_v with $X \in \Omega(1 \times v, k)$ and $\lambda(X) = \alpha + 1$.

In this paper, we shall investigate the sizes of maximum $(u \times v, 4, 1, 3)$-OOCs. We finally determine the exact value of $\Phi(u\times v, 4, 1, 3)$, which is the number of some full orbits in $\Omega(u\times v, 4)$ under the action of Z_v.

§2 Useful results

Let Q be any orbit in $\Omega(u \times v, 4)$ under the action of Z_v. If $X, Y \in Q$, then $\cup_{x\in I_u}\Delta_{xx}(X) = \cup_{x\in I_u}\Delta_{xx}(Y)$ and $\lambda(X) = \lambda(Y)$. Without loss of generality, we can assume that a representative X of orbit Q is one of the following types:

Type 1: $X = \{(x, 0), (y, a), (z, b), (w, c)\}$, where $x, y, z, w \in I_u$ are distinct and $a, b, c \in Z_v$;

Type 2: $X = \{(x, 0), (x, a), (y, b), (z, c)\}$, where $x, y, z \in I_u$ are distinct, $a, b, c \in Z_v$ and $a \neq 0$;

Type 3: $X = \{(x, 0), (x, a), (y, b), (y, c)\}$, where $x \neq y \in I_u$, $a \in Z_v \setminus \{0\}$ and $b \neq c \in Z_v$;

Type 4: $X = \{(x, 0), (x, a), (x, b), (y, c)\}$, where $x \neq y \in I_u$, $a \neq b \in Z_v \setminus \{0\}$ and $c \in Z_v$;

Type 5: $X = \{(x, 0), (x, a), (x, b), (x, c)\}$, where $x \in I_u$ and $a, b, c \in Z_v \setminus \{0\}$ are distinct.

The following lemma follows immediately.

Lemma 2.1. *Let $X \in \Omega(u\times v, 4)$. Then if $\lambda(X) = 3$, X is of Type 4 or Type 5; and if $\lambda(X) = 2$, X is not of Type 1.*

We quote the following result in [15, 16] for later use.

Lemma 2.2. [15, Theorem 3.4] *Let v be a positive integer.*

(1) *If v is odd, then*

$$\Phi(v, 4, 2, 3) = \begin{cases} \frac{(v-6)(v^2-1)}{24}, & (v, 15) = 1, \\ \frac{(v-6)(v^2-9)}{24}, & (v, 15) = 3, \\ \frac{(v+2)(v-3)(v-5)}{24}, & (v, 15) = 5, \\ \frac{v^3-6v^2-9v+78}{24}, & (v, 15) = 15. \end{cases}$$

(2) *If v is even, then*

$$\Phi(v, 4, 2, 3) = \begin{cases} \frac{(v-6)(v^2-4)}{24}, & (v, 60) = 2, \\ \frac{(v-6)(v^2-12)}{24}, & (v, 60) = 6, \\ \frac{(v-4)(v^2-2v-12)}{24}, & (v, 60) = 4, 10, \\ \frac{v^3-6v^2-12v+96}{24}, & (v, 60) = 12, 30, \\ \frac{v^3-6v^2-4v+72}{24}, & (v, 60) = 20, \\ \frac{v^3-6v^2-12v+120}{24}, & (v, 60) = 60. \end{cases}$$

Lemma 2.3. [15, Theorem 4.8] *Let v be an odd positive integer. Then*

$$\Phi(v, 4, 1, 3) = \begin{cases} \frac{(v-1)(v-9)(v-11)}{24} - \Delta_1, & (v, 105) = 1, 5, 7, 35, \\ \frac{(v-3)(v-9)^2}{24} - \Delta_2, & (v, 105) = 3, 15, 21, 105, \end{cases}$$

where

$$\Delta_1 = \begin{cases} 0, & (v,105)=1,\\ 4, & (v,105)=5,\\ 2, & (v,105)=7,\\ 6, & (v,105)=35, \end{cases} \quad \Delta_2 = \begin{cases} 0, & (v,105)=3,\\ 4, & (v,105)=15,\\ 2, & (v,105)=21,\\ 6, & (v,105)=105. \end{cases}$$

