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μ-Separations in generalized topological spaces

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Abstract. This paper takes some investigations on generalized topological spaces with some μ-separations. Some characterizations of μT_i-spaces for $i = 0, 1, 2, 3, 4$, μT_D-spaces and μR_0-spaces are obtained and some relations among these spaces are established.

§1 Introduction

Let X be a set. A subset μ of $\exp X$ is called a *generalized topology* on X and (X, μ) is called a generalized topological space[4], if μ has the following properties:

(1) $\emptyset \in \mu$.

(2) Any union of elements of μ belongs to μ.

Let $\mathcal{B} \subset \exp X$ and $\emptyset \in \mathcal{B}$. Then $\mathcal{B}$ is called a *base*[4] for μ if $\{\cup\mathcal{B}' : \mathcal{B}' \subset \mathcal{B}\} = \mu$. We also say that μ is generated by $\mathcal{B}$.

Generalized topological space is an important generalization of topological spaces and many interesting results have been obtained. In this paper, we investigate separations in generalized topological spaces and give some characterizations of separations and some relations among them.

Throughout this paper, a space (X, μ), or simply X for short, will always mean a strong generalized topological space with the strong generalized topology μ unless otherwise explicitly stated. Here, a generalized topology μ is said to be *strong*[4] if $X \in \mu$. Separation in (X, μ) is called *μ-separation*. A subset B of X is called *μ-open* (or *μ-closed*) if $B \in \mu$ (or $X - B \in \mu$). For $B \subset X$, let $I(B)$ be the largest μ-open subset of B. Equivalently, $I(B)$ is the union of all μ-open subsets of B. Let $C(B)$ be the smallest μ-closed subset which contains B. Equivalently, $C(B)$ is the intersection of all μ-closed subsets which contain B. A point $x \in X$ is called a *μ-cluster point* of B if $U \cap (B - \{x\}) \neq \emptyset$ for each $U \in \mu$ with $x \in U$. The set of all μ-cluster points of B is denoted by $d(B)$.

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§2 Preliminaries

Notation. Let X be a space. Throughout this paper, we use the following notation for $x \in X$ and $\mathcal{F} \subset \exp X$.

(1) $\cap\mathcal{F} = \cap\{F : F \in \mathcal{F}\}$.

(2) $\cup\mathcal{F} = \cup\{F : F \in \mathcal{F}\}$.

(3) $\mu_x = \{U : x \in U \in \mu\}$.

(4) $C(\mu_x) = \{C(U) : U \in \mu_x\}$.

(5) $N(x) = \cap\mu_x$.

(6) $\overline{N(x)} = \cap C(\mu_x)$.

The following μT_i-space for $i = 0, 1, 2, 3, 4$, μT_D-space and μR_0-space are generalizations of T_i-space[6] for $i = 0, 1, 2, 3, 4$, T_D-space[1] and R_0-space[5,7], respectively.

Definition 2.1. Let X be a strong generalized topological space. X is called a μT_0*-space* (resp. μT_1*-space,* μT_2*-space,* μT_3*-space,* μT_4*-space,* μT_D*-space,* μR_0*-space*) if X satisfies the following μT_0-separation (resp. μT_1-separation, μT_2-separation, μT_3-separation, μT_4-separation, μT_D-separation, μR_0-separation) condition.

(1) μT_0-*separation*: If $x, y \in X$ and $x \neq y$, then there is $U \in \mu$ such that either $U \cap \{x, y\} = \{x\}$ or $U \cap \{x, y\} = \{y\}$.

(2) μT_1-*separation*: If $x, y \in X$ and $x \neq y$, then there are $U_x, U_y \in \mu$ such that $U_x \cap \{x, y\} = \{x\}$ and $U_y \cap \{x, y\} = \{y\}$.

(3) μT_2-*separation*: If $x, y \in X$ and $x \neq y$, then there are $U_x \in \mu_x$ and $U_y \in \mu_y$ such that $U_x \cap U_y = \emptyset$.

(4) μT_3-*separation*: If $x \notin F$ with F μ-closed, then there are $U_x \in \mu_x$ and $U_F \in \mu$ such that $F \subset U_F$ and $U_x \cap U_F = \emptyset$.

(5) μT_4-*separation*: If F_1 and F_2 are μ-closed and $F_1 \cap F_2 = \emptyset$, then there are $U_1, U_2 \in \mu$ with $F_1 \subset U_1$ and $F_2 \subset U_2$ such that $U_1 \cap U_2 = \emptyset$.

(6) μT_D-*separation*: If $x \in X$, then there are μ-open U and μ-closed F such that $\{x\} = U \cap F$.

(7) μR_0-*separation*: If $x \in U \in \mu$, then $C(\{x\}) \subset U$.

Remark 2.1. (1) If μ is a topology on X, then μ-separations in X are separations in topological space X.

