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f-Harmonic maps of doubly warped product manifolds

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Abstract. In this paper, we study f-harmonicity of some special maps from or into a doubly warped product manifold. First we recall some properties of doubly twisted product manifolds. After showing that the inclusion maps from Riemannian manifolds M and N into the doubly warped product manifold $M \times_{(\mu,\lambda)} N$ can not be proper f-harmonic maps, we use projection maps and product maps to construct nontrivial f-harmonic maps. Thus we obtain some similar results given in [21], such as the conditions for f-harmonicity of projection maps and some characterizations for non-trivial f-harmonicity of the special product maps. Furthermore, we investigate non-trivial f-harmonicity of the product of two harmonic maps.

§1 Introduction

In many physical important examples of Einstein metric, we usually meet the warped product metric (WPMe). The concept of WPMe plays an important role in differential geometry as well as in theoretical physics. It was first introduced by Kruǒkoviǒ in 1961 (see [15]) and first treated by Bishop and O'Neill [7] in 1969 in their study of manifolds with negative curvature.

Since then, in Riemannian geometry, WPMes have offered new examples of Riemannian manifolds with special curvature properties. Beem, Ehrlich and Powell [4] pointed out that many exact solutions to Einstein's field equation can be expressed in terms of Lorentzian WPMe. Furthermore, Beem and Ehrlich [6] concluded that causality and completeness of WPMes can be related to causality and completeness of their components. O'Neill [18] discussed WPMes and explored curvature formulas in terms of curvatures of their components. The same author also examined Robertson-Walker, static, Schwarzchild and Kruskal space-time as warped product manifold (WPMa).

In general, doubly WPMa can be considered as a generalization of singly WPMa. Beem and Powell [5] considered these product manifolds for Lorentzian manifolds. Allison [1-2] considered causality and global hyperbolicity of doubly WPMa and null pseudoconvexity of Lorentzian

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doubly WPMa. Conformal properties of doubly WPMa were studied by Gebarowski (cf. [13] or references therein).

On the other hand, the study of f-harmonic maps comes from a physical motivation, since in physics, f-harmonic map can be viewed as stationary solution of the inhomogeneous Heisenberg spin system [16]. In addition, the interaction of f-harmonicity with curvature conditions looks promising and justifies their study in order to deduce information on weighted manifolds and gradient Ricci solitons [23].

f-Harmonic maps $\phi : (M, g) \to (N, h)$ with a positive function $f \in C^\infty(M)$ between two Riemannian manifolds M, N, as a generalization of harmonic maps, were first introduced and studied by Lichnerowicz [17] (see also Section 10.20 in Eells-Lemaire's report [11]). It is a critical point of f-energy

$$E_f(\phi) = \frac{1}{2}\int_\Omega f|d\phi|^2 dv_g.$$

The Euler-Lagrange equation gives the f-tension field equation

$$\tau_f(\phi) = f\tau(\phi) + d\phi(\text{grad } f) = 0,$$

where $\tau(\phi) = Tr_g \nabla d\phi$ ([12]) is the tension field of ϕ.

Since an f-harmonic map with $f = const. > 0$ is nothing but a harmonic map, we are interested in non-trivial f-harmonic maps with non-constant function f which are called *proper (*or *non-trivial) f-harmonic maps*. To this end we want to find or construct such examples. Unfortunately, since the examples of proper f-harmonic maps are rather sparse in common spaces, constructing such examples becomes very difficult. Our strategy is to extend the domain manifolds or target manifolds and to construct some special Riemannian metrics.

Motivated by [21] and [3], we discover that the WPMa (either singly or doubly warped) is a good candidate that allows us to construct some new examples of f-harmonic maps.

In their search for biharmonic maps, Perktaş and Kiliç [21] studied biharmonic maps between doubly WPMas, based on generalizing the study of biharmonic maps between (singly) WPMas in [4]. In the same paper, the authors investigated biharmonicity of the inclusion $i_{y_0} : M \to M \times_{(\mu,\lambda)} N$ (resp. $i_{x_0} : N \to M \times_{(\mu,\lambda)} N$) of a Riemannian manifold M (resp. N) into doubly WPMa $M \times_{(\mu,\lambda)} N$. Also in [21] some characterizations for biharmonicity of the projection from $M \times_{(\mu,\lambda)} N$ into the first factor M (resp. the second factor N) are given. Further the authors in [21] obtained two new classes of proper biharmonic maps by using product of harmonic maps and warping metric.

In this paper, following the ideas in [21], we investigate proper f-harmonicity of the maps between doubly WPMas. More precisely, we use the methods provided in [3,21] to study another new type of harmonicity. Our main results are included in Sections 4 and 6. In particular, we also construct some examples of non-trivial f-harmonic maps in Section 6, Example 6.1.

The organization of this paper is as follows. In Section 2, we review the concepts of f-harmonic map. In Section 3, we recall the definitions of doubly twisted product manifold (TPM) and doubly WPMa, respectively, and give a more explicit expression for the Levi-Civita connection $\bar{\nabla}$ on a doubly TPM. Section 4 is devoted to analyzing the conditions for both of the leaves $\{x_0\} \times N$ and $M \times \{y_0\}$ to be f-harmonic as a submanifold and we show that both of the leaves $\{x_0\} \times N$ and $M \times \{y_0\}$ can not be proper f-harmonic as a submanifold of doubly WPM $M \times_{(\mu,\lambda)} N$. In Section 5, we give some characterizations for projection maps $\bar{\pi}_1 : M \times_{(\mu,\lambda)} N \to M$ and $\bar{\pi}_2 : M \times_{(\mu,\lambda)} N \to N$ to be f-harmonic in terms of some particular functions. We also construct an example of non-trivial f-harmonic map. In the last section, we consider that under imposing certain conditions on the two warping functions, the product maps $\overline{\Psi} = \overline{Id_M \times \phi_N}$, $\widetilde{\Psi} = \widetilde{\phi_M \times Id_N}$ and $\widehat{\Psi} = \widehat{Id_M \times \phi_N}$ and more generally $\bar{\Phi} = \varphi_M \times \varphi_N : (M \times_{(\mu,\lambda)} N, \bar{g}) \to (M \times N, g \oplus h)$ may remain f-harmonic maps.