Lemma 2.4. [15, Theorem 4.9] *Let v be an even positive integer. Then*

$$\Phi(v,4,1,3) = \begin{cases} \frac{(v-2)(v^2-25v+162)}{24} - \Delta_3, & (v,840)=2,10,14,70,\\ \frac{(v-4)(v^2-23v+144)}{24} - \Delta_4, & (v,840)=4,8,20,28,40,\\ & \qquad 56,140,280,\\ \frac{(v-6)(v^2-21v+102)}{24} - \Delta_5, & (v,840)=6,30,42,210,\\ \frac{(v-12)(v^2-15v+72)}{24} - \Delta_6, & (v,840)=12,24,60,84,120,\\ & \qquad 168,420,840, \end{cases}$$

where

$$\Delta_3 = \begin{cases} 0, & (v,840)=2,\\ 4, & (v,840)=10,\\ 2, & (v,840)=14,\\ 6, & (v,840)=70, \end{cases} \qquad \Delta_5 = \begin{cases} 0, & (v,840)=6,\\ 4, & (v,840)=30,\\ 2, & (v,840)=42,\\ 6, & (v,840)=210, \end{cases}$$

$$\Delta_4 = \begin{cases} 0, & (v,840)=4,\\ 4, & (v,840)=8,20,\\ 2, & (v,840)=28,\\ 8, & (v,840)=40,\\ 6, & (v,840)=56,140,\\ 10, & (v,840)=280, \end{cases} \qquad \Delta_6 = \begin{cases} 0, & (v,840)=12,\\ 4, & (v,840)=24,60,\\ 2, & (v,840)=84,\\ 8, & (v,840)=120,\\ 6, & (v,840)=168,420,\\ 10, & (v,840)=840. \end{cases}$$

Lemma 2.5. [16, Theorem 3.4] *Let u and v be positive integers.*

(1) *If v is odd, then*

$$\Phi(u\times v,4,2,3) = \begin{cases} \frac{u[(uv-1)(uv-2)(uv-3)-12(v-1)]}{24}, & (v,15)=1,\\ \frac{u(u^3v^3-6u^2v^2+3uv-12v+54)}{24}, & (v,15)=3,\\ \frac{u[(uv-1)(uv-2)(uv-3)-12(v-3)]}{24}, & (v,15)=5,\\ \frac{u(u^3v^3-6u^2v^2+3uv-12v+78)}{24}, & (v,15)=15. \end{cases}$$

(2) *If v is even, then*

$$\Phi(u\times v,4,2,3) = \begin{cases} \frac{u[uv(uv-2)(uv-4)-12(v-2)]}{24}, & (v,60)=2,\\ \frac{u[uv(uv-2)(uv-4)-12(v-4)]}{24}, & (v,60)=4,10,\\ \frac{u[u^2v^2(uv-6)-12(v-6)]}{24}, & (v,60)=6,\\ \frac{u[u^2v^2(uv-6)-12(v-8)]}{24}, & (v,60)=12,30,\\ \frac{u[uv(uv-2)(uv-4)-12(v-6)]}{24}, & (v,60)=20,\\ \frac{u[u^2v^2(uv-6)-12(v-10)]}{24}, & (v,60)=60. \end{cases}$$

§3 Exact value of $\Phi(u \times v, 4, 1, 3)$

In this section we will determine the exact value of $\Phi(u \times v, 4, 1, 3)$. By Lemma 1.1, we only need to compute the size of $\Theta(1)$, where $\Theta(1)$ is the set of all full orbits $O(X)$ in $\Omega(u \times v, 4)$ under the action of Z_v with $X \in \Omega(u \times v, 4)$ and $\lambda(X) = 2$. By Lemma 2.1, $\Theta(1)$ can be written as the union of disjoint sets Θ_i, $3 \le i \le 6$, that is $\Theta(1) = \bigcup_{i=3}^{6} \Theta_i$, where

$\Theta_3 = \{O(X) : X = \{(x,0),(x,a),(y,b),(z,c)\},\ x, y, z \in I_u$ are distinct, $a, b, c \in Z_v$, $a \neq 0$ and $\lambda(X) = 2\}$,