(2) A μT_1-space is called a μ*-regular space* (or μ*-normal space*) if it is a μT_3-space (or μT_4-space).

(3) It is clear that μ-normal space $\Longrightarrow$ μ-regular space $\Longrightarrow$ μT_2-space $\Longrightarrow$ μT_1-spaces $\Longrightarrow$ μT_0-spaces.

(4) None of the implications in (3) can be reversed.

(5) μ-normal space (or μ-regular space) in (3) cannot be relaxed to μT_4-space (or μT_3-regular space).

The following proposition is known.

Proposition 2.1. *Let B be a subset of a space X. Then the followings hold.*

(1) $I(B) \subset B \subset C(B)$.

(2) $I(I(B)) = I(B)$ *and* $C(C(B)) = C(B)$.

(3) *If* $B' \subset B$, *then* $I(B') \subset I(B)$, $C(B') \subset C(B)$ *and* $d(B') \subset d(B)$.

(4) $I(B) = B$ *iff* B *is* μ*-open.*

(5) $C(B) = B$ *iff* B *is* μ*-closed.*

(6) $C(B) = X - I(X - B)$ *and* $I(B) = X - C(X - B)$.

(7) $x \in C(B)$ *iff* $U \cap B \neq \emptyset$ *for each* $U \in \mu_x$.

(8) $x \in I(B)$ *iff* $U \subset B$ *for some* $U \in \mu_x$.

(9) $C(B) = B \cup d(B)$.

(10) $x \notin d(\{x\})$ *for each* $x \in X$.

Remark 2.2. The equalities $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$ and $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$ do not hold. In fact, let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{c, a\}, X\}$. Then $I(\{a, b\} \cap \{b, c\}) = I(\{b\}) = \emptyset \neq \{b\} = \{a, b\} \cap \{b, c\} = I(\{a, b\}) \cap I(\{b, c\})$ and $C(\{a\} \cup \{b\}) = C(\{a, b\}) = X \neq \{a, b\} = \{a\} \cup \{b\} = C(\{a\}) \cup C(\{b\})$.

We now give a sufficient condition such that $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$ and $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$ hold. The following definition come from [2,8].

Definition 2.2. Let X be a space.

(1) Let $x \in X$ and $U \in \mu_x$. Then x is called a *representative element* of U if $U \subset V$ for each $V \in \mu_x$.

(2) A space X is called a C_0*-space* if $C_0 = X$, where C_0 is the set of all representative elements of sets of μ.

(3) Let $x \in X$. The set $Md(x) = \{U \in \mu_x : U \supset V \in \mu_x \Longrightarrow V = U\}$ is called the *minimal description* of x.

Remark 2.3. Let X be a space and $x \in X$. If μ_x is finite, then $x \notin C_0$ iff $|Md(x)| > 1$ in which $|Md(x)|$ is the cardinality of $Md(x)$. (See [2].)

Proposition 2.2. *Let B_1 and B_2 be subsets of a C_0-space X. Then*

(1) $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$.

(2) $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$.

Proof. (1) $I(B_1 \cap B_2) \subset I(B_1) \cap I(B_2)$ from Proposition 2.1(3). If $x \in I(B_1) \cap I(B_2)$, then there are $U_1, U_2 \in \mu_x$ such that $U_1 \subset B_1$ and $U_2 \subset B_2$. Since X is a C_0-space, $x \in C_0$. So there is $U \in \mu_x$ such that x is a representative element of U, and hence $U \subset U_1$ and $U \subset U_2$. Consequently, $x \in U \subset U_1 \cap U_2 \subset B_1 \cap B_2$. It follows that $x \in I(B_1 \cap B_2)$. Thus, $I(B_1) \cap I(B_2) \subset I(B_1 \cap B_2)$. This proves that $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$.

(2) By (1) and Proposition 2.1(6), $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$. □

Proposition 2.3. *Let X be a space. If μ is finite, then the followings are equivalent.*

(1) X *is a* C_0*-space.*

(2) $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$ *for each* $B_1, B_2 \in \exp X$.

(3) $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$ *for each* $B_1, B_2 \in \exp X$.

Proof. (1) $\Longrightarrow$ (2) It follows from Proposition 2.2.

(2) $\Longrightarrow$ (1) If X is not a C_0-space, then there is $x \in X$ such that $x \notin C_0$. By Remark 2.3, $|Md(x)| > 1$. So there are $U_1, U_2 \in Md(x)$ such that $U_1 \neq U_2$, hence $x \in U_1 \cap U_2 = I(U_1) \cap I(U_2)$. On the other hand, for each $U \in \mu_x$, $U \not\subset U_1 \cap U_2$ because $U_1, U_2 \in Md(x)$. So $x \notin I(U_1 \cap U_2)$. This contradicts $I(U_1 \cap U_2) = I(U_1) \cap I(U_2)$.