§2 f-harmonic maps

Recall that the energy of a smooth map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is defined by integral $E(\phi) = \int_\Omega e(\phi) dv_g$ for every compact domain $\Omega \subset M$, where $e(\phi) = \frac{1}{2}|d\phi|^2$ is energy density. Then ϕ is called *harmonic* if it's a critical point of energy. From the first variation formula for the energy, the Euler-Lagrange equation is given by the vanishing of the *tension field* $\tau(\phi) = Tr_g \nabla d\phi$ (see [12]). As the generalizations of harmonic maps, biharmonic maps and f-harmonic maps are defined as follows.

Definition 2.1. (i) *Biharmonic maps* $\phi : (M, g) \to (N, h)$ between Riemannian manifolds are critical points of the bienergy functional

$$E^2(\phi) = \frac{1}{2} \int_\Omega |\tau(\phi)|^2 dv_g$$

for any compact domain $\Omega \subset M$.

(ii) An *f-harmonic map with a positive function $f \in C^\infty(M)$* is a critical point of f-energy functional

$$E_f(\phi) = \frac{1}{2} \int_{\Omega \Subset M} f |d\phi|^2 dv_g. \tag{1}$$

The first variation formula for the bienergy which is derived in [14] shows that the Euler-Lagrange equation for bienergy is

$$\tau^2(\phi) = Tr\big(\nabla^\phi_\cdot \nabla^\phi_\cdot \tau(\phi) - \nabla^\phi_{^M\nabla_\cdot \cdot} \tau(\phi)\big) - Tr_g(R^N(d\phi, \tau(\phi))d\phi)) = 0, \tag{2}$$

where $R^N(X, Y)$ is the curvature operator on N. $\tau^2(\phi)$ is called the bitension field of ϕ. The Euler-Lagrange equation for the f-energy is given by $\tau_f(\phi)$ (see [9-10],[19-20])

$$\tau_f(\phi) = f\tau(\phi) + d\phi(\mathrm{grad}_g f) = 0. \tag{3}$$

Clearly any harmonic map ϕ is f-harmonic map with $\mathrm{grad} f \in \mathrm{Ker}(d\phi)$. We call the non-harmonic f-harmonic maps with nonconstant function f as proper (or non-trivial) f-harmonic maps.

§3 Riemannian structure of doubly WPMa

First we refer to [24] and give the definition of doubly WPMa.

Definition 3.1. Let (M, g) and (N, h) be Riemannian manifolds of dimensions m and n respectively and let $\lambda : M \to (0, +\infty)$ and $\mu : N \to (0, +\infty)$ be smooth functions. A *doubly* *warped product manifold (WPMa)* $G = M \times_{(\mu,\lambda)} N$ is the product manifold $M \times N$ endowed with the doubly warped product metric (WPMe) $\bar{g} = \mu^2 g \oplus \lambda^2 h$ defined by

$$\bar{g}(X,Y) = (\mu\circ\pi_1)^2 g(d\pi_1(X), d\pi_1(Y)) + (\lambda\circ\pi_2)^2 h(d\pi_2(X), d\pi_2(Y))$$

for all $X, Y \in T_{(x,y)}(M\times N)$, where $\pi_1 : M\times N \to M$ and $\pi_2 : M\times N \to N$ are the canonical projections. The functions λ and μ are called the *warping functions*.

If either $\mu = 1$ or $\lambda = 1$ but not both we obtain a *(singly) WPMa*. If both $\mu = 1$ and $\lambda = 1$ then we have a *direct product manifold*. If neither μ nor λ is constant, then we have a *non-trivial doubly WPMa*.

Generalizing the idea of WPMa, Chen [8] defined the twisted product manifold (TPM) to construct a family of totally umbilical submanifolds with various properties.

Definition 3.2. The *doubly twisted product manifold* (TPM) of Riemannian manifolds (M, g) and (N, h) with twisting functions $\lambda, \mu : M\times N \to (0, +\infty)$ is the Riemannian manifold $(M\times N, (\mu\circ\pi_1)^2\pi_1^* g + (\lambda\circ\pi_2)^2\pi_2^* h)$, denoted by $M\times_{(\mu,\lambda)} N$, where $\pi_1 : M\times N \to M$ and $\pi_2 : M\times N \to N$ are the canonical projections, and $(\mu\circ\pi_1)^2\pi_1^* g \oplus (\lambda\circ\pi_2)^2\pi_2^* h$ is simply denoted by $\bar{g} = \mu^2 g\oplus\lambda^2 h$.

If λ and μ synchronously depend on the points of both M and N, then we have a non-trivial doubly TPM. When either $\mu\equiv 1$ or $\mu = \mu(x)$ we have a (simply) TPM with twisting function $\lambda(x, y)$. When either $\mu\equiv 1$ or $\mu = \mu(x)$ and λ depends only on the points on M we have a (simply) WPMa with the warping function $\lambda(x)$. For more study on the geometry of doubly TPM, we refer to [22].

For the Levi-Civita connection of doubly TPM $G = M\times_{(\mu,\lambda)} N$, we have

Proposition 3.1. *Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of the direct product manifold $M\times N$ and doubly TPM G respectively, where λ, $\mu : M\times N \to (0, +\infty)$ are smooth maps. Then we get the Levi-Civita connection of doubly TPM G as follows:*

$$\begin{aligned}
&\bar{\nabla}_{(X_1,Y_1)}(X_2,Y_2)\\
=\;& \nabla_{(X_1,Y_1)}(X_2,Y_2)\\
&+\tfrac{1}{2}[X_2(\log\mu^2)+Y_2(\log\mu^2)](X_1,0_2)+\tfrac{1}{2}[X_1(\log\mu^2)+Y_1(\log\mu^2)](X_2,0_2)\\
&+\tfrac{1}{2}[X_2(\log\lambda^2)+Y_2(\log\lambda^2)](0_1,Y_1)+\tfrac{1}{2}[X_1(\log\lambda^2)+Y_1(\log\lambda^2)](0_1,Y_2)\\
&-\tfrac{1}{2}\mu^2 g(X_1,X_2)(\mathrm{grad}_g\log\mu^2,0_2)-\tfrac{1}{2}\lambda^2 h(Y_1,Y_2)(\mathrm{grad}_g\log\lambda^2,0_2)\\
&-\tfrac{1}{2}\mu^2 g(X_1,X_2)(0_1,\mathrm{grad}_h\log\mu^2)-\tfrac{1}{2}\lambda^2 h(Y_1,Y_2)(0_1,\mathrm{grad}_h\log\lambda^2)
\end{aligned}\tag{4}$$

