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Mean size formula of wavelet subdivision tree on Heisenberg group

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Abstract. The purpose of this paper is to investigate the mean size formula of wavelet packets (wavelet subdivision tree) on Heisenberg group. The formula is given in terms of the p-norm joint spectral radius. The vector refinement equations on Heisenberg group and the subdivision tree on the Heisenberg group are discussed. The mean size formula of wavelet packets can be used to describe the asymptotic behavior of norm of the subdivision tree.

§1 Introduction

(Orthogonal)wavelet bases are powerful tools for time-frequency localization. Interest in them has grown at a explosive rate in the last twenty years. Wavelets appeal to scientists and engineers among different backgrounds. There are exciting applications in signal analysis, statistical estimation, numerical analysis and so on.

However, wavelet bases can not do excellent localization analysis for high frequency signals. In order to obtain better localization for high frequency components in the wavelet decomposition, Coifman et al[1] introduced another kind of bases called wavelet packets. Wavelet packets in connection with orthonormal wavelets provide a best basic algorithm in terms of the entropy estimates(see [2]).

There are many possible ways to measure/define the frequency localization of wavelet packets. One much used measure is the L_1 norm of Fourier transform of the wavelets, see, e.g., [3]. This measure is closely related to the L_∞ norm of wavelet packets, and it is thus reasonable to use the L_p norm of wavelet packets for large p to measure their frequency localization. So the mean size of the L_p norm of wavelet packets on a given scale will then correspond to the mean frequency localization of the functions.

Indeed, the mean size of the L_p norm plays an important role in describing the behavior of wavelet packets. Recently, Nielsen[4] showed that wavelet packets of some well-known

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Daubechies' orthogonal wavelets do not form Schauder bases for L_p when p is sufficiently large. These negative results are all based on estimating the L_p norms of wavelet packets.

In [5], Nielsen and Zhou studied the mean size of wavelet packets in L_p. An asymptotic formula for the mean size was given in terms of the p-norm spectral radius. They also gave some applications of this formula. Li et al[6] generalized their results to wavelet packets generated by multiple wavelets when the dilation matrix is a general integer expanding matrix.

Wavelet analysis can go beyond the Euclidean space. Many studies on the wavelet analysis of Heisenberg group have been carried out in these years. The theory of admissible (continuous) wavelets on the Heisenberg group was established in [7]. Yang[8] gave a result on the existence of orthogonal (discrete) wavelets on Heisenberg group and showed that at most countably many solvable Lie groups are suitable for building multiresolution analysis. In [9], Peng constructed two kinds of orthogonal wavelets on Heisenberg group. Liu et al[10] studied the $L^p(H^s)$ convergence of cascade sequence on Heisenberg group in terms of the p-norm joint spectral radius and gave a sufficient condition for the $L^p(H^s)$ convergence of cascade sequence.

Just as on the Euclidean space, if one wants to enhance high frequency localization capability to signals on Heisenberg group, wavelet packets on Heisenberg group should be introduced. Then a natural question appears: how to measure/define the frequency localization capability of the wavelet packets on Heisenberg group? One natural way is the mean size of the L_p norm of wavelet packets. The purpose of this paper is to study the mean size of wavelet packets on Heisenberg group. We also give a formula for the mean size in terms of the p-norm spectral radius on Heisenberg group. These works can be considered as some generalizations of both results of [5] and [6]. In some sense, our results can also be seen extensions of the conclusions of [10]. To our best knowledge, the theory of wavelet packets on Heisenberg group was not considered before, so our results are new. We believe that the results obtained in this paper will provide some theoretic tools for the analysis on Heisenberg group.

Our general result will be stated for multivariate vector refinement equations on Heisenberg group. Here is the outline of this paper: in §2, we give some basic facts of Heisenberg group. The multivariate vector refinement equations on Heisenberg group and the subdivision tree on Heisenberg group are defined in this section. Then §3 is devoted to establishing some norm estimates of the subdivision tree. We give our main conclusions and their proofs in §4.