(2) $\Longleftrightarrow$ (3) It holds by Proposition 2.1(6). □

Remark 2.4. "Finite" in Proposition 2.3 cannot be omitted. In fact, let X be the closed interval $[0, 1]$. Then $I(B_1 \cap B_2) = I(B_1) \cap I(B_2)$ and $C(B_1 \cup B_2) = C(B_1) \cup C(B_2)$ for each $B_1, B_2 \in \exp X$, but X is not a C_0-space.

§3 The main results

Lemma 3.1. *Let X be a space and $x \in X$. Then $C(\{x\}) = X - \cup(\mu - \mu_x)$.*

Proof. Let $y \in C(\{x\}) = X - I(X - \{x\})$. Then $y \notin I(X - \{x\})$. If $U \in \mu_y$, then $U \not\subset X - \{x\}$, and hence $x \in U$, i.e., $U \in \mu_x$. So $\mu_y \subset \mu_x$, and hence $\mu - \mu_x \subset \mu - \mu_y$. It follows that $\cup(\mu - \mu_x) \subset \cup(\mu - \mu_y)$. It is easy to see that $y \notin \cup(\mu - \mu_y)$. So $y \notin \cup(\mu - \mu_x)$. Consequently, $y \in X - \cup(\mu - \mu_x)$. On the other hand, let $y \in X - \cup(\mu - \mu_x)$. Then we have $y \in C(\{x\})$ by reversing the proof above. This proves that $C(\{x\}) = X - \cup(\mu - \mu_x)$. □

Proposition 3.1. *Let X be a space and $x, y \in X$. Then the following are equivalent.*

(1) $U \cap \{x, y\} = \{x, y\}$ *for each* $U \in \mu_x$.

(2) $y \in N(x)$.

(3) $\mu_x \subset \mu_y$.

(4) $N(y) \subset N(x)$.

(5) $C(\{x\}) \subset C(\{y\})$.

(6) $x \in C(\{y\})$.

Proof. (1) $\Longrightarrow$ (2) Let $U \cap \{x, y\} = \{x, y\}$ for each $U \in \mu_x$. So $y \in U$ for each $U \in \mu_x$. Consequently, $y \in \cap\mu_x = N(x)$.

(2) $\Longrightarrow$ (3) Let $y \in N(x) = \cap\mu_x$. If $U \in \mu_x$, then $y \in U$, and so $U \in \mu_y$. Consequently, $\mu_x \subset \mu_y$.

(3) $\Longrightarrow$ (4) Let $\mu_x \subset \mu_y$. Then $N(y) = \cap\mu_y \subset \cap\mu_x = N(x)$.

(4) $\Longrightarrow$ (1) Let $N(y) \subset N(x)$. Then $y \in N(y) \subset N(x)$. So, for each $U \in \mu_x$, $y \in U$. Note that $x \in U$. It follows that $U \cap \{x, y\} = \{x, y\}$.

(3) $\Longrightarrow$ (5) Let $\mu_x \subset \mu_y$. Then $\mu - \mu_y \subset \mu - \mu_x$, and hence $\cup(\mu - \mu_y) \subset \cup(\mu - \mu_x)$. So $X - \cup(\mu - \mu_x) \subset X - \cup(\mu - \mu_y)$. By Lemma 3.1, $C(\{x\}) \subset C(\{y\})$.

(5) $\Longrightarrow$ (6) Let $C(\{x\}) \subset C(\{y\})$. Then $x \in C(\{x\}) \subset C(\{y\})$.

(6) $\Longrightarrow$ (3) Let $x \in C(\{y\}) = X - \cup(\mu - \mu_y)$. Then $x \notin \cup(\mu - \mu_y)$. So, if $U \in \mu - \mu_y$, then $x \notin U$. Consequently, if $U \in \mu_x$, then $U \in \mu_y$. This proves that $\mu_x \subset \mu_y$. □

Corollary 3.1. *Let X be a space, $x, y \in X$ and $x \neq y$. Then the following are equivalent.*

(1) $U \cap \{x, y\} = \{x\}$ *for some* $U \in \mu_x$.

(2) $\mu_x \not\subset \mu_y$.

(3) $x \notin C(\{y\})$.

Theorem 3.1. *The following are equivalent for a space X.*

(1) *X is a μT_0-space.*

(2) *If $x, y \in X$ and $x \neq y$, then $\mu_x \not\subset \mu_y$ or $\mu_y \not\subset \mu_x$.*

(3) *If $x, y \in X$ and $x \neq y$, then $x \notin C(\{y\})$ or $y \notin C(\{x\})$.*

(4) *If $x, y \in X$ and $x \neq y$, then $\mu_x \neq \mu_y$.*

(5) *If $x, y \in X$ and $x \neq y$, then $C(\{x\}) \neq C(\{y\})$.*

Proof. (1) $\iff$ (2) $\iff$ (3) They follow from Corollary 3.1.

(2) $\iff$ (4) It is obvious.