or

$$\begin{aligned}
\bar{\nabla}_{(X_1,Y_1)}(X_2,Y_2) =\;& \nabla_{(X_1,Y_1)}(X_2,Y_2)\\
&+\tfrac{1}{2\mu^2}[X_2(\mu^2)+Y_2(\mu^2)](X_1,0_2)\\
&+\tfrac{1}{2\mu^2}[X_1(\mu^2)+Y_1(\mu^2)](X_2,0_2)\\
&+\tfrac{1}{2\lambda^2}[X_2(\lambda^2)+Y_2(\lambda^2)](0_1,Y_1)\\
&+\tfrac{1}{2\lambda^2}[X_1(\lambda^2)+Y_1(\lambda^2)](0_1,Y_2)\\
&-\tfrac{1}{2}g(X_1,X_2)(\mathrm{grad}_g\mu^2,0_2)-\tfrac{1}{2}\mu^2 h(Y_1,Y_2)(\mathrm{grad}_g\lambda^2,0_2)\\
&-\tfrac{1}{2}g(X_1,X_2)(0_1,\mathrm{grad}_h\mu^2)-\tfrac{1}{2}h(Y_1,Y_2)(0_1,\mathrm{grad}_h\lambda^2)
\end{aligned}\tag{5}$$

for any (X_1, Y_1), $(X_2, Y_2) \in \Gamma(TG)$, $X_1, X_2 \in \Gamma(TM)$ *and* $Y_1, Y_2 \in \Gamma(TN)$, *where* (X_i, Y_i) *is identified with* $(X_i, 0_2) + (0_1, Y_i)$, $i = 1, 2$, *and* $0_1 \in T_pM$ *and* $0_2 \in T_qN$ *are null vectors.*

Since in the whole paper we mainly focus on doubly WPMa, here we omit the proof of Proposition 3.1.

If the doubly TPM G becomes the doubly WPMa, i.e., $\lambda(x, y) = \lambda(x)$ and $\mu(x, y) = \mu(y)$, or $\mathrm{grad}_g\mu = 0$ and $\mathrm{grad}_h\lambda = 0$, then we have

Corollary 3.1. *Let* (M, g) *and* (N, h) *be Riemannian manifolds with Levi-Civita connections* ${}^M\nabla$ *and* ${}^N\nabla$ *respectively and let* ∇ *and* $\bar{\nabla}$ *be the Levi-Civita connections of the product manifold* $M \times N$ *and doubly WPMa* G *respectively, where* $\lambda : M \to (0, +\infty)$ *and* $\mu : N \to (0, +\infty)$ *are smooth maps,* $\bar{g} = \mu^2 g \oplus \lambda^2 h$. *Then the Levi-Civita connection of doubly WPMa* G *is of the form:*

$$\begin{aligned}
&\bar{\nabla}_{(X_1,Y_1)}(X_2, Y_2)\\
= \;& \nabla_{(X_1,Y_1)}(X_2, Y_2) + \tfrac{1}{2\lambda^2}X_1(\lambda^2)(0_1, Y_2) + \tfrac{1}{2\lambda^2}X_2(\lambda^2)(0_1, Y_1)\\
&+\tfrac{1}{2\mu^2}Y_1(\mu^2)(X_2, 0_2) + \tfrac{1}{2\mu^2}Y_2(\mu^2)(X_1, 0_2)\\
&-\tfrac{1}{2}g(X_1, X_2)(0_1, \mathrm{grad}_h\mu^2) - \tfrac{1}{2}h(Y_1, Y_2)(\mathrm{grad}_g\lambda^2, 0_2)\\
= \;& ({}^M\nabla_{X_1}X_2 + \tfrac{1}{2\mu^2}Y_1(\mu^2)X_2 + \tfrac{1}{2\mu^2}Y_2(\mu^2)X_1 - \tfrac{1}{2}h(Y_1, Y_2)\mathrm{grad}_g\lambda^2, 0_2)\\
&+(0_1, {}^N\nabla_{Y_1}Y_2 + \tfrac{1}{2\lambda^2}X_1(\lambda^2)Y_2 + \tfrac{1}{2\lambda^2}X_2(\lambda^2)Y_1 - \tfrac{1}{2}g(X_1, X_2)\mathrm{grad}_h\mu^2)
\end{aligned} \tag{6}$$

for any (X_1, Y_1), $(X_2, Y_2) \in \Gamma(TG)$, $X_1, X_2 \in \Gamma(TM)$ *and* $Y_1, Y_2 \in \Gamma(TN)$.

Remark 3.1. In fact, Eq.(6) agrees with the already known result Eq.(3.3) of [21].

Next we shall discuss some special maps and aim at constructing some examples of proper f-harmonic maps.

§4 f-harmonicity of the inclusion maps

In this section, we consider the inclusion map of M

$$\begin{aligned} i_{y_0} : (M, g) &\to M \times_{(\mu,\lambda)} N\\ x &\mapsto (x, y_0)\end{aligned}$$

at the point $y_0 \in N$ level in $M \times_{(\mu,\lambda)} N$ and the inclusion map of N

$$\begin{aligned} i_{x_0} : (N, h) &\to M \times_{(\mu,\lambda)} N\\ y &\mapsto (x_0, y)\end{aligned}$$

at the point $x_0 \in M$ level in $M \times_{(\mu,\lambda)} N$. We obtain some non-existence results for the f-harmonicity of inclusion maps i_{y_0} of M and i_{x_0} of N.