§2 Some facts about Heisenberg group and the subdivision tree

The Heisenberg group H^s is a Lie group with the underlying manifold $\mathbf{R}^s \times \mathbf{R}^s \times \mathbf{R}$ and the multiplication

$$(x, y, t)(x', y', t') := (x + x', y + y', t + t' + 2x'y - 2xy'),$$

where

$$x = (x_1, \cdots, x_s), x' = (x'_1, \cdots, x'_s), y = (y_1, \cdots, y_s), y' = (y'_1, \cdots, y'_s) \in \mathbf{R}^s,$$

and $t', t \in \mathbf{R}$.

The Lebesgue measure on $\mathbf{R}^s \times \mathbf{R}^s \times \mathbf{R}$ gives the bi-invariant Haar measure on H^s.

Let $\Gamma = \{(m,n,l) \in H^s : m,n \in \mathbf{Z}^s, l \in \mathbf{Z}\}$ be a discrete subgroup of H^s satisfying $\tau\Gamma \subset \Gamma$,where τ is the homogeneous dilation of Heisenberg group defined by $\tau(x,y,t) := (2x,2y,4t)$. Let E be a complete representative of distinct right cosets of $\Gamma/\tau(\Gamma)$. It is assumed that E contains 0. It is easy to show that

$$\Gamma/\tau(\Gamma) \cong E = \{\eta = (m,n,l) \in H^s : m_j, n_j = 0,1, j = 1,2,\cdots,d; l = 0,1,2,3\}$$

and $E = 2^Q$, where $Q = 2s+2$ is the homogeneous dimension of Heisenberg group.

The discrete subgroup Γ acts on H^s by

$$U_\gamma f(x,y,t) = U_{(m,n,l)} f(x,y,t) := f(\gamma^{-1}(x,y,t)) = f(x-m, y-n, t-l-2(nx-my)),$$

where $\gamma = (m,n,l) \in \Gamma$ and $f \in L^p(H^s)$.

The dilation τ acts on $L^p(H^s)$ by

$$\tau f(x,y,t) := f(\tau(x,y,t)) = f(2x,2y,4t), f \in L^p(H^s).$$

It is not difficult to verify that $\tau((x,y,t)(x',y',t')) = \tau(x,y,t)\tau(x',y',t')$. Also, it is obvious that the operator U_γ and τ are not commutative, but we have $U_\gamma \tau = \tau U_{\tau(\gamma)}$.

The homogeneous norm on Heisenberg group is defined as follows:

$$|(x,y,t)| := \max((|x|^2+|y|^2)^{1/2}, |t|^{1/2}), (x,y,t) \in H^s.$$

For $1 \le p \le \infty$, we denote by $(L_p(H^s))^r$ the linear space of all vectors $f = (f_1,\cdots,f_r)^T$ such that $||f||_p < \infty$, where

$$||f||_p := \left(\sum_{j=1}^r \int_{H^s} |f_j|^p dq\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty,$$

and $||f||_\infty$ is the essential supremum of $\max_{1\le j\le r} |f_j|$ on H^s. It is easy to show that when $1 \le p \le \infty$, $||\cdot||_p$ is a norm. Equipped with this norm, $(L_p(H^s))^r$ is a Banach space.

For $1 \le p \le \infty$, we denote by $\ell_p(\Gamma)$ the linear space of all sequences c for which $||c||_p < \infty$, the ℓ_p norm is defined by

$$||c||_p := \left(\sum_{\alpha\in\Gamma} |c(\alpha)|^p\right)^{1/p}, \quad 1 \le p < \infty,$$

and $||c||_\infty$ is the supremum of $|c|$ on H^s. It is easy to show that $||\cdot||_p$ is a norm.