(3) $\implies$ (5) Let $x, y \in X$ and $x \neq y$. Without loss of generality, assume that $x \notin C(\{y\})$. Since $x \in C(\{x\})$ by Proposition 2.1(1), $C(\{x\}) \neq C(\{y\})$.

(5) $\implies$ (2) If for $x, y \in X$ and $x \neq y$, $C(\{x\}) \neq C(\{y\})$. Without loss of generality, assume that there is $z \in C(\{x\})$ and $z \notin C(\{y\})$. By Proposition 3.1 and Corollary 3.1, $\mu_z \subset \mu_x$ and $\mu_z \not\subset \mu_y$. So $\mu_x \not\subset \mu_y$. □

Theorem 3.2. *The followings are equivalent for a space X.*

(1) *X is a μT_1-space.*

(2) *If $x, y \in X$ and $x \neq y$, then $\mu_x \not\subset \mu_y$ and $\mu_y \not\subset \mu_x$.*

(3) *If $x, y \in X$ and $x \neq y$, then $x \notin C(\{y\})$ and $y \notin C(\{x\})$.*

(4) *$\{x\} = C(\{x\})$ for each $x \in X$. So each singleton of X is μ-closed.*

(5) *$\{x\} = N(x)$ for each $x \in X$.*

(6) *If $x, y \in X$ and $x \neq y$, then $C(\{x\}) \cap C(\{y\}) = \emptyset$.*

Proof. (1) $\iff$ (2) $\iff$ (3) They follow from Corollary 3.1.

(3) $\implies$ (4) Let $x \in X$. If $y \in X$ and $y \neq x$, then $y \notin C(\{x\})$. So $C(\{x\}) \subset \{x\}$. On the other hand, $\{x\} \subset C(\{x\})$ by Proposition 2.1(1). It follows that $\{x\} = C(\{x\})$.

(4) $\implies$ (5) Let $x \in X$ and $y \in N(x)$. If $y \neq x$, then $x \notin \{y\} = C(\{y\}) = X - I(X - \{y\})$, and hence $x \in I(X - \{y\})$. So there is $U \in \mu_x$ such that $U \subset X - \{y\}$, i.e., $y \notin U$. This contradicts $y \in N(x) = \cap\mu_x$. So $y = x$. This proves that $N(x) \subset \{x\}$. On the other hand, it is clear that $\{x\} \subset N(x)$. So $N(x) = \{x\}$.

(5) $\implies$ (1) Let $x, y \in X$ and $x \neq y$. Then $y \notin N(x)$, and hence there is $U_x \in \mu_x$ such that $y \notin U_x$. Thus $U_x \cap \{x, y\} = \{x\}$. In the same way, there is $U_y \in \mu_y$ such that $U_y \cap \{x, y\} = \{y\}$. Consequently, X is a μT_1-space.

(4) $\implies$ (6) Let $x, y \in X$ and $x \neq y$. Then $C(\{x\}) = \{x\}$ and $C(\{y\}) = \{y\}$. So $C(\{x\}) \cap C(\{y\}) = \{x\} \cap \{y\} = \emptyset$.

(6) $\implies$ (4) Let $x \in X$. If $y \in X$ and $y \neq x$, then $C(\{x\}) \cap C(\{y\}) = \emptyset$. It follows that $y \notin C(\{x\})$ because $y \in C(\{y\})$. So $C(\{x\}) \subset \{x\}$. On the other hand, it is known that $\{x\} \subset C(\{x\})$. Consequently, $\{x\} = C(\{x\})$. □

It is well-known that a topological space X is a T_1-space iff each finite subset of X is closed. Can "singleton" in Theorem 3.2(4) be replaced by "finite subset"? The following example shows that the answer of this question is negative.

Example 3.1. There is a μT_1-space X with a finite subset F of X such that F is not μ-closed. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a, b\}, \{b, c\}, \{c, a\}, X\}$. Then (X, μ) is a μT_1-space, but $F = \{a, b\}$ is not μ-closed.

Proposition 3.2. *Let F be a subset of a C_0-, μT_1-space X. Then the followings hold.*

(1) $d(F) = \emptyset$.

(2) *F is μ-closed.*

Proof. (1) Let $x \in X$. Since X is a C_0-space, there is $U \in \mu_x$ such that x is a representative element of U, and hence $U \subset V$ for each $V \in \mu_x$. It follows that $U \subset \cap\mu_x = N(x)$. By Theorem 3.2, $N(x) = \{x\}$. Therefore, $U = \{x\}$. It is clear that $U \cap (F - \{x\}) = \emptyset$. So $x \notin d(F)$. This proves that $d(F) = \emptyset$.