Theorem 4.1. *Let* $f : M \to (0, +\infty)$ *be a smooth function. The inclusion map of the manifold* (M, g) *into the non-trivial doubly WPMa* G *is never a proper* f*-harmonic map.*

Proof. Let $\{e_j\}_{j=1}^m$ be an orthonormal frame on (M, g). By using the tension field of i_{y_0}, we have

$$\begin{aligned}\tau(i_{y_0}) &= Tr_g \nabla di_{y_0} = \sum_{j=1}^{m} (\nabla_{e_j}^{i_{y_0}^{-1}TG} di_{y_0})(e_j) \\ &= \sum_{j=1}^{m} (\bar{\nabla}_{(e_j,0_2)}(e_j,0_2) - ({}^{M}\nabla_{e_j} e_j, 0_2) \\ &\overset{\text{by (6)}}{=} -\tfrac{m}{2}(0_1,\ \mathrm{grad}_h \mu^2) \mid_{i_{y_0}} .\end{aligned} \tag{7}$$

Here it is obvious from the expression of the tension field of i_{y_0} that i_{y_0} is harmonic if and only if $\mathrm{grad}_h \mu^2 \mid_{i_{y_0}} = 0$. From (3), the f-tension field of i_{y_0} is

$$\begin{aligned}\tau_f(i_{y_0}) &= -\frac{m}{2} f(x)(0_1,\ \mathrm{grad}_h \mu^2(y)) + di_{y_0}(\mathrm{grad}_g f(x)) \\ &= -\frac{m}{2} f(x)(0_1,\ \mathrm{grad}_h \mu^2(y)) + (\mathrm{grad}_g f(x), 0_2).\end{aligned} \tag{8}$$

Therefore, the inclusion map $i_{y_0} : (M,g) \to G$ for $y_0 \in N$ is a proper f-harmonic map if and only if $\mathrm{grad}_g f = 0$ and $\mathrm{grad}_h \mu^2 = 0$. Since $\tau(i_{y_0}) \neq 0$ implies that $\mathrm{grad}_h \mu^2 \mid_{i_{y_0}} \neq 0$, it can be seen from (8) that f must be a constant function. But this is in contradiction with that G is a non-trivial doubly WPMa. Thus, we obtain Theorem 4.1. □

In the same way, we have

Theorem 4.2. *Let $f : (N,h) \to (0,+\infty)$ be a smooth function. The inclusion map of the manifold (N,h) into the non-trivial doubly WPMa G is never a proper f-harmonic map.*

Remark 4.1. (i) Observe that f-tension field $\tau_f(i_{y_0})$ of the inclusion map i_{y_0} is not of the form

$$\begin{aligned}&-\frac{m}{2} f(x)(\mathrm{grad}_g \lambda^2(x), 0_2) + (\mathrm{grad}_g f(x), 0_2) \\ =&\big(-\frac{m}{2} f(x)\mathrm{grad}_g \lambda^2(x) + \mathrm{grad}_g f(x), 0_2\big).\end{aligned}$$

Neither is $\tau_f(i_{x_0})$ of the form

$$\begin{aligned}&-\frac{n}{2} f(y)(0_1, \mathrm{grad}_h \mu^2(y)) + (0_1, \mathrm{grad}_h\, f(y)) \\ =&\big(0_1, -\frac{n}{2} f(y)\mathrm{grad}_h \mu^2(y) + \mathrm{grad}_h f(y)\big).\end{aligned}$$

So in the case of doubly WPMa, neither the inclusion map i_{y_0} nor i_{x_0} can be proper f-harmonic map.

(ii) When $\mu = 1$, the doubly WPMa $M \times_{(\mu,\lambda)} N$ becomes a singly WPMa $M \times_\lambda N$. Since the inclusion $i_{y_0} : (M,g) \to M \times_\lambda N$ of M at the level $y_0 \in N$ is always totally geodesic (or from (7) we see that $\mathrm{grad_h} \mu^2 = 0$ implies $\tau_{i_{y_0}} = 0$), then it is f-harmonic for any warping function $\lambda \in C^\infty(M)$. So in the case of WPMa, f-harmonicity of the inclusion map i_{y_0} is trivial.

§5 f-harmonicity of the projection maps

Firstly, we study the f-harmonicity of the projection $\bar{\pi}_1 : M \times_{(\mu,\lambda)} N \to M$. We have

Theorem 5.1. *The projection $\bar{\pi}_1 : M^m \times_{(\mu,\lambda)} N^n \to (M^m, g)$, $\bar{\pi}_1(x,y) = x$, of a doubly WPMa onto its first factor is an f-harmonic map with $f : M \times N \to (0,+\infty)$ if and only if $\lambda^n(x) f^{\frac{1}{\mu^2}}(x,y) = \tilde{f}(y)$ for some function $\tilde{f} : N \to (0,+\infty)$.*

Proof. Let $\{e_j\}_{j=1}^m$ be a local orthonormal frame on (M,g) and $\{\bar{e}_\alpha\}_{\alpha=1}^n$ be an orthonormal frame on (N,h). Then $\{(\frac{1}{\mu}e_j, 0_2), (0_1, \frac{1}{\lambda}\bar{e}_\alpha)\}_{\substack{j=1,\dots,m,\\ \alpha=1,\dots,n}}$ is a local orthonormal frame on the doubly WPMa $M^m \times_{(\mu,\lambda)} N^n$. By (6), we have

$$\begin{aligned}\tau(\bar{\pi}_1) &= Tr_{\bar{g}}\nabla d\bar{\pi}_1 = \sum_{j=1}^m \big(\tfrac{1}{\mu^2}\, {}^M\nabla_{d\bar{\pi}_1(e_j,0_2)} d\bar{\pi}_1(e_j,0_2) - \tfrac{1}{\mu^2} d\bar{\pi}_1(\bar{\nabla}_{(e_j,0_2)}(e_j,0_2))\big)\\ &+ \sum_{\alpha=1}^n \big(\tfrac{1}{\lambda^2}\, {}^N\nabla_{d\bar{\pi}_1(0_1,\bar{e}_\alpha)} d\bar{\pi}_1(0_1,\bar{e}_\alpha) - \tfrac{1}{\lambda^2} d\bar{\pi}_1(\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha))\big)\\ &= -\tfrac{1}{\lambda^2}\sum_{\alpha=1}^n d\bar{\pi}_1\big((-\tfrac{1}{2}h(\bar{e}_\alpha,\bar{e}_\alpha)\mathrm{grad}_g\lambda^2,\, 0_2) + (0_1, \bar{\nabla}_{\bar{e}_\alpha}\bar{e}_\alpha)\big)\\ &= \tfrac{n}{2\lambda^2}\mathrm{grad}_g\lambda^2 \circ \bar{\pi}_1\\ &= \mathrm{grad}_g \log\lambda^n \circ \bar{\pi}_1.\end{aligned} \tag{9}$$