It is well known that the stability concept is important in wavelet theory and approximation theory. We say that the shifts of compactly supported functions vector $f = (f_1,\cdots,f_r)^T \in (L_p(H^s))^r$ are to be L_p-stable if there exist two positive constants C_1 and C_2 such that, for arbitrary $b_1,\cdots,b_r \in \ell_p(\Gamma)$(Liu et al[11] defined the L_p-stability when $r=1$ in the name of Riesz sequence. Here we extend this concept to the vector case),

$$C_1(\sum_{j=1}^r ||b_j||_p^p)^{1/p} \le ||\sum_{j=1}^r \sum_{\alpha\in\Gamma} b_j(\alpha) f_j(\alpha^{-1}q)||_p \le C_2(\sum_{j=1}^r ||b_j||_p^p)^{1/p}.$$

The central equation in the multiresolution analysis on Heisenberg group is the following refinement equation:

$$\psi = \sum_{\gamma\in\Gamma} a(\gamma)\tau U_\gamma \psi.$$

The nonzero solution $\psi = (\psi_1,\cdots,\psi_r)^T \in (L_p(H^s))^r$ of this equation is referred to as (τ,a)

refinable function, $a = (a(\gamma))_{\gamma\in\Gamma}$ on Γ is called a refinement sequence and each $a(\gamma)$ is an $r \times r$ matrix. In this paper, we assume that the refinable function ψ is compactly supported and satisfies $\int_{H^s} \psi(q)dq = 1$. Also, we assume that the refinement sequence a is a finitely supported matrix sequence on Γ.

Now it is time to define the subdivision tree on Heisenberg group. Let $1 \le p \le \infty, s, r$, be positive integers, κ be a positive real number and $\psi = (\psi_1, \cdots, \psi_r)^T$ be a vector of functions in $(L_p(H^s))^r$ supported on $\{q \in H^s : |q| \le \kappa\}$. Denote by D the masks index set. Let $\mathcal{M} = \{a_\varepsilon : \varepsilon \in D\}$ be a finite set of sequences of $r \times r$ matrices supported on $\{\gamma \in \Gamma : |\gamma| \le \kappa\}$(For example, when $D = \{0, 1\}$, a_0 is the refinement mask and a_1 is the wavelet mask). Define the subdivision tree $\mathcal{T}(\mathcal{M}, \psi)$ associated with $\mathcal{M}$ and ψ to be an infinite tree of vectors of functions

$$\{\psi_{\varepsilon_1,\cdots,\varepsilon_n} : n \in \mathbf{N}, \varepsilon_1, \cdots, \varepsilon_n \in D\}, \tag{2.1}$$

where

$$\psi_{\varepsilon_1,\cdots,\varepsilon_n}(q) := \sum_{\alpha\in\Gamma} a_{\varepsilon_1}(\alpha)\psi_{\varepsilon_2,\cdots,\varepsilon_n}(\alpha^{-1}\tau(q)), \quad q \in H^s,$$

with $\psi_{\varepsilon_1,\cdots,\varepsilon_n}(x) := \psi$ when $n = 0$.

Our main conclusions in this paper are about the mean size of L_p-norms of a subdivision tree for $1 \le p \le \infty$. We want to investigate the limit

$$M_p(\mathcal{M}, \psi) := \lim_{n\to\infty} \{ \sum_{\varepsilon_1,\cdots,\varepsilon_n\in D} ||\psi_{\varepsilon_1,\cdots,\varepsilon_n}||_p^p \}^{\frac{1}{np}}. \tag{2.2}$$

§3 Norm estimates of subdivision tree on Heisenberg group

In this paper, we will show that the limit (2.2) always exists for $1 \le p \le \infty$ and equals the p-norm joint spectral radius of a finite set of operators. Now let us review the concept of p-norm joint spectral radius. Let $\mathcal{A}$ be a finite collection of linear operators on a finite dimensional vector space V. A vector norm $|| \cdot ||$ on V induces a norm on the linear operator A on V as follows:

$$||A|| := \max_{||v||=1} \{||Av||\}.$$

A subspace of V is said to be $\mathcal{A}$-invariant if it is invariant under every operator in $\mathcal{A}$. For $v \in V$, we call the intersection of all $\mathcal{A}$-invariant subspaces of V containing v the minimal $\mathcal{A}$ -invariant subspace generated by v, which we denote by $V(v)$. For a positive integer n we denote by $\mathcal{A}^n$ the Cartesian power of $\mathcal{A}$,

$$\mathcal{A}^n := \{(A_1, \cdots, A_n) : A_1, \cdots, A_n \in \mathcal{A}\}.$$

When $n = 0$, we interpret $\mathcal{A}^0$ as the set $\{I\}$, where I is the identity map on V.