$\Theta_4 = \{O(X) : X = \{(x,0),(x,a),(y,b),(y,c)\},\ x \neq y \in I_u,\ a \in Z_v \setminus \{0\},\ b \neq c \in Z_v$ and $\lambda(X) = 2\}$,

$\Theta_5 = \{O(X) : X = \{(x,0),(x,a),(x,b),(y,c)\},\ x \neq y \in I_u,\ a \neq b \in Z_v \setminus \{0\},\ c \in Z_v$ and $\lambda(X) = 2\}$,

$\Theta_6 = \{O(X) : X = \{(x,0),(x,a),(x,b),(x,c)\},\ x \in I_u,\ a, b, c \in Z_v \setminus \{0\}$ are distinct and $\lambda(X) = 2\}$.

Hence, we have $|\Theta(1)| = \sum_{i=3}^{6} |\Theta_i|$. Next we will compute the size of Θ_i for each $3 \le i \le 6$.

Lemma 3.1. *Let u and v be positive integers. Then* $|\Theta_3| = \frac{u(u-1)(u-2)v^2}{4}$ *if* $2|v$.

Proof. Let Q be any orbit of Θ_3. By the definition of Θ_3, Q can be written as $Q = O(X)$ where $X = \{(x,0),(x,a),(y,b),(z,c)\}$, $x, y, z \in I_u$ are distinct, $a, b, c \in Z_v$, $a \neq 0$ and $\lambda(X) = 2$. Then $\Delta_{xx}(X) = \{\pm a\}$. Hence $\lambda(X) = 2$ if and only if $2a \equiv 0\ (mod\ v)$, i.e., $2|v$ and $a = v/2$. If $2|v$, then Q can be rewritten as $Q = O(Z)$, where $Z = \{(x,0),(x,\frac{v}{2}),(y,b),(z,c)\}$, $x, y, z \in I_u$ are distinct, $y < z$ and $b, c \in Z_v$. Let $A = \{Z : Z = \{(x,0),(x,\frac{v}{2}),(y,b),(z,c)\}, x, y, z \in I_u$ are distinct, $y < z$, and $b, c \in Z_v\}$. Then $\Theta_3 = \{O(Z) :\ Z \in A\}$. Define a mapping σ from A to Θ_3 by $Z \to O(Z)$ for any $Z \in A$. Obviously σ is a surjection from A onto Θ_3. For any given $Q \in \Theta_3$, we compute the cardinality of $\sigma^{-1}(Q) \subseteq A$. Suppose that $Q = O(Z)$ for some $Z \in A$. If $Z' = \{(x',0),(x',\frac{v}{2}),(y',b'),(z',c')\} \in \sigma^{-1}(Q)$ where $x', y', z' \in I_u$ are distinct, $y' < z'$ and $b', c' \in Z_v$, then $O(Z) = O(Z')$, i.e., $Z = Z' + d$ for some $d \in Z_v$. This gives $\{(x,0),(x,\frac{v}{2})\} = \{(x',d),(x',\frac{v}{2}+d)\}$ and $\{(y,b),(z,c)\} = \{(y',b'+d),(z',c'+d)\}$, which implies $(x,y,z) = (x',y',z')$ and $d = 0, v/2$. If $d = 0$, then $b' = b$ and $c' = c$, i.e., $Z' = Z$. If $d = v/2$, then $b' = b - v/2$ and $c' = c - v/2$, i.e., $Z' = Y$ where $Y = \{(x,0),(x,\frac{v}{2}),(y,b-v/2),(z,c-v/2)\}$. Hence, $\sigma^{-1}(Q) \subseteq \{Y, Z\}$. It is obvious that $\{Y, Z\} \subseteq \sigma^{-1}(Q)$. Therefore, we have $\sigma^{-1}(Q) = \{Y, Z\}$ and $|\sigma^{-1}(Q)| = 2$ for any $Q \in \Theta_3$. Hence $|\Theta_3| = |A|/2 = \frac{u(u-1)(u-2)v^2}{4}$ if $2|v$. $\square$