Since we have

$$\begin{aligned}d\bar{\pi}_1\big(\mathrm{grad}_{\bar{g}} f(x,y)\big) &= d\bar{\pi}_1\Big(\sum_{j=1}^m (\frac{1}{\mu}e_j, 0_2)(f(x,y))(\frac{1}{\mu}e_j,0_2) + \sum_{\alpha=1}^n \frac{1}{\lambda}(0_1,\bar{e}_\alpha)(f(x,y))(0_1,\frac{1}{\lambda}\bar{e}_\alpha)\Big)\\ &= \frac{1}{\mu^2}\mathrm{grad}_g f(x,y)\circ\bar{\pi}_1.\end{aligned}$$

Putting these facts together, we get

$$\begin{aligned}\tau_f(\bar{\pi}_1) &= f(x)\mathrm{grad}_g \log(\lambda^n(x))\circ\bar{\pi}_1 + \tfrac{1}{\mu^2(y)}\mathrm{grad}_g f(x,y)\circ\bar{\pi}_1\\ &= f(x,y)\mathrm{grad}_g \log\big((\lambda^n(x) f^{\frac{1}{\mu^2}}(x,y)\big)\circ\bar{\pi}_1.\end{aligned} \tag{10}$$

Thus f-harmonic of $\bar{\pi}_1$ implies that $\lambda^n f^{\frac{1}{\mu^2}}$ does not depend on the points in M, that is, there exists some function $\tilde{f}: N \to (0,+\infty)$ such that $\lambda^n(x) f^{\frac{1}{\mu^2}}(x,y) = \tilde{f}(y)$. □

For the projection $\bar{\pi}_2 : M \times_{(\mu,\lambda)} N \to N$, a similar proof gives

$$\tau_f(\bar{\pi}_2) = f\,\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}})\circ\bar{\pi}_2. \tag{11}$$

Therefor we have

Theorem 5.2. *The projection $\bar{\pi}_2 : (M^m \times_{(\mu,\lambda)} N^n, \bar{g}) \to (N^n, h)$, $\bar{\pi}_2(x,y) = y$, of a doubly WPMa onto its second factor is a f-harmonic map with $f: M\times N \to (0,+\infty)$ if and only if $\mu^m(y) f^{\frac{1}{\lambda^2}}(x,y) = \hat{f}(x)$ for some function $\hat{f}: M \to (0,+\infty)$.*

§6 f-harmonicity of the product maps

In this section we give a method to construct proper f-harmonic maps of product type.

6.1 The special product maps

A straightforward verification shows that $\Psi = Id_M \times \varphi_N : (M\times N, g\oplus h) \to (M\times N, g\oplus h)$ is a harmonic map if $Id_M : (M,g) \to (M,g)$ is an identity map and $\varphi_N : (N,h) \to (N,h)$ is a harmonic map. It would be interesting to know whether the product map renders some non-trivial f-harmonic maps by replacing the product metric in the domain or the target manifold by a doubly WPMe.

We first take into account the product map

$$\overline{\Psi} = \overline{Id_M \times \varphi_N} : M^m \times_{(\mu,\lambda)} N^n \to (M \times N, g \oplus h), \overline{\Psi}(x,y) = (x, \varphi_N(y)). \tag{12}$$

We obtain

Theorem 6.1. *Let (M^m, g) and (N^n, h) be Riemannian manifolds, and let $\lambda \in C^\infty(M)$, $\mu \in C^\infty(N)$ and $f \in C^\infty(M \times N)$ be three positive functions. Assume that $\lambda^n(x)\mu^m(y)f^{(\frac{1}{\mu^2}+\frac{1}{\lambda^2})}(x,y) \neq const.$ and $\varphi_N : N \to N$ is a harmonic map. Then the product map $\overline{\Psi} = \overline{Id_M \times \varphi_N}$ defined by (12) is a proper f-harmonic map if and only if λ and μ are non-constant solutions of*

$$\mathrm{grad}_g \lambda^n \log f^{\frac{1}{\mu^2}} = 0, \tag{13}$$

and

$$d\varphi_N\big(\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}})\big) = 0. \tag{14}$$

More precisely, $\overline{\Psi}$ is a proper f-harmonic map if and only if there exists a positive function $\bar{\mu} \in C^\infty(N)$ such that

$$\lambda^n(x) f^{\frac{1}{\mu^2}}(x,y) = \bar{\mu}(y) \tag{15}$$

and

$$\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}}) \in \mathrm{Ker}(d\varphi_N).$$

