Appl. Math. J. Chinese Univ.
2009, 24(3): 269-281

On weak solutions for an image denoising-deblurring model

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Abstract. A new denoising-deblurring model in image restoration is proposed, in which the regularization term carries out anisotropic diffusion on the edges and isotropic diffusion on the regular regions. The existence and uniqueness of weak solutions for this model are proved, and the numerical model is also testified. Compared with the TV diffusion, this model preferably reduces the staircase appearing in the restored images.

§1 Introduction

The mathematical model of image restoration usually possesses the following form:

$$u_0 = \mathcal{K}u + n, \tag{1}$$

where u and u_0 are the true image and the observed one respectively. $\mathcal{K}$ is a blur operator, n is an additive noise, Gaussian white noise is often used. The goal of image restoration is to recover u as well as possible from the observed image u_0. This is so-called the image denoising-deblurring problem which is ill-posed. Therefore, some forms of regularization must be applied to accurately reconstruct u. A general strategy is to design a minimizing problem

$$\min \int_\Omega [\phi(|\nabla u|) + \lambda |\mathcal{K}u - u_0|^2] dx, \tag{2}$$

which can be reduced to the evolution problem

$$u_t - \operatorname{div}\,(\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}) + \lambda \mathcal{K}^*(\mathcal{K}u - u_0) = 0, \text{ in } \Omega, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad u(0) = u_0, \text{ on } \bar{\Omega}, \tag{3}$$

where $\mathcal{K}^*$ is the adjoint of $\mathcal{K}$, and $\phi: \mathbf{R} \to \mathbf{R}$ is an increasing function which governs the regularization behavior according to the gradient magnitude $|\nabla u|$.

There are various researches on image restoration by choosing a proper ϕ to propose certain local behavior (ref. [1-4] and references therein). In fact, the divergence in (3) can be

Received: 2008-05-30
MR Subject Classification: 49J40, 35K65, 45K05
Keywords: image restoration, deblurring-denoising, integro-differential equation, weak solution
Digital Object Identifier(DOI): 10.1007/s11766-009-2083-6
Supported by the National Natural Science Foundation of China (10531040)

decomposed into two simultaneously oriented one-dimensional diffusions,

$$\text{div}\,(\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}) = \frac{\phi'(|\nabla u|)}{|\nabla u|}u_{\xi\xi} + \phi''(|\nabla u|)u_{\eta\eta},$$

where $u_{\eta\eta}$ and $u_{\xi\xi}$ denote the second order derivatives of u in orthogonal directions $\eta = \dfrac{\nabla u}{|\nabla u|}$ and $\xi = \eta^{\perp}$, respectively. So the smoothing process is performed along the tangent direction ξ to the isophote lines with the weight $\dfrac{\phi'(|\nabla u|)}{|\nabla u|}$, and along the normal direction η to the isophote lines with the weight $\phi''(|\nabla u|)$.

Traditionally, $\phi = s^2/2$, the so-called H^1 regularization, which is related to the isotropic diffusion, $\text{div}\,(\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}) = \Delta u = u_{\xi\xi} + u_{\eta\eta}$, the weight along the tangent direction ξ equals the one along the normal direction η. So the solution is smooth and the edges are damaged.

In 1992, Rudin, Osher and Fatemi[1] first used $\phi(s) = s$, the TV regularization which is the famous ROF model. Afterwards, Rudin and Osher[5], Vese[6], Chen[7] and Vogel[8] studied the BV (Bounded Variation) solutions of the problem (2) with $\phi(s) = s$ or its several versions. The distinct advantage of BV solution allows jumps(one kind of discontinuity), so the edges of images can be preserved perfectly. But the TV regularization corresponds to the anisotropic diffusion, $\text{div}\,(\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}) = \frac{1}{|\nabla u|}u_{\xi\xi}$. The diffusion only along the tangent direction ξ inevitably results in the staircasing effect appearing in the recovered images(see more details in [9]). One way of reducing the staircasing in reconstruction is to introduce higher order derivatives into the regularization (see [9-10]), that causes the computation complexity and increases the computational expense.

In order to carry out anisotropic diffusion on the edges and isotropic diffusion in regular regions, it is well known[3] that the function ϕ should satisfy

A $\phi'(s) \geq 0$, $\phi''(s) \geq 0$, $\lim_{s\to\infty} \phi''(s) = \lim_{s\to\infty} \dfrac{s\phi''(s)}{\phi'(s)} = 0$,

$\lim_{s\to 0} \phi''(s) = \lim_{s\to 0} \dfrac{\phi'(s)}{s} = c > 0$.

A simple choice is the function chosen by Vese[6],

$$\phi(s) = \begin{cases} s^2/(2s_0), & \text{if } s < s_0, \\ s - s_0/2, & \text{if } s \geq s_0. \end{cases}$$

More refined functions ϕ are introduced by Chen[7], Vogel[8], and others. These researches have improved the quality of restored images more or less, but ϕ functions seem too complicated. Recently, Wang and Zhou[11] discussed the weak solutions of nonlinear parabolic equation (3) with $\phi(s) = s\log(1+s)$ but without the data fidelity term. Apparently, $\phi(s) = s\log(1+s)$ has three distinct advantages: First of all, it satisfies the hypothesis A. Secondly, $\phi(s) \sim s^{1+\varepsilon}$, as $s \to \infty$, for any small $\varepsilon > 0$ the slightly stronger convexity for large s may be helpful to reduce the staircasing effects during image reconstruction. Thirdly, it's obviously simpler than those subsection functions.

In this paper, we prove the existence and uniqueness of weak solution for problem (3) with $\phi(s) = s\log(1+s)$ and $\mathcal{K}$ satisfying the following assumptions:

B $\mathcal{K}$ is a linear operator on $L^2(\Omega)\bigcap W^{1,1}(\Omega)$ and continuous in $L^1(\Omega)$-norm, satisfying the

DC(direct current)-condition: $\mathcal{K}1 = 1$ and the consistence condition: $\int_\Omega \mathcal{K}v dx = \int_\Omega v dx$, namely, $\mathcal{K}^*1 = 1$.