Lemma 3.2. *Let u and v be positive integers. Then*

$$|\Theta_4| = \begin{cases} \frac{uv(u-1)(v-2)}{2}, & v \equiv 0\ (mod\ 2), \\ \frac{uv(u-1)(v-1)}{4}, & v \equiv 1\ (mod\ 2). \end{cases}$$

Proof. Let Q be any orbit of Θ_4. By the definition of Θ_4, Q can be written as $Q = O(X)$, $X = \{(x,0),(x,a),(y,b),(y,c)\}$, where $x \neq y \in I_u$, $a \in Z_v \setminus \{0\}$, $b \neq c \in Z_v$ and $\lambda(X) = 2$. Then $\Delta_{xx}(X) \cup \Delta_{yy}(X) = \{\pm a,\ \pm(c-b)\}$. If $2|v$ and $v/2 \in \Delta_{xx}(X) \cup \Delta_{yy}(X)$, then $a = v/2$ and $b - c \neq v/2$ or $b - c = v/2$ and $a \neq v/2$, in this case Q can be rewritten as $Q = O(Z)$, $Z = \{(x,0),(x,\frac{v}{2}),(y,b),(y,c)\}$, where $x \neq y \in I_u$, $b \neq c \in Z_v$ and $c - b \neq v/2$. If $v/2 \notin \Delta_{xx}(X) \cup \Delta_{yy}(X)$, $\lambda(X) = 2$ is equivalent to $a \equiv \pm(c-b) \pmod v$ and $2a \not\equiv 0 \pmod v$; in this case Q can be rewritten as $Q = O(Z)$, $Z = \{(x,0),(x,a),(y,b),(y,b-a)\}$, where $x, y \in I_u$, $x < y$, $a, b \in Z_v$ and $2a \not\equiv 0 \pmod v$. We define two sets as below.

$B_1 = \{Z : Z = \{(x,0),(x,\frac{v}{2}),(y,b),(y,c)\},\ x \neq y \in I_u,\ b \neq c \in Z_v,\ c - b \neq v/2\}$ if $2|v$,

$B_2 = \{Z : Z = \{(x,0),(x,a),(y,b),(y,b-a)\}, x, y \in I_u, x < y, a, b \in Z_v,\ 2a \not\equiv 0 \pmod v)\}$.

Then Θ_4 is the union of two disjoint subsets, i.e., $\Theta_4 = \Theta_{41} \bigcup \Theta_{42}$, where $\Theta_{4i} = \{O(Z) : Z \in B_i\}$, $i = 1, 2$. To determine the cardinality of Θ_4, we just need to compute $|\Theta_{41}|$ and $|\Theta_{42}|$.

We also define mapping σ_i from B_i to Θ_{4i} by $Z \to O(Z)$ for any $Z \in B_i$, $i = 1, 2$. Then σ_i is clearly a surjection from B_i onto Θ_{4i}. For given $Q \in \Theta_{4i}$, we will compute the cardinality of $\sigma_i^{-1}(Q) \subseteq B_i$ for $i = 1, 2$.

Suppose that $Q = O(Z)$ for some $Z \in B_1$. If $Z' = \{(x',0),(x',\frac{v}{2}),(y',b'),\ (y',c')\} \in \sigma_1^{-1}(Q)$ where $x' \neq y' \in I_u$, $b' \neq c' \in Z_v$ and $c' - b' \neq v/2$, then $O(Z) = O(Z')$, i.e., $Z = Z' + d$ for some $d \in Z_v$. This gives $\{(x,0),(x,\frac{v}{2})\} = \{(x',d),(x',\frac{v}{2}+d)\}$ and $\{(y,b),(y,c)\} = \{(y',b'+d),(y',c'+d)\}$, which implies $(x,y) = (x',y')$ and $d = 0, v/2$. Moreover, we have $(b',c') \in \{(b,c),(c,b),(b-v/2,c-v/2),(c-v/2,b-v/2)\}$. It is not difficult to show that $\sigma_1^{-1}(Q) = \{\{(x,0),(x,\frac{v}{2}),(y,b'),(y,c')\} : (b',c') \in \{(b,c),(c,b),(b-v/2,c-v/2),(c-v/2,b-v/2)\}\}$. Therefore, we have $|\sigma_1^{-1}(Q)| = 4$ for any orbit $Q \in \Theta_{41}$. Hence $|\Theta_{41}| = |B_1|/4 = \frac{uv(u-1)(v-2)}{4}$ if $2|v$.