Proof. By calculation similar to (9) and (10), together with the assumption that φ_N is harmonic, i.e., $\tau(\varphi_N) = 0$, we have

$$\begin{aligned}
&\tau(\overline{\Psi})\\
= \ & \sum_{j=1}^{m}\Big(\tfrac{1}{\mu^2}\,{}^M\nabla_{d\overline{\Psi}(e_j,0_2)} d\overline{\Psi}(e_j,0_2) - \tfrac{1}{\mu^2} d\overline{\Psi}(\bar{\nabla}_{(e_j,0_2)}(e_j,0_2))\Big)\\
&+\sum_{\alpha=1}^{n}\big(\tfrac{1}{\lambda^2}\,{}^N\nabla_{Id_{TM}\times d\varphi_N(0_1,\bar{e}_\alpha)} Id_{TM}\times d\varphi_N(0_1,\bar{e}_\alpha) - \tfrac{1}{\lambda^2} Id_{TM}\times d\varphi_N(\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha))\big)\\
= \ & n(\mathrm{grad}_g \log\lambda, 0_2) + m(0_1, d\varphi_N(\mathrm{grad}_h \log\mu)) + \tfrac{1}{\lambda^2}(0_1, \tau(\varphi_N))\\
= \ & n(\mathrm{grad}_g \log\lambda, 0_2) + m(0_1, d\varphi_N(\mathrm{grad}_h \log\mu))
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
\tau_f(\overline{\Psi}) = \ & nf(\mathrm{grad}_g \log\lambda, 0_2) + mf(0_1, d\varphi_N(\mathrm{grad}_h \log\mu))\\
&+Id_{TM}\times d\varphi_N(\mathrm{grad}_{\mu^2 g} f, \mathrm{grad}_{\lambda^2 h} f)\\
= \ & (nf\,\mathrm{grad}_g \log\lambda + \tfrac{1}{\mu^2}\mathrm{grad}_g f, 0_2)\\
&+(0_1, mf\, d\varphi_N(\mathrm{grad}_h \log\mu) + d\varphi_N(\tfrac{1}{\lambda^2}\mathrm{grad}_h f))\\
= \ & f\Big(\mathrm{grad}_g \log(\lambda^n f^{\frac{1}{\mu^2}}),\ d\varphi_N\big(\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}})\big)\Big).
\end{aligned} \tag{17}$$

Thus, the f-harmonicity of Ψ implies that

$$\mathrm{grad}_g \log(\lambda^n f^{\frac{1}{\mu^2}}) = 0$$

and

$$d\varphi_N\big(\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}})\big) = 0,$$

which amount to $\lambda^n(x) f^{\frac{1}{\mu^2}}(x,y) = \bar{\mu}(y)$ for some positive function $\bar{\mu} \in C^\infty(N)$, and

$$\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}}) \in \mathrm{Ker}(d\varphi_N)$$

since $\lambda^n(x) f^{\frac{1}{\mu^2}}(x,y)$ must be independent of x. Hence we obtain the first and the last statements. □

Example 6.1. Let $M=\mathbb{R}$, $N=\mathbb{R}^2$ and $\varphi_N:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $\varphi_N(y,z)=(y-z,0)$. Then one can easily check that $\tau(\varphi_N)=0$ and φ_N is a usual harmonic map. Given $\lambda(x)=e^x$ and $\mu(y,z)=e^{y+z}$, (13) and (14) yield

$$\frac{\partial(f^{e^{-2y-2z}}(x,y,z)e^{nx})}{\partial x}\frac{\partial}{\partial x}=0, \tag{18}$$

$$\frac{\partial(f^{e^{-2x}}(x,y,z)e^{my+mz})}{\partial y}\frac{\partial}{\partial y}-\frac{\partial(f^{e^{-2x}}(x,y,z)e^{my+mz})}{\partial z}\frac{\partial}{\partial z}=0, \tag{19}$$

respectively, which imply that

$$f^{e^{-2x}}(x,y,z)e^{my+mz}=f_1(x)e^{\alpha(y,z)}$$

and

$$f^{e^{-2y-2z}}(x,y,z)e^{nx}=f_2(y),$$

where $f_1(x),f_2(y)$ are arbitrary positive functions and $\alpha=\alpha(y,z)$ is any function satisfying $\alpha_y=\alpha_z$. So, we can find

$$f(x,y,z)=[f_1(x)f_2(y)e^{\alpha(x,y)-nx-m(y+z)}]^{\frac{1}{e^{-2x}+e^{-2y-2z}}}.$$

For instance, we take $f_1(x)=e^x, f_2(y)=e^y$, $\alpha(x,y)=y+z$, then we have two non-constant warping functions $\lambda(x)=e^x$ and $\mu(y,z)=e^{y+z}$ that could guarantee that

$\overline{\Psi}=\overline{Id_M\times\varphi_N}:(\mathbb{R}\times_{(\mu,\lambda)}\mathbb{R}^2,\ \bar{g}=e^{y+z}dx^2\oplus e^x(dy^2+dz^2))\to(\mathbb{R}\times\mathbb{R}^2,\ g=dx^2\oplus(dy^2+dz^2))$ is proper f-harmonic map with $f(x,y,z)=[e^{(1-n)x+(2-m)y+(1-m)z}]^{\frac{1}{e^{-2x}+e^{-2y-2z}}}$.

By using f-tension fields of the projections $\bar{\pi}_1$ and $\bar{\pi}_2$ given by (10) and (11) respectively, (17) has the expression

$$\tau_f(\overline{\Psi})=(\tau_f(\bar{\pi}_1),d\varphi_N(\tau_f(\bar{\pi}_2))). \tag{20}$$

So we have

Corollary 6.1. *The product map $\overline{\Psi}:M^m\times_{(\mu,\lambda)}N^n\to M\times N$, $\overline{\Psi}(x,y)=(x,\varphi_N(y))$ with harmonic map φ_N is a proper f-harmonic map if the projections $\bar{\pi}_1$ and $\bar{\pi}_2$ are all proper f-harmonic maps.*

Remark 6.1. Corollary 6.1 provides an approach that can be applied to construct some examples of non-trivial f-harmonic maps. Of course, by (20) we know that the sufficient condition "$\tau_f(\bar{\pi}_1)=0=\tau_f(\bar{\pi}_2)$" in Corollary 6.1 can change to the necessary and sufficient condition "$\tau_f(\bar{\pi}_1)=0$ and $\tau_f(\bar{\pi}_2)\in\mathrm{Ker}(\mathrm{d}\varphi_\mathrm{N})$".