A general linear blur is in the form of $\mathcal{K}v(x) = k * v(x),\ x \in \Omega = [0,a] \times [0,b] \subset \mathbf{R}^2$, where k is called the blur kernel function. In fact, a convolution is only to mess up the grey value on pixel by local (weighted) average uniformly, so the DC condition and the consistence condition hold. For example, the motion blur kernel generally takes the following form[12]:

$$k(x) = \frac{1}{L} 1_{[0,L]}(x \cdot t) \times \delta(x \cdot n), \qquad x = (x_1, x_2) \in \Omega, \tag{4}$$

where L is the travel distance, $1_{[0,L]}(s)$ is the characteristic function of interval $[0,L]$, t denotes the unit vector of motion velocity and n is the unit normal perpendicular to t. Accordingly

$$\mathcal{K}u(x) = \frac{1}{L}\int_0^L u(x - st) ds. \tag{5}$$

Another example is the out-of-focus blur[12], where the Gaussian function is often used,

$$k_\sigma(x) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x_1^2 + x_2^2}{2\sigma^2}\right) \qquad x = (x_1, x_2) \in \Omega.$$

Next, we define the weak solution of problem (3) on the cylinder $Q_T = \Omega \times [0,T]$ for any $T > 0$. Without loss of generality, we set $\lambda = 1$ in the theoretical analysis.

Definition 1. A function u: $\bar{Q}_T \to \mathbf{R}$ is called a weak solution of problem (3), if $u \in C([0,T]; L^2(\Omega)) \bigcap L(0,T; W^{1,1}(\Omega))$ with $\nabla u \in L\log L(Q_T)$ and satisfies

$$-\int_\Omega u_0(x)\varphi(x,0)dx + \int_0^T \int_\Omega [-u\varphi_t + \phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|} \cdot \nabla\varphi] dxdt$$
$$+ \int_0^T \int_\Omega (\mathcal{K}u - u_0)\mathcal{K}\varphi dxdt = 0, \tag{6}$$

for any $\varphi \in C^1(\bar{Q}_T)$, with $\varphi(\cdot, T) = 0$.

Here $\nabla u \in L\log L(Q_T)$ means $\nabla u \in [L^1(Q_T)]^2$ and $\int_0^T \int_\Omega |\nabla u| \log(1 + |\nabla u|) dxdt < +\infty$. The space $L\log L(Q_T)$ is very useful in this study as in [11], a crucial fact is that the bounded set in $L\log L(Q_T)$ is relatively weakly compact in $[L^1(Q_T)]^2$. The main result of this paper is the following theorem.

Theorem 1. If $u_0 \in L^2(\Omega)$, then for any $T > 0$ there is a unique weak solution u of the initial-boundary value problem (3), $u \in C([0,T]; L^2(\Omega)) \bigcap L^1(0,T; W^{1,1}(\Omega))$, with $\nabla u \in L\log L(Q_T)$.

In the following the letter c will represent a generic positive constant that may change from line to line even in the same inequality.

This paper is organized as follows: in §2 we construct an approximate solution sequence for the problem (3) by difference and variation techniques; then in §3, with a prior estimates and subsequential arguments, we establish the existence and uniqueness of weak solution for problem (3). Finally, numerical experimental results are presented in §4.

§2 Approximate solutions

At the beginning of this section we introduce some useful lemmas.

Lemma 1.[11] Suppose that $\phi : [0, +\infty) \to [0, +\infty)$ is a C^2 convex function. Then for all $\xi_1, \xi_2 \in \mathbf{R}^N$, we have $\left[\frac{\phi'(|\xi_1|)}{|\xi_1|}\xi_1 - \frac{\phi'(|\xi_2|)}{|\xi_2|}\xi_2\right] \cdot (\xi_1 - \xi_2) \geq 0$.

Lemma 2.[11] Let $D \subset \mathbf{R}^d$ be measurable with finite Lebesgue measure $|D|$ and suppose that $\{f_j\} \subset [L^1(D)]^N$ satisfies $\int_D |f_j|\log(1+|f_j|)dx \le c$, then there exist a subsequence $\{f_i\} \subset \{f_j\}$ and a function $f \in [L^1(D)]^N$ such that $f_i \rightharpoonup f$ weakly in $[L^1(D)]^N$ as $i \to \infty$ with $\int_D |f|\log(1+|f|)dx \le c$.

In order to construct the approximate solutions for problem (3), we discrete the time variable interval $[0,T]$. Let n be a positive integer. Denote $h = T/n$. With the first equation of (3) we get the following integro-differential equations

$$\frac{u_k - u_{k-1}}{h} - \nabla \cdot (\phi'(| \nabla u_k |)\frac{\nabla u_k}{| \nabla u_k |}) + \mathcal{K}^*\mathcal{K}u_k - \mathcal{K}^*u_0 = 0, \quad \text{in } \Omega, \tag{7}$$

where $k = 1, \cdots, n$. To solve these equations step by step, we only need to prove the existence and uniqueness of the solution for the following problem, for fixed $h > 0$ and u_0, $\tilde{u} \in L^2(\Omega)$,

$$\frac{u - \tilde{u}}{h} - \nabla \cdot (\phi'(| \nabla u |)\frac{\nabla u}{| \nabla u |}) + \mathcal{K}^*\mathcal{K}u - \mathcal{K}^*u_0 = 0, \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega. \tag{8}$$

Definition 2. A function $u \in L^2(\Omega)\bigcap W^{1,1}(\Omega)$ with $\nabla u \in L\log L(Q_T)$ is called a weak solution of problem (8), if for any $\varphi \in C^1(\bar{\Omega})$, the following equation holds.

$$\int_\Omega \frac{u - \tilde{u}}{h}\varphi dx + \int_\Omega \phi'(| \nabla u |)\frac{\nabla u}{| \nabla u |} \cdot \nabla\varphi dx + \int_\Omega (\mathcal{K}u - u_0)\mathcal{K}\varphi dx = 0. \tag{9}$$

Especially, when $\varphi \equiv 1$, since $\mathcal{K}1 = 1$ and $\int_\Omega \mathcal{K}udx = \int_\Omega udx$, we get

$$\int_\Omega \frac{u - \tilde{u}}{h}dx + \int_\Omega (u - u_0)dx = 0,$$

namely $\int_\Omega udx = \frac{1}{1+h}\int_\Omega \tilde{u}dx + \frac{h}{1+h}\int_\Omega u_0dx$.