Similarly, suppose that $Q = O(Z)$ for some $Z \in B_2$. If $Z' = \{(x',0),(x',a'),\ (y',b'),(y',b'-a')\} \in \sigma_2^{-1}(Q)$, where $x', y' \in I_u$, $x' < y'$, $a', b' \in Z_v$ and $2a' \not\equiv 0 \pmod v$, then $O(Z) = O(Z')$, i.e., $Z = Z' + d$ for some $d \in Z_v$. This gives $\{(x,0),(x,a)\} = \{(x',d),(x',a'+d)\}$ and $\{(y,b),(y,b-a)\} = \{(y',b'+d),(y',b'-a'+d)\}$, which implies $(x,y) = (x',y')$, $\{0,a\} = \{d,a'+d\}$ and $\{b,b-a\} = \{b'+d,b'-a'+d\}$. If $d = 0$, we have $a' = a$, then $b' = b$. Otherwise, $b' = b - a$ and $b = b' - a$, that is $2a \equiv 0\ (mod\ v)$, it is impossible. In this case we have $(x',y',a',b') = (x,y,a,b)$. If $d = a$, then $a' = -a$ and $\{b,b-a\} = \{b'+a,b'+2a\}$, which gives $(x',y',a',b') = (x,y,-a,b-2a)$. Hence, we have $\sigma_2^{-1}(Q) \subseteq \{\{(x,0),(x,a'),(y,b'),(y,b'-a')\} : (a',b') \in \{(a,b),(-a,b-2a)\}\}$. It is easy to check that $\{(x,0),(x,a'),(y,b'),(y,b'-a')\} \in \sigma_2^{-1}(Q)$ for $(a',b') \in \{(a,b),(-a,b-2a)\}$. Therefore, $|\sigma_2^{-1}(Q)| = 2$ for any orbit $Q \in \Theta_{42}$. Hence, $|\Theta_{42}| = |B_2|/2 = \frac{uv(u-1)}{2}\lfloor\frac{v-1}{2}\rfloor$. Since $|\Theta_4| = |\Theta_{41}| + |\Theta_{42}|$, we obtain the conclusion immediately. $\square$

Lemma 3.3. *Let u and v be positive integers.*

(1) *If v is odd, then*

$$|\Theta_5| = \begin{cases} \frac{uv(u-1)(v-3)}{2}, & v \equiv 3\ (mod\ 6), \\ \frac{uv(u-1)(v-1)}{2}, & v \equiv 1, 5\ (mod\ 6). \end{cases}$$

(2) *If v is even, then*

$$|\Theta_5| = \begin{cases} uv(u-1)(v-2), & v \equiv 2, 10 \ (mod\ 12), \\ uv(u-1)(v-3), & v \equiv 4, 6, 8 \ (mod\ 12), \\ uv(u-1)(v-4), & v \equiv 0 \ (mod\ 12). \end{cases}$$

Proof. Let Q be any orbit of Θ_5. By the definition of Θ_5, Q can be written as $Q = O(X)$, $X = \{(x,0),(x,a),(x,b),(y,c)\}$, where $x \neq y \in I_u$, $a \neq b \in Z_v \backslash \{0\}$, $c \in Z_v$ and $\lambda(X) = 2$. Then $\Delta_{xx}(X) = \{\pm a, \pm b, \pm(b-a)\}$. If $2|v$ and $v/2 \in \Delta_{xx}(X)$, then $a = v/2$ and $b \neq 0, v/2$, or $b = v/2$ and $a \neq 0, v/2$, or $b - a = v/2$ and $a \neq 0, v/2$, in this case Q can be rewritten as $Q = O(Z)$, $Z = \{(x,0),(x,a),(x,\frac{v}{2}),(y,c)\}$, where $x \neq y \in I_u$, $a, c \in Z_v$ and $a \neq 0, v/2$. If $v/2 \notin \Delta_{xx}(X)$, $\lambda(X) = 2$ is equivalent to $b \equiv -a$ $(mod\ v)$ or $b \equiv 2a$ $(mod\ v)$ satisfying that $ra \not\equiv 0$ (mod v) for $r = 3, 4$; in this case Q can be rewritten as $Q = O(Z)$, $Z = \{(x,0),(x,a),(x,2a),(y,c)\}$, where $x \neq y \in I_u$, $a, c \in Z_v$ and $ra \not\equiv 0$ (mod v) for $r = 3, 4$. We define two sets as below.