In Theorem 6.1, if we allow φ_N to be a general map (not necessary harmonic), then we have

Theorem 6.2. *Let (M^m,g) and (N^n,h) be Riemannian manifolds, and let $\lambda\in C^\infty(M)$, $\mu\in C^\infty(N)$ and $f\in C^\infty(M\times N)$ be three positive functions. Then the product map $\overline{\Psi}=\overline{Id_M\times\varphi_N}$ is a proper f-harmonic map if and only if λ is a non-constant solution of*

$$\mathrm{grad}_g\log(\lambda^n f^{\frac{1}{\mu^2}})=0 \tag{21}$$

and μ is a non-constant solution to

$$d\varphi_N\big(\mathrm{grad}_h\log(\mu^m f^{\frac{1}{\lambda^2}})\big)+\frac{1}{\lambda^2}\tau(\varphi_N)=0. \tag{22}$$

A proof similar to that of Theorem 6.1 can be obtained by simply noting that $\tau(\overline{\Psi})$ in (16) contains the term $\lambda^{-2}(0_1, \tau(\varphi_N))$.

As for the product map

$$\widetilde{\Psi} = \widetilde{\varphi_M \times Id_N} : M^m \times_{(\mu,\lambda)} N^n \to (M \times N, g \oplus h),\ \widetilde{\Psi}(x,y) = (\varphi_M(x), y) \tag{23}$$

with harmonic map $\varphi_M : M \to M$ and identity map $Id_N : N \to N$, we easily obtain analogous results to Theorem 6.1 and Corollary 6.1.

Finally we turn to consider the case of the product map

$$\widehat{\Psi} = \widehat{Id_M \times \varphi_N} : (M \times N, g \oplus h) \to M^m \times_{(\mu,\lambda)} N^n,\ \widehat{\Psi}(x,y) = (x, \varphi_N(y)), \tag{24}$$

that is, the case of the product metric on the codomain is doubly warped metric. We see that the energy density $e(\varphi_N)$ plays an important role for the f-harmonicity of the product maps $\widehat{\Psi} = \widehat{Id_M \times \varphi_N}$. We have

Theorem 6.3. *Let (M,g) and (N,h) be Riemannian manifolds of dimensions m and n respectively, and let $\lambda \in C^\infty(M)$, $\mu \in C^\infty(N)$ and $f \in C^\infty(M \times N)$ be three positive functions. Suppose that $Id_M : M \to M$ is an identity map and $\varphi_N : N \to N$ is a harmonic map. Then the product map $\widehat{\Psi} = \widehat{Id_M \times \varphi_N}$ defined by (24) is a proper f-harmonic map if and only if*

$$e(\varphi_N) f\, \mathrm{grad}_g \lambda^2 - \mathrm{grad}_g f = 0 \tag{25}$$

and

$$-\frac{m}{2} f\, \mathrm{grad}_h \mu^2 + d\varphi_N(\mathrm{grad}_h f) = 0. \tag{26}$$

Proof. Let $\{(e_j, 0_2), (0_1, \bar{e}_\alpha)\}_{\substack{j=1,\ldots,m,\\ \alpha=1,\ldots,n}}$ be a local orthonormal frame on the usual product manifold $M^m \times N^n$, where $\{e_j\}_{j=1}^m$ and $\{\bar{e}_\alpha\}_{\alpha=1}^n$ respectively denote orthonormal frames on (M,g) and on (N,h). By $\tau(\varphi_N) = 0$, the tension field of $\widehat{\Psi}$ is given by

$$\begin{aligned}
\tau(\widehat{\Psi}) &= Tr_{g\oplus h} \nabla d\widehat{\Psi} \\
&= \sum_{j=1}^m \big(\bar{\nabla}_{d\widehat{\Psi}(e_j,0_2)} d\widehat{\Psi}(e_j, 0_2) - d\widehat{\Psi}(\nabla_{(e_j,0_2)}(e_j, 0_2))\big) \\
&\quad + \sum_{\alpha=1}^n \big(\bar{\nabla}_{Id_{TM}\times d\varphi_N(0_1,\bar{e}_\alpha)} Id_{TM} \times d\varphi_N(0_1, \bar{e}_\alpha) - Id_{TM} \times d\varphi_N(\nabla_{(0_1,\bar{e}_\alpha)}(0_1, \bar{e}_\alpha))\big) \\
&= -\frac{m}{2}(0_1, \mathrm{grad}_h \mu^2) + (0_1, \tau(\varphi_N)) - \frac{1}{2}\sum_{\alpha=1}^n \varphi_N^* h(\bar{e}_\alpha, \bar{e}_\alpha)(\mathrm{grad}_g \lambda^2, 0_2) \\
&= (0_1, -\frac{m}{2}\mathrm{grad}_h \mu^2) - (e(\varphi_N)\mathrm{grad}_g \lambda^2, 0_2),
\end{aligned} \tag{27}$$

thus we get

$$\begin{aligned}
\tau_f(\widehat{\Psi}) &= (0_1, -\tfrac{m}{2} f\, \mathrm{grad}_h \mu^2) + (-\, e(\varphi_N) f\, \mathrm{grad}_g \lambda^2, 0_2) \\
&\quad + Id_{TM} \times d\varphi_N(\mathrm{grad}_g f, \mathrm{grad}_h f) \\
&= \big(0_1, -\tfrac{m}{2} f\, \mathrm{grad}_h \mu^2 + d\varphi_N(\mathrm{grad}_h f)\big) \\
&\quad + \big(-\, e(\varphi_N) f\, \mathrm{grad}_g \lambda^2 + \mathrm{grad}_g f, 0_2\big).
\end{aligned} \tag{28}$$

It follows that $\tau_f(\widehat{\Psi}) = 0$ if and only if the following two equations hold

$$e(\varphi_N) f\, \mathrm{grad}_g \lambda^2 - \mathrm{grad}_g f = 0$$

and

$$-\frac{m}{2}f\,\mathrm{grad}_h\mu^2 + d\varphi_N(\mathrm{grad}_h f) = 0.$$

Therefore, the theorem is proved. □

6.2 The general product maps

In this subsection, we consider the more general product case, i.e., we replace the identity map Id_M by a harmonic map $\varphi_M : (M, g) \to M$ in Theorem 6.1, then we obtain the following result.