With the variance approach to prove the existence of weak solution for problem (8), let us consider the minimizing problem

$$\inf_{u\in\mathcal{V}} J(u), \tag{10}$$

where $\mathcal{V} = \overline{\mathcal{W}}^{L^2(\Omega)\cap W^{1,1}(\Omega)}$,

$$\mathcal{W} = \{u \in C^\infty(\bar{\Omega}) \mid \nabla u \in L\log L(\Omega), \quad \int_\Omega udx = \frac{1}{1+h}\int_\Omega \tilde{u}dx + \frac{h}{1+h}\int_\Omega u_0dx\},$$

and when $\forall u \in \mathcal{V}$ the functional J is defined as

$$J(u) = \frac{1}{2h}\int_\Omega (u - \tilde{u})^2dx + \int_\Omega \phi(| \nabla u |)dx + \frac{1}{2}\int_\Omega (\mathcal{K}u - u_0)^2dx. \tag{11}$$

It is obvious that (8) is just the Euler-Lagrange equations of the functional J.

Theorem 2. There exists a unique weak solution of problem (8).

Proof. First we confirm that $J(u)$ has a minimizer u in $\mathcal{V}$.

Let $u_1 = \dfrac{1}{|\Omega|(1+h)}\displaystyle\int_\Omega (\tilde{u} + hu_0)dx \in \mathcal{V}$. Because of

$$0 \le \inf_{u\in\mathcal{V}} J(u) \le J(u_1) = \frac{1}{2h}\int_\Omega (u_1 - \tilde{u})^2dx + \frac{1}{2}\int_\Omega (u_1 - u_0)^2dx,$$

there exists a minimizing sequence $\{u_n\} \subset \mathcal{V}$ such that $\lim_{n\to\infty} J(u_n) = \inf_{u\in\mathcal{V}} J(u)$.

Since $\{J(u_n)\}$ is convergent, there exists a constant $c > 0$ such that $0 < J(u_n) \le c$. Therefore,

$$\int_\Omega u_n^2dx + 2h\int_\Omega \phi(| \nabla u_n |)dx + h\int_\Omega (\mathcal{K}u_n - u_0)^2dx \le c,$$

which shows that $\{u_n\}$ is bounded in $L^2(\Omega)$ and $| \nabla u_n |$ is bounded in $L\log L(\Omega)$. Since

$L\log L(\Omega)$ is relatively weakly compact in $L^1(\Omega)$, then as Lemma 2 has mentioned, there exists a subsequence $\{u_m\}\subseteq\{u_n\}$ and a function $u\in L^2(\Omega)\bigcap W^{1,1}(\Omega)$ such that $u_m\to u$ in $L^2(\Omega)$ weakly and $\nabla u_m\to\nabla u$ in $L^1(\Omega)$ weakly, which results in $u_m\to u$ in $L^1(\Omega)$ strongly, and $u_m\to u$, a.e. $x\in\Omega$. Since $\mathcal{K}$ is continuous in $L^1(\Omega)$, thus

$$\mathcal{K}u_m\to\mathcal{K}u \quad \text{in } L^1(\Omega) \quad \text{strongly and then} \quad \mathcal{K}u_m\to\mathcal{K}u, \text{ a.e. } \quad x\in\Omega.$$

Therefore applying the Fatou Lemma, we obtain

$$\int_\Omega(\mathcal{K}u-u_0)^2dx\le\liminf_{m\to\infty}\int_\Omega(\mathcal{K}u_m-u_0)^2dx,\qquad \int_\Omega(u-\tilde u)^2dx\le\liminf_{m\to\infty}\int_\Omega(u_m-\tilde u)^2dx,$$

$$\int_\Omega\phi(|\nabla u|)dx\le\liminf_{m\to\infty}\int_\Omega\phi(|\nabla u_m|)dx,\qquad \int_\Omega udx=\lim_{m\to\infty}\int_\Omega u_mdx.$$

Thus, $u\in\mathcal{V}$ and $J(u)\le\liminf_{m\to\infty}J(u_m)=\inf_{u\in\mathcal{V}}J(u)$, consequently u is the minimizer of $J(u)$. Moreover, with $u_0\in L^2(\Omega)$, we get $\mathcal{K}u\in L^2(\Omega)$.

Next we show that the minimizer u is the weak solution of (8). For any $\varphi\in C^1(\bar\Omega)$, denote $\varphi_\Omega=\dfrac{1}{|\Omega|}\displaystyle\int_\Omega\varphi(x)dx$, then $u+t(\varphi-\varphi_\Omega)\in\mathcal{V}$, for all t. Let $j(t)=J(u+t(\varphi-\varphi_\Omega))$, then $j(0)\le j(t)$, which results in $j'(0)=(\delta J(u),\varphi-\varphi_\Omega)=0$, namely,

$$\int_\Omega\frac{u-\tilde u}{h}(\varphi-\varphi_\Omega)dx+\int_\Omega\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla(\varphi-\varphi_\Omega)dx+\int_\Omega(\mathcal{K}u-u_0)(\mathcal{K}\varphi-\mathcal{K}\varphi_\Omega)dx=0.$$

Since $\int_\Omega\dfrac{u-\tilde u}{h}\varphi_\Omega dx+\displaystyle\int_\Omega(\mathcal{K}u-u_0)\mathcal{K}\varphi_\Omega dx=0$ by $u\in\mathcal{V}$, then

$$\int_\Omega\frac{u-\tilde u}{h}\varphi dx+\int_\Omega\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla\varphi dx+\int_\Omega(\mathcal{K}u-u_0)\mathcal{K}\varphi dx=0,$$

which shows that u is a weak solution of (8).