Let

$$||\mathcal{A}^n|_{V(v)}||_\infty := \max\{||A_1 \cdots A_n|_{V(v)}|| : (A_1, \cdots, A_n) \in \mathcal{A}^n\},$$

where $A_1 \cdots A_n|_{V(v)}$ is the restriction of $A_1 \cdots A_n$ on the subspace $V(v)$.

Then the uniform joint spectral radius of $\mathcal{A}|_{V(v)}$ is defined by the following equality:

$$\rho_\infty(\mathcal{A}|_{V(v)}) := \lim_{n\to\infty} ||\mathcal{A}^n|_{V(v)}||_\infty^{1/n}. \tag{3.1}$$

It is well known that the uniform joint spectral radius was introduced by Rota and Strang in [12] and applied to the investigation of wavelets by many papers such as [13] and [14].

The p-norm joint spectral radius of $A|_{V(v)}$ is defined as

$$\rho_p(\mathcal{A}|_{V(v)}) := \lim_{n\to\infty} ||\mathcal{A}^n|_{V(v)}||_p^{1/n}, \tag{3.2}$$

where

$$||\mathcal{A}^n|_{V(v)}||_p := \left(\sum_{(A_1,\cdots,A_n)\in\mathcal{A}^n} ||A_1\cdots A_n|_{V(v)}||^p\right)^{1/p}, 1 \le p < \infty.$$

It is well known that this limit exists and equals the infimum:

$$\lim_{n\to\infty} ||\mathcal{A}^n|_{V(v)}||_p^{1/n} = \inf_{n\in\mathbf{N}} ||\mathcal{A}^n|_{V(v)}||_p^{1/n}. \tag{3.3}$$

It is easily seen that $\rho_p(\mathcal{A}|_{V(v)})$ is independent of the choice of the vector norm on V. The mean joint spectral radius ($p = 1$) was introduced by Wang in [15] for investigating the existence of L_1-solution of refinement equation on the Euclidean space. When $1 < p < \infty$, the p-norm joint spectral radius was introduced by Jia in [16]. This tool plays an important role in the investigation of existence and smoothness of L_p-solution of refinement equation on the Euclidean space (see [17] and [18]). Zhou[19] provided an efficient formula to compute the p-norm joint spectral radius in terms of the spectral radius of some finite matrix when p is an even integer.

For any $v \in V$, we denote, for $1 \le p < \infty$,

$$||\mathcal{A}^n v||_p := \left(\sum\nolimits_{(A_1,\cdots,A_n)\in\mathcal{A}^n} ||A_1\cdots A_n v||_p^p\right)^{1/p},$$

and, for $p = \infty$,

$$||\mathcal{A}^n v||_\infty := \max\{||A_1\cdots A_n v||_\infty : (A_1,\cdots,A_n)\in\mathcal{A}^n\}.$$

It follows from [20] that there exists a positive constant C such that

$$||\mathcal{A}^n|_{V(v)}||_p/C \le ||\mathcal{A}^n v||_p \le C||\mathcal{A}^n|_{V(v)}||_p, \quad n \in N, \quad 1 \le p \le \infty.$$

Therefore,

$$\rho_p(\mathcal{A}|_{V(v)}) = \lim_{n\to\infty} ||\mathcal{A}^n v||_p^{1/n}, \quad 1 \le p \le \infty.$$