Theorem 6.4. *Let (M^m, g) and (N^n, h) be Riemannian manifolds, and let $\lambda \in C^\infty(M)$, $\mu \in C^\infty(N)$ and $f \in C^\infty(M \times N)$ be three positive functions. Suppose that $\lambda^n(x)\mu^m(y)f^{\frac{1}{\mu^2}+\frac{1}{\lambda^2}}(x, y)$ $\neq$ const. and $\varphi_M : (M, g) \to M$, $\varphi_N : N \to N$ are two harmonic maps. Then the product map $\overline{\Phi} = \overline{\varphi_M \times \varphi_N} : M^m \times_{(\mu,\lambda)} N^n \to (M \times N, g \oplus h)$ defined by $\overline{\Phi}(x, y) = (\varphi_M(x), \varphi_N(y))$ is a proper f-harmonic map if and only if λ, μ are non-constant solutions of*

$$d\varphi_M\big(\mathrm{grad}_g \log(\lambda^n f^{\frac{1}{\mu^2}})\big) = 0 \tag{29}$$

and

$$d\varphi_N\big(\,\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}})\big) = 0. \tag{30}$$

More precisely, $\overline{\Phi}$ is a proper f-harmonic map if and only if $\lambda^n(x)f^{\frac{1}{\mu^2}}(x, y) \in \ker(d\varphi_M)$ and $\mu^m(y)f^{\frac{1}{\lambda^2}}(x, y) \in \mathrm{Ker}(d\varphi_N)$.

Proof. By calculation similar to (9) and (10), together with the assumption $\tau(\varphi_M) = \tau(\varphi_N) = 0$, we have

$$\begin{aligned}
&\tau(\overline{\Phi})\\
=&\sum_{j=1}^{m}\left(\frac{1}{\mu^2}\nabla_{d\overline{\Phi}(e_j,0_2)}d\overline{\Phi}(e_j,0_2) - \frac{1}{\mu^2}d\overline{\Phi}(\,\bar{\nabla}_{(e_j,0_2)}(e_j,0_2)\,)\right)\\
&\quad+\sum_{\alpha=1}^{n}\left(\frac{1}{\lambda^2}\nabla_{d\varphi_M\times d\varphi_N(0_1,\bar{e}_\alpha)}d\varphi_M\times d\varphi_N(0_1,\bar{e}_\alpha) - \frac{1}{\lambda^2}d\varphi_M\times d\varphi_N(\,\bar{\nabla}_{(0_1,\bar{e}_\alpha)}(0_1,\bar{e}_\alpha)\,)\right)\\
=&\frac{1}{\mu^2}\big(d\varphi_M\big(\tau(\varphi_M)\big),0_2\big) + n\big(d\varphi_M(\mathrm{grad}_g\log\lambda),0_2\big)\\
&\quad+m\big(\,0_1,d\varphi_N(\mathrm{grad}_h\log\mu)\big) + \frac{1}{\lambda^2}\big(0_1,d\varphi_N\big(\tau(\varphi_N)\big)\big)\\
=&n\big(d\varphi_M(\mathrm{grad}_g\log\lambda),0_2\big) + m\big(\,0_1,d\varphi_N(\mathrm{grad}_h\log\mu)\big)
\end{aligned} \tag{31}$$

and

$$\begin{aligned}
\tau_f(\overline{\Phi}) &= nf\big(d\varphi_M(\mathrm{grad}_g\log\lambda),0_2\big) + mf\big(0_1,d\varphi_N(\mathrm{grad}_h\log\mu)\big)\\
&\quad+d\varphi_M\times d\varphi_N(\mathrm{grad}_{\mu^2 g}f,\mathrm{grad}_{\lambda^2 h}f)\\
&= \big(f\,d\varphi_M(\mathrm{grad}_g\log\lambda^n) + d\varphi_M(\tfrac{1}{\mu^2}\mathrm{grad}_g f),0_2\big)\\
&\quad+\big(0_1,f\,d\varphi_N(\,\mathrm{grad}_h\log\mu^m) + d\varphi_N(\tfrac{1}{\lambda^2}\mathrm{grad}_h f)\big)\\
&= f\,\Big(d\varphi_M\big(\mathrm{grad}_g\log(\lambda^n f^{\frac{1}{\mu^2}})\big),\,d\varphi_N\big(\,\mathrm{grad}_h\log(\mu^m f^{\frac{1}{\lambda^2}})\big)\Big).
\end{aligned} \tag{32}$$

Thus f-harmonicity of $\overline{\Phi}$ implies that

$$d\varphi_M\big(\mathrm{grad}_g \log(\lambda^n f^{\frac{1}{\mu^2}})\big) = 0$$

and

$$d\varphi_N\big(\,\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}})\big) = 0,$$

which amount to $\mathrm{grad}_g \log(\lambda^n f^{\frac{1}{\mu^2}}) \in \ker(d\varphi_M)$ and $\mathrm{grad}_h \log(\mu^m f^{\frac{1}{\lambda^2}}) \in \ker d\varphi_N$. Hence we conclude the first statement and the last statement. □

By using f-tension fields of the projections $\bar{\pi}_1$ and $\bar{\pi}_2$ given by (10) and (11) respectively, (32) has the expression

$$\tau_f(\overline{\Phi}) = \big(d\varphi_M\big(\tau_f(\bar{\pi}_1)\big), d\varphi_N\big(\tau_f(\bar{\pi}_2)\big)\big)\,. \tag{33}$$

So we have

Corollary 6.2. *The product map* $\overline{\Phi} = \overline{\varphi_M \times \varphi_N} : M^m \times_{(\mu,\lambda)} N^n \to M \times N$, $\overline{\Phi}(x,y) = (\varphi_M(x), \varphi_N(y))$ *with harmonic maps* φ_M *and* φ_N *is a proper* f*-harmonic map if the projections* $\bar{\pi}_1$ *and* $\bar{\pi}_2$ *are all proper* f*-harmonic maps.*

Remark 6.2. It is easily seen from (33) that the necessary and sufficient conditions of Corollary 6.2 is $\tau_f(\bar{\pi}_1) \in \mathrm{Ker}(\mathrm{d}\varphi_\mathrm{M})$ and $\tau_f(\bar{\pi}_2) \in \mathrm{Ker}(\mathrm{d}\varphi_\mathrm{N})$.

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