Finally, we prove the uniqueness of the weak solution. Suppose there exists another weak solution v, then it satisfies

$$\int_\Omega\frac{u-v}{h}\varphi dx+\int_\Omega\left[\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}-\phi'(|\nabla v|)\frac{\nabla v}{|\nabla v|}\right]\cdot\nabla\varphi dx+\int_\Omega\mathcal{K}(u-v)\mathcal{K}\varphi dx=0.$$

If we choose $\varphi=u-v$, then

$$\int_\Omega\frac{(u-v)^2}{h}dx+\int_\Omega[\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}-\phi'(|\nabla v|)\frac{\nabla v}{|\nabla v|}]\cdot(\nabla u-\nabla v)dx+\int_\Omega[\mathcal{K}(u-v)]^2dx=0.$$

It follows from Lemma 1 that $\int_\Omega(u-v)^2dx\le0$, consequently, $u=v$, a.e. in Ω.

Remark. For each $h=T/n$, by Theorem 2, we construct a sequence $\{u_k\}\in L^2(\Omega)\bigcap W^{1,1}(\Omega)$ of the solutions of equation (7) for $k=1,2,\cdots,n$, and then the approximate solution u_h of problem (3) can be defined by $u_h(x,t)=u_j(x)$, where $(j-1)h<t\le jh$ $j=1,2,\cdots,n$.

§3 Existence and uniqueness of the weak solution

Proof of Theorem 1. First we prove the uniqueness of weak solution for problem (3). Suppose u,v are two weak solutions, set $\varphi=u-v$, then

$$\frac12\int_\Omega(u-v)^2dx+\int_0^t\int_\Omega[\phi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}-\phi'(|\nabla v|)\frac{\nabla v}{|\nabla v|}]\cdot[\nabla u-\nabla v]dxdt$$
$$+\int_0^t\int_\Omega[\mathcal{K}(u-v)]^2dxdt=0.$$

With Lemma 1 it is easy to deduce that $u=v$ a.e. in Q_T.

Next we will compress a weak solution from the above approximate solutions.

It follows from (7) that $\forall\varphi\in C^1(\bar{\Omega})$,

$$\int_\Omega \frac{u_k-u_{k-1}}{h}\varphi dx+\int_\Omega \phi^{'}(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k|}\cdot\nabla\varphi dx+\int_\Omega(\mathcal{K}u_k-u_0)\mathcal{K}\varphi dx=0. \tag{12}$$

By approximation we can choose $\varphi=u_k$ to get

$$\frac{1}{2}\int_\Omega u_k^2dx+h\int_\Omega\phi^{'}(|\nabla u_k|)\,|\nabla u_k|\,dx+\frac{h}{2}\int_\Omega(\mathcal{K}u_k-u_0)^2dx\le\frac{1}{2}\int_\Omega u_{k-1}^2dx+\frac{h}{2}\int_\Omega u_0^2dx.$$

After summing k from 1 to j, as $t=jh$, we get that

$$\begin{aligned}&\frac{1}{2}\int_\Omega u_h(x,t)^2dx+\int_0^t\int_\Omega\phi^{'}(|\nabla u_h|)|\nabla u_h|dxdt+\frac{1}{2}\int_0^t\int_\Omega(\mathcal{K}u_h-u_0)^2dxdt\\&\le\frac{1+t}{2}\int_\Omega u_0^2(x)dx\le\frac{1+T}{2}\int_\Omega u_0^2(x)dx.\end{aligned} \tag{13}$$

Therefore $\int_\Omega u_h(x,t)^2dx\le c$, and $\int_0^T\int_\Omega\phi(|\nabla u_h|)dxdt\le c$ because of $\phi(s)\le\phi^{'}(s)s$ for all $s>0$.

By Lemma 2 we obtain a subsequence of $\{u_h\}$ (for simplicity, denote it by the original) and a function $u\in L^\infty(0,T;L^2(\Omega))\bigcap L^1(0,T;W^{1,1}(\Omega))$ satisfies $\int_0^T\int_\Omega\phi(|\nabla u|)dxdt\le c$, namely $\nabla u\in L\ln L(Q_T)$ such that $u_h\rightharpoonup u$ weakly star in $L^\infty(0,T;L^2(\Omega))$ and weakly in $L(0,T;W^{1,1}(\Omega))$. We want to show that u is just a weak solution of problem (3).

Denote $\xi_h=\phi^{'}(|\nabla u_h|)\frac{\nabla u_h}{|\nabla u_h|}$. Since $\phi'(s)=\log(1+s)+\frac{s}{1+s}\le\log(1+s)+1$, we have

$$|\xi_h|^2\le 2[\log(1+|\nabla u_h|)]^2+2\le 2\phi(|\nabla u_h|)+2,$$

then $\{\xi_h\}$ is bounded in $[L^2(Q_T)]^2$, which implies that there exists a subsequence of $\{\xi_h\}$ (for simplicity, denote it by the original) and a function $\xi\in[L^2(Q_T)]^2$ such that $\xi_h\to\xi$ in $[L^2(Q_T)]^2$ weakly.

Meanwhile, because of $|\xi_h|\exp(|\xi_h|)\le e(1+|\nabla u_h|)[\log(1+|\nabla u_h|)+1]\le 2e\phi(|\nabla u_h|)+3e$, then $\int_0^T\int_\Omega|\xi|\exp(|\xi|)dxdt\le\liminf_{h\to0}\int_0^T\int_\Omega|\xi_h|\exp(|\xi_h|)dxdt\le c$, so with the inequality $ab\le ae^a+b\log(1+b)$, $\forall a,\ b\ge0$, we get

$$|\xi\cdot\nabla u|\le|\xi||\nabla u|\le|\xi|\exp(|\xi|)+|\nabla u|\log(1+|\nabla u|),$$

which means $\xi\cdot\nabla u\in L^1(Q_T)$.