(2) By Proposition 2.1(9) and (1) of this proposition , $C(F) = F \cup d(F) = F \cup \emptyset = F$. It follows that F is μ-closed by Proposition 2.1(5). □

Lemma 3.2. *Let X be a space and $U, V \in \mu$. Then $U \cap V = \emptyset$ iff $C(U) \cap V = \emptyset$.*

Proof. Sufficiency is clear, we only need to prove necessity. If $U \cap V = \emptyset$, then $U \subset X - V$ and $X - V$ is μ-closed. By (3) and (5) of Proposition 2.1, $C(U) \subset C(X - V) = X - V$. So $C(U) \cap V = \emptyset$. □

Theorem 3.3. *The following are equivalent for a space X.*

(1) *X is a μT_2-space.*

(2) *If $x, y \in X$ and $x \neq y$, then there are $U_x \in \mu_x$ and $U_y \in \mu_y$ such that $C(U_x) \cap U_y = \emptyset$.*

(3) *If $x, y \in X$ and $x \neq y$, then there is $U_x \in \mu_x$ such that $y \notin C(U_x)$.*

(4) *If $x, y \in X$ and $x \neq y$, then there is $U_x \in \mu_x$ such that $x \in U_x \subset C(U_x) \subset X - \{y\}$.*

(5) *$\{x\} = \overline{N(x)}$ for each $x \in X$.*

Proof. (1) $\Longrightarrow$ (2) It follows from Lemma 3.2.

(2) $\Longrightarrow$ (3) It is clear.

(3) $\Longrightarrow$ (4) Let $x, y \in X$ and $x \neq y$. Then there is $U_x \in \mu_x$ such that $y \notin C(U_x)$. So $C(U_x) \subset X - \{y\}$. Consequently, $x \in U_x \subset C(U_x) \subset X - \{y\}$.

(4) $\Longrightarrow$ (5) Let $x \in X$. It is clear that $\{x\} \subset \overline{N(x)}$. On the other hand, if $y \in X$ and $y \neq x$, then there is $U_x \in \mu_x$ such that $x \in U_x \subset C(U_x) \subset X - \{y\}$. So $y \notin C(U_x)$, and hence $y \notin \overline{N(x)}$. This proves that $\overline{N(x)} \subset \{x\}$. Consequently, $\{x\} = \overline{N(x)}$.

(5) $\Longrightarrow$ (1) Let $x, y \in X$ and $x \neq y$. Then $y \notin \{x\} = \overline{N(x)}$. So there is $U_x \in \mu_x$ such that $y \notin C(U_x)$. By Proposition 2.1(7), there is $U_y \in \mu_y$ such that $U_y \cap C(U_x) = \emptyset$. It follows that $U_y \cap U_x = \emptyset$. So X is a μT_2-space. □

Theorem 3.4. *The following are equivalent for a space X.*

(1) *X is a μT_3-space.*

(2) *If $x \notin F$ with F μ-closed, then there are $U_x \in \mu_x$ and $U_F \in \mu$ such that $F \subset U_F$ and $C(U_x) \cap U_F = \emptyset$.*

(3) *If $x \notin F$ with F μ-closed, then there is $U_x \in \mu_x$ such that $C(U_x) \cap F = \emptyset$.*

(4) *If $x \in X$ and $U_x \in \mu_x$, then there is $V_x \in \mu_x$ such that $x \in V_x \subset C(V_x) \subset U_x$.*

(5) *$F = \cap\{C(U) : F \subset U \in \mu\}$ for each μ-closed subset F of X.*

Proof. (1) $\Longrightarrow$ (2) It follows from Lemma 3.2.

(2) $\Longrightarrow$ (3) It is clear.

(3) $\Longrightarrow$ (4) Let $x \in X$ and $U_x \in \mu_x$. Then $X - U_x$ is μ-closed and $x \notin X - U_x$. So there is $V_x \in \mu_x$ such that $C(V_x) \cap (X - U_x) = \emptyset$, hence $C(V_x) \subset U_x$. It follows that $x \in V_x \subset C(V_x) \subset U_x$.

(4) $\Longrightarrow$ (5) Let F be a μ-closed subset of X. It is clear that $F \subset \cap\{C(U) : F \subset U \in \mu\}$. On the other hand, if $x \notin F$, then $x \in X - F$. So there is $U_x \in \mu_x$ such that $x \in U_x \subset C(U_x) \subset X - F$. Put $U_F = X - C(U_x)$, then $F \subset U_F \in \mu$. Since $U_F \subset C(U_F) \subset X - U_x$, $x \notin U_F$, and so $x \notin \cap\{C(U) : F \subset U \in \mu\}$. This proves that $\cap\{C(U) : F \subset U \in \mu\} \subset \{x\}$. Consequently, $\{x\} = \cap\{C(U) : F \subset U \in \mu\}$.

(5) $\Longrightarrow$ (1) Let $x \notin F$ with F μ-closed. Then $x \notin \cap\{C(U) : F \subset U \in \mu\}$. So there is $U \in \mu$ such that $U \supset F$ and $x \notin C(U)$. Put $U_x = X - C(U)$. It is clear that $x \in U_x$ and $U_x \cap U = \emptyset$. So X is a μT_3-space. □

With proofs similar to those of Theorem 3.4, we have the following theorem.