$C_1 = \{Z : Z = \{(x,0),(x,a),(x,\frac{v}{2}),(y,c)\}, x \neq y \in I_u, a, c \in Z_v, a \neq 0, v/2\}$ if $2|v$,

$C_2 = \{Z : Z = \{(x,0),(x,a),(x,2a),(y,c)\}, x \neq y \in I_u, a, c \in Z_v, ra \not\equiv 0$ (mod v) for $r = 3, 4\}$.

Then Θ_5 is the union of the following two disjoint sets, i.e., $\Theta_5 = \Theta_{51} \bigcup \Theta_{52}$, where $\Theta_{5i} = \{O(Z) : Z \in C_i\}$ for $i = 1, 2$. Hence $|\Theta_5| = |\Theta_{51}| + |\Theta_{52}|$ and the rest of the lemma is to compute the value of $|\Theta_{51}|$ and $|\Theta_{52}|$. For this purpose, we define a surjection τ_i from C_i to Θ_{5i} by $Z \to O(Z)$ for any $Z \in C_i$, $i = 1, 2$. For any $Q \in \Theta_{5i}$, we will compute the cardinality of $\tau_i^{-1}(Q) \subseteq C_i$ where $i = 1, 2$.

Suppose that $Q = O(Z)$ for some $Z \in C_1$. If $Z' = \{(x',0),(x',a'),(x',\frac{v}{2}),(y',c')\} \in \tau_1^{-1}(Q)$, where $x' \neq y' \in I_u$, $a', c' \in Z_v$ and $a' \neq 0, v/2$, then $O(Z) = O(Z')$, i.e., $Z = Z' + d$ for some $d \in Z_v$. This gives $\{(x,0),(x,a),(x,\frac{v}{2})\} = \{(x',d),(x',a'+d),(x',\frac{v}{2}+d)\}$ and $(y,c) = (y',c'+d)$, which implies $(x',y') = (x,y)$, $c' = c-d$ and $\{0,a,\frac{v}{2}\} = \{d, a'+d, \frac{v}{2}+d\}$. Moreover, we obtain $(a',c') \in \{(a,c),(a-v/2,c-v/2)\}$. It is not difficult to show that

$$\tau_1^{-1}(Q) = \{\{(x,0),(x,a'),(x,\frac{v}{2}),(y,c')\} : (a',c') \in \{(a,c),(a-v/2,c-v/2)\}\}.$$

Therefore, for any $Q \in \Theta_{51}$, we have $|\tau_1^{-1}(Q)| = 2$. Hence $|\Theta_{51}| = |C_1|/2 = \frac{uv(u-1)(v-2)}{2}$ if $2|v$.

Similarly, suppose that $Q = O(Z)$ for some $Z \in C_2$. If $Z' = \{(x',0),(x',a'),(x',2a'),(y',c')\} \in \tau_2^{-1}(Q)$, where $x' \neq y' \in I_u$, $a', c' \in Z_v$ and $ra' \not\equiv 0$ (mod v) for $r = 3, 4$, then $O(Z) = O(Z')$, i.e., $Z = Z' + d$ for some $d \in Z_v$. This gives $\{(x,0),(x,a),(x,2a)\} = \{(x',d),(x',a'+d),(x'+2a'+d)\}$ and $(y,c) = (y',c'+d)$, which implies $(x',y') = (x,y)$, $c' = c-d$ and $\{0,a,2a\} = \{d, a'+d, 2a'+d\}$. Further discussion shows that $(a',c') \in \{(a,c),(-a,c-2a)\}$. It is easy to check that

$$\tau_2^{-1}(Q) = \{\{(x,0),(x,a'),(x,2a'),(y,c')\} : (a',c') \in \{(a,c),(-a,c-2a)\}\}.$$