We consider (7) in the weak formulation,

$$\begin{aligned}&\int_\Omega\frac{u_k(x)-u_{k-1}(x)}{h}\varphi(x,kh)dx+\int_\Omega\phi^{'}(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k(x)|}\cdot\nabla\varphi(x,kh)dx\\&+\int_\Omega(\mathcal{K}u_k-u_0)\mathcal{K}\varphi(x,kh)dx=0,\end{aligned}$$

where $\varphi\in C^1(\bar{Q_T}),\ \varphi(\cdot,T)=0$. Summing k from 1 to n, we get

$$\begin{aligned}&-\frac{1}{h}\int_\Omega u_0(x)\varphi(x,h)dx+\sum_{k=1}^{n-1}\int_\Omega u_k(x)\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}dx\\&+\sum_{k=1}^{n}\int_\Omega\phi^{'}(|\nabla u_k|)\frac{\nabla u_k}{|\nabla u_k(x)|}\cdot\nabla\varphi(x,kh)dx+\sum_{k=1}^{n}\int_\Omega(\mathcal{K}u_k-u_0)\mathcal{K}\varphi(x,kh)dx=0,\end{aligned}$$

which can be transformed into

$$
\begin{aligned}
&-\int_\Omega u_0(x)\varphi(x,h)dx+\int_0^T\int_\Omega -u_h(x,t)\varphi_t(x,t)dxdt\\
&+\int_0^T\int_\Omega \phi^{'}(|\nabla u_h|)\frac{\nabla u_h}{|\nabla u_h|}\cdot\nabla\varphi dxdt+\int_0^T\int_\Omega(\mathcal{K}u_h-u_0)\mathcal{K}\varphi dxdt\\
&+\sum_{k=1}^{n-1}\int_{kh}^{(k+1)h}\int_\Omega u_h(x,kh)(\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}+\varphi_t(x,t))dxdt\\
&+\sum_{k=1}^{n-1}\int_{kh}^{(k+1)h}\int_\Omega \phi^{'}(|\nabla u_h|)\frac{\nabla u_h}{|\nabla u_h(x)|}\cdot(\nabla\varphi(x,kh)-\nabla\varphi(x,t))dxdt\\
&+\sum_{k=1}^{n-1}\int_{kh}^{(k+1)h}\int_\Omega(\mathcal{K}u_h-u_0)\mathcal{K}(\varphi(x,kh)-\varphi(x,t))dxdt=0.
\end{aligned}
$$

Let $h\to 0$, we get

$$
\begin{aligned}
&-\int_\Omega u_0(x)\varphi(x,0)dx-\int_0^T\int_\Omega u(x,t)\varphi_t(x,t)dxdt\\
&+\int_0^T\int_\Omega \xi\cdot\nabla\varphi dxdt+\int_0^T\int_\Omega(\mathcal{K}u-u_0)\mathcal{K}\varphi dxdt=0. \qquad (14)
\end{aligned}
$$

It has to be shown that $\xi=\phi^{'}(|\nabla u|)\frac{\nabla u}{|\nabla u|}$. With the inequality

$$\int_0^T\int_\Omega(\phi^{'}(|\nabla u_h|)\frac{\nabla u_h}{|\nabla u_h|}-\phi^{'}(|\nabla v|)\frac{\nabla v}{|\nabla v|})\cdot(\nabla u_h-\nabla v)dxdt\ge 0$$

for all $v\in C_0^1(Q_T)$, which follows from Lemma 1, and the inequality

$$\frac{1}{2}\int_\Omega u_h(x,t)^2dx+\int_0^t\int_\Omega\phi^{'}(|\nabla u_h|)\,|\nabla u_h|\,dxdt+\int_0^t\int_\Omega(\mathcal{K}u_h-u_0)\mathcal{K}u_h dxdt\le\frac{1}{2}\int_\Omega u_0^2dx$$

which is deduced from (12) in the same way as we have done for (13), we obtain the estimate

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega u_h(x,T)^2dx+\int_0^T\int_\Omega\phi^{'}(|\nabla u_h|)\frac{\nabla u_h}{|\nabla u_h|}\cdot\nabla vdxdt\\
&+\int_0^T\int_\Omega\phi^{'}(|\nabla v|)\frac{\nabla v}{|\nabla v|}\cdot\nabla u_hdxdt-\int_0^T\int_\Omega\phi^{'}(|\nabla v|)\frac{\nabla v}{|\nabla v|}\cdot\nabla vdxdt\\
&+\frac{1}{2}\int_0^T\int_\Omega(\mathcal{K}u_h-u_0)\mathcal{K}u_hdxdt\le\frac{1}{2}\int_\Omega u_0^2dx.
\end{aligned}
$$

Passing to limits, as $h\to 0$, we get

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega u(x,T)^2dx+\int_0^T\int_\Omega\xi\cdot\nabla vdxdt+\int_0^T\int_\Omega\phi^{'}(|\nabla v|)\frac{\nabla v}{|\nabla v|}\cdot\nabla udxdt\\
&-\int_0^T\int_\Omega\phi^{'}(|\nabla v|)\frac{\nabla v}{|\nabla v|}\cdot\nabla vdxdt+\frac{1}{2}\int_0^T\int_\Omega(\mathcal{K}u-u_0)\mathcal{K}u\,dxdt\le\frac{1}{2}\int_\Omega u_0^2dx.
\end{aligned}
$$

From the above inequality and (14)with $\varphi=u$ it follows that

$$\int_0^T\int_\Omega(\phi^{'}(|\nabla v|)\frac{\nabla v}{|\nabla v|}-\xi)\cdot(\nabla u-\nabla v)dxdt\le 0.$$

By approximation we can take $v=u+\alpha w$ for any $\alpha>0$ and $w\in L^2(0,T;H^1(\Omega))$, it follows from the above inequality that

$$\phi^{'}(\mid \nabla u \mid)\frac{\nabla u}{\mid \nabla u \mid} = \xi, \text{ a.e. in } Q_T.$$

Therefore u satisfies (6). It remains to prove that $u \in C(0,T;L^2(\Omega))$.