In our application of joint spectral radius, we consider the space $V = (\ell_0(\Gamma))^r$, the space of all finitely supported sequences of $r \times 1$ vectors on Γ. Let E be a complete representative of distinct right cosets of $\Gamma/\tau(\Gamma)$. It is assumed that E contains 0. We denote by $(\ell_0(\Gamma))^{r\times r}$ the linear space of all finitely supported sequences of $r \times r$ matrices on Γ, on which the norm $c = (c_{jl}(\alpha))_{1\le j\le r, 1\le l\le r} \in (\ell_0(\Gamma))^{r\times r}$ is defined by

$$||c||_p := \left(\sum_{l=1}^{r}\sum_{j=1}^{r}\sum_{\alpha\in\Gamma} |c_{jl}(\alpha)|^p\right)^{1/p},$$

and $||c||_\infty$ is the supremum of $\max_{1\le j\le r, 1\le l\le r} |c_{jl}|$ on Γ. For $\varepsilon \in D, \eta \in E$ and $a_\varepsilon \in (\ell_0(\Gamma))^{r\times r}$, we define the linear operators A_ε^η on $(\ell_0(\Gamma))^r$ as

$$A_\varepsilon^{(\eta)} u(\alpha) := \sum_{\beta\in\mathbf{Z}^s} a_\varepsilon(\tau(\alpha)(\eta)\beta^{-1})u(\beta), \qquad \alpha \in \Gamma, u \in (\ell_0(\Gamma))^r. \tag{3.4}$$

Under these conditions, for any $u \in (\ell_0(\Gamma))^r$, the minimal $\mathcal{A}$-invariant subspace $V(u)$ is always finite dimensional(see [10,Lemma 4]).

The p-norm joint spectral radius of these linear operators will be used to estimate the norms concerning the subdivision sequences which appear in (2.1). For the set $\mathcal{M}$, we define the subdivision sequence$\{(a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha))_{\alpha\in\Gamma} : n \in \mathbf{N}, \varepsilon_1, \cdots, \varepsilon_n \in E\}$ as

$$a_{\varepsilon_1,\cdots\varepsilon_n}(\alpha) := \sum_{\beta\in\Gamma} a_{\varepsilon_1}(\beta) a_{\varepsilon_2,\cdots\varepsilon_n}((\tau^{n-1}(\beta))^{-1}\alpha), \quad \alpha \in \Gamma. \tag{3.5}$$

The subdivision sequence has an equivalent form as follows.

Lemma 3.1. Let $\mathcal{M} := \{a_\varepsilon : \varepsilon \in D\}$ be a finite set of finitely supported sequences of $r \times r$ matrices on Γ. Define the subdivision sequence by (3.5). Then for $n \in \mathbf{N}$ and $\varepsilon_1, \cdots, \varepsilon_n \in D$, we have

$$a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha) = \sum_{\beta\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_{n-1}}(\beta) a_{\varepsilon_n}((\tau(\beta))^{-1}\alpha). \tag{3.6}$$

Proof. The proof proceeds by induction on n. The case $n = 2$ follows from (3.5). Suppose $n > 2$ and the conclusion in Lemma 3.1 is true for n. We demonstrate that it is also true for $n + 1$. By (3.5) we have

$$a_{\varepsilon_1,\cdots,\varepsilon_{n+1}}(\alpha) = \sum_{\beta\in\Gamma} a_{\varepsilon_1}(\beta) a_{\varepsilon_2,\cdots,\varepsilon_{n+1}}((\tau^n(\beta))^{-1}\alpha).$$

It follows from induction hypothesis that

$$a_{\varepsilon_2,\cdots,\varepsilon_{n+1}}((\tau^n(\beta))^{-1}\alpha) = \sum_{\gamma\in\Gamma} a_{\varepsilon_2,\cdots,\varepsilon_n}(\gamma) a_{\varepsilon_{n+1}}((\tau(\gamma))^{-1}(\tau^n(\beta))^{-1}\alpha).$$