Theorem 3.5. *The following are equivalent for a space X.*

(1) *X is a μT_4-space.*

(2) *If F_1, F_2 are μ-closed and $F_1 \cap F_2 = \emptyset$, then there are $U_1, U_2 \in \mu$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $C(U_1) \cap U_2 = \emptyset$.*

(3) *If F_1, F_2 are μ-closed and $F_1 \cap F_2 = \emptyset$, then there is $U \in \mu$ such that $F_1 \subset U$ and $C(U) \cap F_2 = \emptyset$.*

(4) *If F is μ-closed and $F \subset U \in \mu$, then there is $V \in \mu$ such that $F \subset V \subset C(V) \subset U$.*

Each T_2-compact topological space is normal. The following example shows that the similar result in strong generalized topological space is not true.

Example 3.2. There is a μT_2-, finite space X such that X is not μ-regular. Let $X = \{a, b, c, d, e\}$ and let μ be the strong generalized topology on X generated by $\{\emptyset, \{a, b\}, \{c\}, \{b, d\}, \{b, e\}, \{a, c, d\}, \{a, c, e\}\}$. It is clear that X is a μT_2-space. Consider $a \in \{a, b\} \in \mu_a$. It is easy to check that there is no $U \in \mu$ such that $a \in U \subset C(U) \subset \{a, b\}$. By Theorem 3.4, X is not μ-regular.

Lemma 3.3. *Let $x \in X$. If $d(\{x\})$ is not μ-closed, then $C(d(\{x\})) = C(\{x\})$.*

Proof. Let $d(\{x\})$ be not μ-closed. Since $d(\{x\}) \subset C(\{x\})$, $C(d(\{x\})) \subset C(C(\{x\})) = C(\{x\})$, so we only need to prove that $C(\{x\}) \subset C(d(\{x\}))$. Note that $C(\{x\}) = \{x\} \cup d(\{x\})$ and $C(d(\{x\})) = d(\{x\}) \cup d(d(\{x\}))$. It suffices to prove that $x \in d(d(\{x\}))$. Since $d(\{x\})$ is not μ-closed, so there is $y \in d(d(\{x\})) - d(\{x\})$. Since y is not any μ-cluster point of $\{x\}$, there is $U \in \mu$ such that $y \in U$ and $U \cap (\{x\} - \{y\}) = \emptyset$. On the other hand, $y \in d(d(\{x\}))$, so $U \cap d(\{x\}) \neq \emptyset$. Pick $z \in U \cap d(\{x\})$, then $U \cap (\{x\} - \{z\}) \neq \emptyset$ because $z \in U \in \mu$ and z is a μ-cluster point of $\{x\}$, and so $U \cap \{x\} \neq \emptyset$. Note that $U \cap (\{x\} - \{y\}) = \emptyset$. So $x = y \in d(d(\{x\}))$. □

Theorem 3.6. *The followings are equivalent for a space X.*

(1) *X is a μT_D-space.*

(2) *$d(\{x\})$ is μ-closed for each $x \in X$.*

(3) *For each $x \in X$, there is $U \in \mu$ such that $\{x\} = C(\{x\}) \cap U$.*

Proof. (1) $\Longrightarrow$ (2) Let X be a μT_D-space and $x \in X$. Then $\{x\} = U \cap F$ for some μ-open subset U and some μ-closed subset F. If $d(\{x\})$ is not μ-closed, then $C(d(\{x\})) = C(\{x\})$ by Lemma 3.3. Since $x \in C(\{x\}) \subset F$, $U \cap C(\{x\}) = \{x\}$, and so $U \cap C(d(\{x\})) = \{x\}$. By Proposition 2.1(9), $C(d(\{x\})) = d(\{x\}) \cup d(d(\{x\}))$. So $(U \cap d(\{x\})) \cup (U \cap d(d(\{x\}))) = U \cap C(d(\{x\})) = \{x\}$. By Proposition 2.1(10), $x \notin d(\{x\})$. Note that $U \cap d(\{x\}) \subset \{x\}$, so $U \cap d(\{x\}) = \emptyset$. It follows that $U \cap d(d(\{x\})) = \{x\}$. Consequently $x \in d(d(\{x\}))$, i.e., x is a μ-cluster point of $d(\{x\})$. This shows that $U \cap (d(\{x\}) - \{x\}) \neq \emptyset$. which contradicts with $U \cap d(\{x\}) = \emptyset$.

(2) $\Longrightarrow$ (3) Let $x \in X$ and $d(\{x\})$ is μ-closed. By Proposition 2.1(10), $x \notin d(\{x\})$. Let $U = X - d(\{x\})$, then $x \in U \in \mu$ and $U \cap d(\{x\}) = \emptyset$. So $U \cap C(\{x\}) = U \cap (\{x\} \cup d(\{x\})) = (U \cap \{x\}) \cup (U \cap d(\{x\})) = U \cap \{x\} = \{x\}$.