Let us take $\varphi(x,0)=0$ in (14), it follows that

$$-\int_0^T\int_\Omega u(x,t)\varphi_t(x,t)dxdt + \int_0^T\int_\Omega \xi\cdot\nabla\varphi dxdt + \int_0^T\int_\Omega(\mathcal{K}u-u_0)\mathcal{K}\varphi dxdt = 0,$$

namely,

$$u_t - \text{div}\xi + \mathcal{K}^*(\mathcal{K}u-u_0) = 0 \text{ in } \mathcal{D}^{'}(Q_T).$$

Since $\xi \in [L^2(Q_T)]^2$ and $\mathcal{K}^*(\mathcal{K}u-u_0) \in L^2(0,T;L^2(\Omega))$, then $u_t \in L^2(0,T;H^{-1}(\Omega))$. With $u \in L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;L^2(\Omega))$ we get $u \in C(0,T;H^{-1}(\Omega))$.

Let $v_h(x,t) = u(x,t+h)$, which also is the weak solution of problem (3) satisfying $v_h(x,0) = u(x,h)$. Denote $w_h(x,t) = u(x,t+h) - u(x,t)$, then

$$\frac{\partial w_h}{\partial t} - \left[\text{div}(\phi^{'}(\mid\nabla v_h\mid)\frac{\nabla v_h}{\mid\nabla v_h\mid}) - \text{div}(\phi^{'}(\mid\nabla u\mid)\frac{\nabla u}{\mid\nabla v\mid})\right] + \mathcal{K}^*\mathcal{K}w_h = 0.$$

Thus, by Lemma 1,

$$\frac{1}{2}\int_\Omega w_h(x,t)^2dx + \int_0^t\int_\Omega(\mathcal{K}w_h)^2dxdt \le \frac{1}{2}\int_\Omega w_h^2(x,0)dx.$$

We get $\int_\Omega(u(x,t+h)-u(x,t))^2dx \le \int_\Omega(u(x,h)-u_0(x))^2dx$.

In order to show $u \in C(0,T;L^2(\Omega))$, we only need to show that

$$\limsup_{h\to 0^+}\int_\Omega(u(x,h)-u_0(x))^2dx = 0. \tag{15}$$

Suppose (15) isn't true. Then there are a positive number $\delta > 0$ and a subsequence h_j with $h_j \to 0$ as $j\to\infty$ such that $\int_\Omega[u(x,h_j)-u_0(x)]^2dx \ge \delta$. From the estimate (13) it follows that $\int_\Omega u(x,h)^2dx \le (1+h)\int_\Omega u_0(x)^2dx \le (1+T)\int_\Omega u_0(x)^2dx$. So it is easy to deduce that

$$\int_\Omega[u_0(x)-u(x,h_j)]u_0(x)dx + \frac{h_j}{4}\int_\Omega u_0(x)^2 \ge \frac{\delta}{2},$$

and that there exits a subsequence of $\{u(x,h_j)\}$ weakly converging to u_0 in $L^2(\Omega)$ because of $u \in C(0,T;H^{-1}(\Omega))$. Passing to limits, a contradictive result $\delta \le 0$ appears. Theorem 1 is thus proved.

§4 Numerical methods and experimental results

Now we test the numerical results of our model (3), in which $\phi(s) = s\log(1+s)$ and the positive parameter λ is selected to control the trade-off between denoising and deblurring. In other words, for heavily blurred image with light noise, we choose big λ, and vice versa.

We partition the domain Ω into $n_x \times n_y$ uniform cells, denote the time step $1/T$ by $\triangle t$. Accordingly, the cell centers are $(x_i,y_j) = (i,j)$, the time layer is $t_k = k\triangle t$, and the value $u(x_i,y_j,t_k)$ is denoted by $u_{i,j}^k$, $i=1,2,\cdots,n_x; j=1,2,\cdots,n_y; k=0,1,\cdots,T$. Then, to save the computational cost, we approximate the problem (3) by an explicit finite difference scheme:

$$u_{ij}^{k+1} = u_{ij}^k - \triangle t[(Au^k)_{ij} + \lambda(\mathcal{K}^*(\mathcal{K}u^k-u_0))_{ij}],$$

where $(Au^k)_{ij} = (\text{div}\,(\phi'(\mid\nabla u^k\mid)\frac{\nabla u^k}{|\nabla u^k|}))_{ij}$. Using u_0 as initial guess, and homogenous Neumann boundary conditions are as follows:

$$u_{0,j} = u_{1,j}, \quad u_{n_x+1,j} = u_{n_x,j}, \quad u_{i,0} = u_{i,1}, \quad u_{i,n_y+1} = u_{i,n_y},$$

with $i = 1, 2, \cdots, n_x$; $j = 1, 2, \cdots, n_y$; $k = 0, 1, \cdots, T-1$.

More precisely, the degenerate diffusion term $(Au^k)_{ij}$ is approximated by a standard five-point finite difference scheme(as displayed in [9]),

$$(\mathrm{div}\,(\phi'(|\nabla u^k|)\tfrac{\nabla u^k}{|\nabla u^k|}))_{i,j} = (D_{i+1/2,j} + D_{i-1/2,j} + D_{i,j+1/2} + D_{i,j-1/2})u^k_{i,j}$$
$$-(D_{i+1/2,j}u^k_{i+1,j} + D_{i-1/2,j}u^k_{i-1,j} + D_{i,j+1/2}u^k_{i,j+1} + D_{i,j-1/2}u^k_{i,j-1}),$$

where

$$D_{i+1/2,j} = \frac{\phi'(|u^k_{i+1,j} - u^k_{i,j} + \varepsilon|)}{|u^k_{i+1,j} - u^k_{i,j} + \varepsilon|} \approx \left(\frac{\phi'(|\nabla u^k|)}{|\nabla u^k|}\right)_{i+1/2,j},$$

and the same structures are taken for $D_{i-1/2,j}, D_{i,j+1/2}, D_{i,j-1/2}$.