Hence we have $|\tau_2^{-1}(Q)| = 2$ for any given $Q \in \Theta_{52}$. Therefore, $|\Theta_{52}| = |C_2|/2 = \frac{uv(u-1)(v-\rho)}{2}$, where ρ denote the number of $a \in Z_v$ such that $ra \equiv 0$ (mod v) for $r \in \{3, 4\}$. Note that $|\Theta_5| = |\Theta_{51}| + |\Theta_{52}|$. The conclusion then follows. $\square$

Lemma 3.4. *Let u and v be a positive integers.*

(1) *If v is odd, then*

$$|\Theta_6| = \begin{cases} \frac{5u(v-1)(v-7)}{8} + u\gamma_1, & (v,105) = 1,5,7,35, \\ \frac{u(v-3)(5v-33)}{8} + u\gamma_2, & (v,105) = 3,15,21,105, \end{cases}$$

where

$$\gamma_1 = \begin{cases} 0, & (v,105) = 1, \\ 5, & (v,105) = 5, \\ 2, & (v,105) = 7, \\ 7, & (v,105) = 35, \end{cases} \qquad \gamma_2 = \begin{cases} 0, & (v,105) = 3, \\ 5, & (v,105) = 15, \\ 2, & (v,105) = 21, \\ 7, & (v,105) = 105. \end{cases}$$

(2) *If v is even, then*

$$|\Theta_6| = \begin{cases} \frac{u(v-2)(7v-58)}{8} + u\gamma_3, & (v,840) = 2,10,14,70, \\ \frac{u(v-4)(7v-52)}{8} + u\gamma_4, & (v,840) = 4,8,20,28,40,56,140,280, \\ \frac{u(v-6)(7v-38)}{8} + u\gamma_5, & (v,840) = 6,30,42,210, \\ \frac{u(7v^2-88v+320)}{8} + u\gamma_6, & (v,840) = 12,24,60,84,120,168, \\ & \qquad 420,840, \end{cases}$$

where

$$\gamma_3 = \begin{cases} 0, & (v,840) = 2, \\ 5, & (v,840) = 10, \\ 2, & (v,840) = 14, \\ 7, & (v,840) = 70, \end{cases} \qquad \gamma_5 = \begin{cases} 0, & (v,840) = 6, \\ 5, & (v,840) = 30, \\ 2, & (v,840) = 42, \\ 7, & (v,840) = 210, \end{cases}$$

$$\gamma_4 = \begin{cases} 0, & (v,840) = 4, \\ 4, & (v,840) = 8, \\ 5, & (v,840) = 20, \\ 2, & (v,840) = 28, \\ 9, & (v,840) = 40, \\ 6, & (v,840) = 56, \\ 7, & (v,840) = 140, \\ 11, & (v,840) = 280, \end{cases} \qquad \gamma_6 = \begin{cases} 0, & (v,840) = 12, \\ 4, & (v,840) = 24, \\ 5, & (v,840) = 60, \\ 2, & (v,840) = 84, \\ 9, & (v,840) = 120, \\ 6, & (v,840) = 168, \\ 7, & (v,840) = 420, \\ 11, & (v,840) = 840. \end{cases}$$

Proof. By Corollary 1.1, $|\Theta_6| = u[\Phi(v,4,2,3) - \Phi(v,4,1,3)]$. Employing Lemmas 2.2-2.4, the conclusion then follows after careful calculations.

After the well-preparation, we are in position to present the main result in this section.