(3) $\Longrightarrow$ (1) Let $x \in X$. There is $U \in \mu$ such that $\{x\} = U \cap C(\{x\})$. Note that $C(\{x\})$ is μ-closed. So X is a μT_D-space. □

Theorem 3.7. *A μT_1-space is a μT_D-space and a μT_D-space is a μT_0-space.*

Proof. Let X be a μT_1-space and $x \in X$. Then $\{x\} = C(\{x\})$ by Theorem 3.2. If $y \neq x$, then $y \notin \{x\} = C(\{x\}) = \{x\} \cup d(\{x\})$, and so $y \notin d(\{x\})$. On the other hand, $x \notin d(\{x\})$ by Proposition 2.1(10). Thus $d(\{x\}) = \emptyset$. Note that $X \in \mu$, so $d(\{x\}) = \emptyset$ is μ-closed. By Theorem 3.6, X is a μT_D-space.

Let X be a μT_D-space. Then $d(\{z\})$ is μ-closed for each $z \in X$ by Theorem 3.6. Let $x, y \in X$ such that $y \neq x$. If $x \notin C(\{y\})$, by Proposition 2.1(7), there is $U \in \mu_x$ such that $U \cap \{y\} = \emptyset$, i.e., $U \cap \{x, y\} = \{x\}$. If $x \in C(\{y\}) = \{y\} \cup d(\{y\})$, then $x \in d(\{y\})$. Let $U = X - d(\{y\})$, then $x \notin U$. $d(\{y\})$ is μ-closed, so $U \in \mu$. Note that $y \notin d(\{y\})$ by Proposition 2.1(10), so $y \in U$. Consequently, $U \cap \{x, y\} = \{y\}$. This proves that X is a μT_0-space. □

Remark 3.1. By Remark 2.1(1), we see that μT_0-space $\not\Longrightarrow$ μT_D-space $\not\Longrightarrow$ μT_1-space.

By Theorems 3.6 and 3.7, $d(\{x\})$ is μ-closed for each $x \in X$ if X is a μT_1-space. The following example shows that for a finite subset F of a μT_1-space, $d(F)$ need not be μ-closed.

Example 3.3. There is a μT_1-space X with a finite subset F of X such that $d(F)$ is not μ-closed. Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then (X, μ) is a μT_1-space. Let $F = \{a, b\}$, then $d(F) = \{c, d\}$ is not μ-closed.

Theorem 3.8. *The followings are equivalent for a space X.*

(1) *X is a μR_0-space.*

(2) *For each $x, y \in X$, if $x \notin C(\{y\})$, then $y \notin C(\{x\})$.*

(3) *For each $x, y \in X$, if $x \in C(\{y\})$, then $y \in C(\{x\})$.*

(4) *For each $x, y \in X$, $C(\{x\}) = C(\{y\})$ or $C(\{x\}) \cap C(\{y\}) = \emptyset$, i.e., $\{C(\{x\}) : x \in X'\}$ is a partition of X for some $X' \subset X$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that X is a μR_0-space. Let $x, y \in U$ and $x \notin C(\{y\})$. Then $x \in X - C(\{y\}) = I(X - \{y\})$. So there is $U \in \mu$ such that $x \in U \subset X - \{y\}$. Since X is a μR_0-space, $C(\{x\}) \subset U \subset X - \{y\}$. It follows that $y \notin C(\{x\})$.

(2) $\Longrightarrow$ (3) It is obvious.

(3) $\Longrightarrow$ (1) Let $x \in U \in \mu$ and $y \in C(\{x\})$. Then $x \in C(\{y\}) = X - I(X - \{y\})$. So $x \notin I(X - \{y\})$. It shows that $U \not\subset X - \{y\}$, and hence $y \in U$. This proves that $C(\{x\}) \subset U$. So X is a μR_0-space.

(3) $\Longrightarrow$ (4) Let $x, y \in X$. If $C(\{x\}) \cap C(\{y\}) \neq \emptyset$, then there is $z \in C(\{x\}) \cap C(\{y\})$. So $z \in C(\{x\})$, and hence $x \in C(\{z\})$. It follows that $C(\{z\}) \subset C(\{x\})$ and $C(\{x\}) \subset C(\{z\})$. Consequently $C(\{z\}) = C(\{x\})$. In the same way, $C(\{z\}) = C(\{y\})$. So $C(\{x\}) = C(\{y\})$.