For the linear term $\mathcal{K}^*(\mathcal{K}u - u_0)$, we turn to the MATLAB's filter method, i.e., using small mask to denote the convolution kernel k. For example, suppose the 3×3 mask has the form $k = (k_{ls})_{3\times 3}$, then the convolution procedure involves computing the sum of products of the coefficients with the gray levels contained in the region encompassed by the mask. That is,

$$(\mathcal{K}u)_{ij} = \sum_{l,s=1}^{3} k_{ls}u_{i+2-l,j+2-s}.$$

Two simple examples are the out-of-focus blur kernel and the motion blur kernel($L = 3, \theta = 45$) as follows respectively:

$$\begin{pmatrix} 0.5/10 & 1/10 & 0.5/10 \\ 1/10 & 4/10 & 1/10 \\ 0.5/10 & 1/10 & 0.5/10 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1/3 & 0 \\ 1/3 & 0 & 0 \end{pmatrix}.$$

Obviously, the DC condition $\sum_{l,s} k_{ls} = 1$ and the consistence condition $\sum_{i,j}(\mathcal{K}u)_{i,j} = \sum_{i,j} u_{i,j}$ hold.

We will use the signal to noise ratio (SNR) of the image u to measure the level of noise, SNR $= \frac{\|u\|_{L^2}}{\|\eta\|_{L^2}}$. Signal to blurring ratio (SBR) is used to measure the level of blurring effect, SBR$= \frac{\|u\|_{L^2}}{\|Ku-u\|_{L^2}}$. Moreover, we adopt improvement in the signal quality (ISR) to measure the goodness of restored image, ISR$= \frac{\|u-u_0\|_{L^2}}{\|u-u_{\mathrm{new}}\|_{L^2}}$, where u_{new} is the restored image. Apparently, the larger the value of ISR, the better the restored image.

Next, we display some main experimental results and give some analysis for the results. Fig. 1 displays two tested images, both are 256×256 and image I is a piecewise constant image "Square", image II is a natural image "Lina".

In Fig. 2 we illustrate the effect of our deblurring model to only blurred images, i.e, SNR=∞. Two kinds of blur operators are tested, one is the out-of-focus blur and the other is the motion blur. For the out-of-focus blur, we set $\sigma^2 = 10$ in its PSF, accordingly, SBR=13.6561 for image I and SBR=11.2045 for image II. As to the motion blur, we choose motion length $L = 20$ and motion direction $\theta = \pi/4$, SBR=10.2301 for image I and SBR=8.6920 for image II. Since there's no noise, we set $\lambda = 5$ in the model equation. Columns II and IV display the restored images, ISR=3.6116 and ISR=3.0274 respectively for column II, and ISR=2.8125 and

Fig.1 Original images of image I (left) and image II (right)

Fig.2 Deblurring results for only blurred images. **Column I**: Out-of-focus blurred images. **Column II**: Restored images of column I. **Column III**: Motion blurred image with motion length $L = 20$ and direction $\theta = \pi/4$. **Column IV**: Restored images of column III.

ISR=2.4103 respectively for column IV. The good results suggest that our method is robust.

In Fig. 3 we test the denoising effect of the model (3) with $\mathcal{K} = \mathcal{I}$ to the noisy images. In other words, SBR=∞. SNR=150 for both images in column I and SNR=75 in column III. Columns II and IV are the restored images, ISR=1.1383 and ISR=0.8293 respectively in column II, and ISR=1.3269 and ISR=0.7182 respectively in column IV. During computation, we set $\lambda = 10^{-8}$. From the results we can see that denoising is basically successful even for heavily noisied image. But the lightness of images seems to be decreased, that deserves to be studied ulteriorly.

Fig. 4 displays the results of our denoising-deblurring model($\phi(s) = s\log(1+s)$) for both blurred and noisy images. During reconstruction we choose λ by use of some parameter selection methods, as given in Chen YM's work[3]. As shown in columns II and IV, the noise is almost gone, and the blurring effects are basically removed, which is consistent with our theory. Imperfectly, there's still some trace of blur, especially near the edges. The reason may be the faultiness in algorithm, we will optimize it in future work. Table 1 displays the computational values including SBR, SNR, iterative numbers and ISR. We test Square and Lina respectively, each with out-of-focus blur and motion blur, and the noise levels are different too. The algorithm stops until the error between u^{v+1} and u^v is less than the given precision. Apparently,

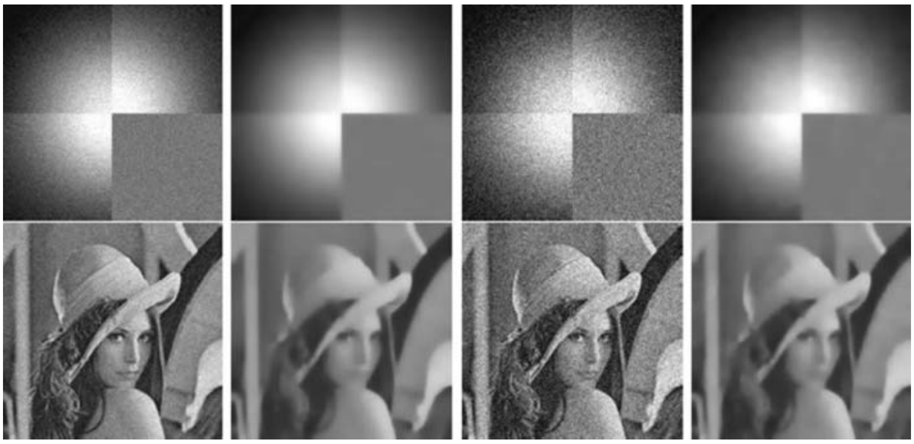

Fig.3 Denoising results for noisy images. **Column I**: Noisy images, with SNR=150. **Column II**: Restored images for column I. **Column III**: Noisy images, with SNR=75. **Column IV**: Restored images for column III.