Theorem 3.1. *Let u and v be positive integers and v odd. Then*

$$\Phi(u \times v, 4, 1, 3)$$
$$= \begin{cases} \frac{u[(uv-1)(uv-2)(uv-3)-3(v-1)(6uv-v-31)]}{24} - u\eta_1, & (v,105) = 1,5,7,35, \\ \frac{u(u^3v^3-6u^2v^2-18uv^2+45uv+3v^2+90v-243)}{24} - u\eta_2, & (v,105) = 3,15,21,105, \end{cases}$$

where

$$\eta_1 = \begin{cases} 0, & (v,105)=1, \\ 4, & (v,105)=5, \\ 2, & (v,105)=7, \\ 6, & (v,105)=35, \end{cases} \qquad \eta_2 = \begin{cases} 0, & (v,105)=3, \\ 4, & (v,105)=15, \\ 2, & (v,105)=21, \\ 6, & (v,105)=105. \end{cases}$$

Proof. From Lemma 1.1, $\Phi(u \times v, 4, 1, 3) = \Phi(u \times v, 4, 2, 3) - |\Theta(1)|$, where $\Theta(1) = \bigcup_{i=3}^{6} \Theta_i$. By Lemmas 3.1-3.4, $|\Theta(1)| = |\Theta_4| + |\Theta_5| + |\Theta_6|$ since v is odd. The conclusion then follows after careful computations.

Theorem 3.2. *Let u and v be positive integers and v even. Then*

$$\Phi(u \times v, 4, 1, 3)$$
$$= \begin{cases} \frac{u(u^3v^3-12u^2v^2-18uv^2+3v^2+80uv+132v-324)}{24} - u\eta_3, & (v,840)=2,10,14,70, \\ \frac{u(u^3v^3-12u^2v^2-18uv^2+3v^2+104uv+132v-576)}{24} - u\eta_4, & (v,840)=4,8,20,28,40, \\ & \qquad 56,140,280, \\ \frac{u(u^3v^3-12u^2v^2-18uv^2+3v^2+96uv+132v-612)}{24} - u\eta_5, & (v,840)=6,30,42,210, \\ \frac{u(u^3v^3-12u^2v^2-18uv^2+3v^2+120uv+132v-864)}{24} - u\eta_6, & (v,840)=12,24,60,84, \\ & \qquad 120,168,420,840, \end{cases}$$

where

$$\eta_3 = \begin{cases} 0, & (v,840)=2, \\ 4, & (v,840)=10, \\ 2, & (v,840)=14, \\ 6, & (v,840)=70, \end{cases} \qquad \eta_5 = \begin{cases} 0, & (v,840)=6, \\ 4, & (v,840)=30, \\ 2, & (v,840)=42, \\ 6, & (v,840)=210, \end{cases}$$

$$\eta_4 = \begin{cases} 0, & (v,840)=4, \\ 4, & (v,840)=8,20, \\ 2, & (v,840)=28, \\ 8, & (v,840)=40, \\ 6, & (v,840)=56,140, \\ 10, & (v,840)=280, \end{cases} \qquad \eta_6 = \begin{cases} 0, & (v,840)=12, \\ 4, & (v,840)=24,60, \\ 2, & (v,840)=84, \\ 8, & (v,840)=120, \\ 6, & (v,840)=168,420, \\ 10, & (v,840)=840, \end{cases}$$

Proof. From Lemma 1.1, $\Phi(u \times v, 4, 1, 3) = \Phi(u \times v, 4, 2, 3) - |\Theta(1)|$, where $\Theta(1) = \bigcup_{i=3}^{6} \Theta_i$. By Lemmas 3.1-3.4, $|\Theta(1)| = \sum_{i=3}^{6} |\Theta_i|$ since v is even. The conclusion then follows by detailed calculations.

§4 Concluding remarks

In the present paper, by Theorems 3.1 and 3.2, we have determined the exact value of $\Phi(u\times v, 4, 1, 3)$. This gives the sizes of maximum 2-D $(u\times v, 4, 1, 3)$-OOCs. So far we completely determine the sizes of maximum 2-D $(u \times v, 4, \lambda, 3)$-OOCs for $1 \leq \lambda \leq 4$ by combining the results in [14, 16]. Determinations of the sizes of maximum 2-D $(u \times v, 4, \lambda_a, \lambda_c)$-OOCs with other parameters λ_a and λ_c are under investigation for further work.

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