(4) $\Longrightarrow$ (3) Let $x, y \in X$. Then $C(\{x\}) = C(\{y\})$ or $C(\{x\}) \cap C(\{y\}) = \emptyset$. If $x \in C(\{y\})$, then $x \in C(\{x\}) \cap C(\{y\}) \neq \emptyset$. So $C(\{x\}) = C(\{y\})$. Consequently $y \in C(\{y\}) = C(\{x\})$. □

Theorem 3.9. *The following are equivalent for a space X.*

(1) *X is a μT_1-space.*

(2) *X is a μT_0- and μR_0-space.*

Proof. (1) $\Longrightarrow$ (2) Let X be a μT_1-space. Then X is a μT_0-space by Remark 2.1(3). For each $x \in X$, $\{x\} = C(\{x\})$ by Theorem 3.2. It follows that $\{C(\{x\}) : x \in X\} = \{\{x\} : x \in X\}$ is a partition of X. So X is a μR_0-space by Theorem 3.8.

(2) $\Longrightarrow$ (1) Suppose that X is a μT_0- and μR_0-space. Let $x, y \in X$ and $x \neq y$. Since X is a μT_0-space, $C(\{x\}) \neq C(\{y\})$ by Theorem 3.1. So $C(\{x\}) \cap C(\{y\}) = \emptyset$ by Theorem 3.8 since X is a μR_0-space. By Theorem 3.2, X is a μT_1-space. □

Remark 3.2. By Remark 2.1(1), T_0-space $\not\Longrightarrow$ R_0-space and R_0-space $\not\Longrightarrow$ T_0-space, so μT_0-space $\not\Longrightarrow$ μR_0-space and μR_0-space $\not\Longrightarrow$ μT_0-space.

§4 Some separations in topological spaces

Á.Császár[3] formulated some separation axioms by replacing open sets by an arbitrary family of subsets of a topological space. In this section, we compare our notions and results with those of *Császár*.

Definition 4.1. [3] Let X be a topological space and $\xi \subset \exp X$.

(1) T_0-*separation*: $x, y \in X$ and $x \neq y$ imply the existence of $K \in \xi$ containing precisely one of x and y.

(2) T_1-*separation*: $x, y \in X$ and $x \neq y$ imply the existence of $K \in \xi$ such that $x \in K$ and $y \notin K$.

(3) T_2-*separation*: $x, y \in X$ and $x \neq y$ imply the existence of $K, K' \in \xi$ such that $x \in K$, $y \in K'$ and $K \cap K' = \emptyset$.

(4) S_1-*separation*: If $x, y \in X$ and there is $K \in \xi$ such that $x \in K$ and $y \notin K$, then there is $K' \in \xi$ satisfying $y \in K'$ and $x \notin K'$.

(5) S_2-*separation*: If $x, y \in X$ and there is $K \in \xi$ such that $x \in K$ and $y \notin K$, then there are $K', K'' \in \xi$ such that $x \in K'$, $y \in K''$ and $K' \cap K'' = \emptyset$.

Remark 4.1. (1) If X is a generalized topological space with generalized topology ξ, then X has T_0-separation (resp. T_1-separation, T_2-separation) in the sense of Definition 4.1(1) (resp. Definition 4.1(2), Definition 4.1(3)) is equivalent to that X is a μT_0-space (resp. μT_1-space, μT_2-space).

(2) T_1-separation (resp. T_2-separation) is equivalent to S_1- and T_0-separation (resp. S_2- and T_0-separation)[3].

Let X be a set and $\xi \subset \exp X$. As in [3], for each $x \in X$, $\kappa\{x\}$ denotes the subset $\{y \in X : y \in K \in \xi \Longrightarrow x \in K\}$ of X.

Remark 4.2. Let $x \in X$. If X is a generalized topological space with generalized topology ξ, then $\kappa\{x\} = C(\{x\})$ by Proposition 2.1(7).

Á.Császár obtained the following results in [3].

Proposition 4.1. [3, Lemma 3.1] *Given $x, y \in X$, $\kappa\{x\} \neq \kappa\{y\}$ if and only if there is $K \in \xi$ containing precisely one of x and y .*

Proposition 4.2. [3, Lemma 3.2] *X has S_1-Separation if and only if $x \in K \in \xi$ implies $\kappa\{x\} \subset K$.*

Proposition 4.3. [3, Proposition 3.3] *Let X be a generalized topological space with generalized topology ξ. If X has S_1-separation, then $\{\kappa\{x\} : x \in X'\}$ is a partition of X for some $X' \subset X$.*

The following remark results the three propositions above to generalized topological spaces.

Remark 4.3. Let X be a generalized topological space with generalized topology ξ.

(1) By Remark 4.2, Proposition 4.1 is the same as (1)$\Longleftrightarrow$(5) in Theorem 3.1.

(2) By Remark 4.2 and Proposition 4.2, X has S_1-separation is equivalent to X has μR_0-separation.

(3) From (2) above, Theorem 3.8 and Remark 4.2, X has S_1-separation if and only if $\{\kappa\{x\} : x \in X'\}$ is a partition of X for some $X' \subset X$, which shows that Proposition 4.3 can be reversed.

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