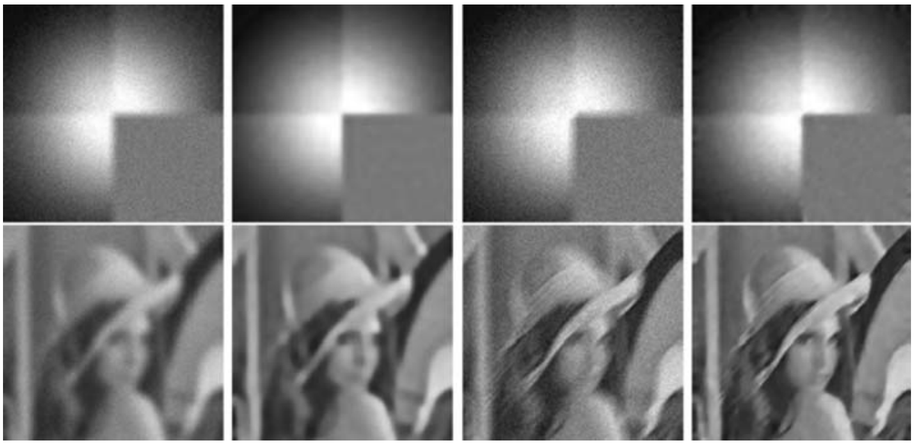

Fig.4 Deblur-debnoise results for blurred, noisy images by using $\phi(s) = s\log(1+s)$.**Column I**: Out-of-focus blurred and noisy image. **Column II**: Restored images for column I. **Column III**: Motion blurred and noisy image. **Column IV**: Restored images for column III.

the ISRs are larger for less degenerated images(larger SBR and SNR).

Table 1 Deblur-denoise results using $\phi = s\log(1+s)$

Images	SBR	SNR	times	ISR
Square	28.31	137.90	38	1.58
	23.45	88.30	100	1.51
Lina	20.38	133.57	44	1.61
	18.27	76.51	113	1.53

Up to now, all our results are based on $\phi(s) = s\log(1+s)$. Next, we compare the effect of denoising and deblurring with different functions ϕ. For simplicity, we denote three ϕ functions as follows:

$$\phi_1(s) = \begin{cases} s - \dfrac{s_0}{2}, & \text{if } s \geq s_0 \\ \dfrac{s^2}{2s_0}, & \text{if } s < s_0 \end{cases}, \ \phi_2(s) = s\log(1+s), \ \phi_3(s) = \frac{1}{2}s^2,$$

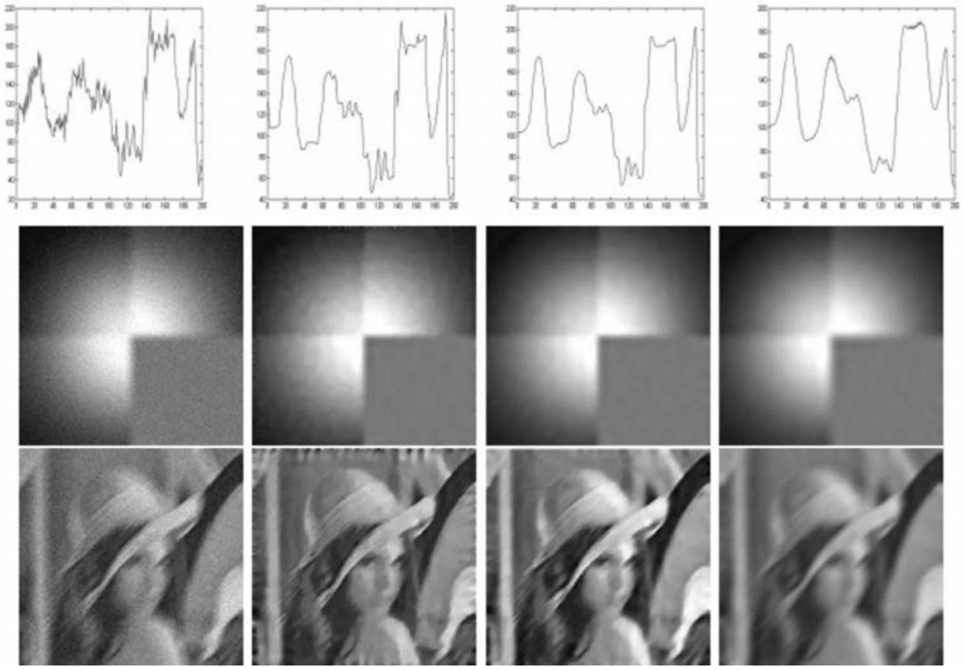

Fig.5 Deblur-denoise results by using different functions ϕ. **Column I**: Blurred noisy image, out-of-focus blur for Square while motion blur for 1-D signal and Lina. **Column II**: Restored images with ϕ_1. **Column III**: Restored images with ϕ_2. **Column IV**: Restored images with ϕ_3.

Table 2 Deblur-denoise results using different ϕ functions

Different ϕ	ϕ_1		ϕ_2		ϕ_3	
	times	ISR	times	ISR	times	ISR
Square	50	1.33	32	1.35	65	1.09
Lina	100	1.27	76	1.31	93	1.12

where $s_0 = \frac{1}{|\Omega|}\int_\Omega |\nabla u_0(\vec{x})|\, d\vec{x}$ stands for the average value of gradient norm of the blurred noisy image u_0. The restoration is illustrated in Fig. 5, and the numerical results are displayed in Table 2. We test out-of-focus blur on Square and motion blur on 1-D signal and Lina. Obviously, the effect is best by using $\phi_2(s) = s\log(1+s)$, which can be detected clearly from the 1-D signal case, the result for ϕ_2 is neither too staircasing like the subsection ϕ_1, nor too fuzzy like ϕ_3. Similarly, as shown in Table 2, the restoration demands least iterative times and gets highest ISR by using $\phi_2(s) = s\log(1+s)$. It suggests that our method is much better.

In this paper we present a new model for image denoising and deblurring. The above numerical experimental results demonstrate the superiority in reducing the staircasing. Here we consider the restoration of the blurred image only for the uniform case, i.e., the motion length and angle are uniformly the same for each pixel in image. Practically, there are lots of nonuniform cases. For example, when a digital camera is kept static, but the catching objects are moving in distinct speeds or toward different directions. This will be our main work in the future.

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