Therefore, we have

$$\begin{aligned} a_{\varepsilon_1,\cdots,\varepsilon_{n+1}}(\alpha) &= \sum_{\beta\in\Gamma} a_{\varepsilon_1}(\beta) \sum_{\gamma\in\Gamma} a_{\varepsilon_2,\cdots,\varepsilon_n}(\gamma) a_{\varepsilon_{n+1}}((\tau(\gamma))^{-1}(\tau^n(\beta))^{-1}\alpha) \\ &= \sum_{\eta\in\Gamma}\sum_{\beta\in\Gamma} a_{\varepsilon_1}(\beta) a_{\varepsilon_2,\cdots,\varepsilon_n}(\tau^{n-1}(\beta))^{-1}\eta) a_{\varepsilon_{n+1}}((\tau(\eta))^{-1}\alpha) \\ &= \sum_{\eta\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\eta) a_{\varepsilon_{n+1}}((\tau(\eta))^{-1}\alpha). \end{aligned}$$

This completes the proof of Lemma 3.1.

Remark 3.2. We remark that Lemma 3.1 has the corresponding result in the scalar case when $\mathcal{M}$ has only one element[10]. Nielsen and Zhou[5] gave similar result for the Euclidean space.

By induction on n and using Lemma 3.1, it is easy to show that the subdivision tree defined by (2.1) can be written as combinations of scaled shifts of ψ with subdivision sequence coefficients.

Lemma 3.3. Let $\mathcal{T}(\mathcal{M}, \psi)$ be a subdivision tree defined by (2.1). Define the subdivision sequence by (3.5). Then for $n \in \mathbf{N}$ and $\varepsilon_1, \cdots, \varepsilon_n \in D$, we have

$$\psi_{\varepsilon_1,\cdots\varepsilon_n}(q) = \sum_{\alpha\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(\alpha^{-1}\tau^n(q)). \tag{3.7}$$

Remark 3.4. Lemma 3.3 was established by Nielsen and Zhou in the Euclidean wavelet packets analysis[5]. This lemma is also the generalization of the result in [10].

To study the mean size of wavelet packets in L_p norm, one needs the relation between subdivision sequence and linear operator defined by (3.4).

Lemma 3.5. Let $\varepsilon_1, \cdots, \varepsilon_n \in D$ and $\alpha \in \Gamma$. If $\alpha = \tau^n(\gamma)\tau^{n-1}(\eta_1)\cdots\tau(\eta_{n-1})\eta_n$ with $\eta_1, \cdots, \eta_n \in E, \gamma \in \Gamma$. Then for $v \in (\ell_0(\Gamma))^r$, we have

$$a_{\varepsilon_1,\cdots,\varepsilon_n} * v(\alpha) = A_{\varepsilon_1}^{(\eta_1)} \cdots A_{\varepsilon_n}^{(\eta_n)} v(\gamma), \tag{3.8}$$

where $a_{\varepsilon_1,\cdots,\varepsilon_n} * v$ is the convolution given by

$$a_{\varepsilon_1,\cdots,\varepsilon_n} * v(\alpha) := \sum_{\beta\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\beta)v(\beta^{-1}\alpha), \quad \alpha \in \Gamma.$$

Proof. For $n = 1$ and $\alpha = \tau(\gamma)\eta_1$, it follows from (3.4) that

$$a_{\varepsilon_1} * v(\alpha) = \sum_{\beta\in\Gamma} a_{\varepsilon_1}(\beta)v(\beta^{-1}\alpha) = \sum_{\beta\in\Gamma} a_{\varepsilon_1}(\tau(\gamma)\eta_1\beta^{-1})v(\beta) = A^{(\eta_1)}_{\varepsilon_1} v(\gamma).$$

Suppose $n > 1$ and the above lemma has been verified for n. We now show that this lemma is also true for $n+1$. For $\alpha = \tau(\alpha_1)\eta_{n+1}$, where $\alpha_1 = \tau^n(\gamma)\tau^{n-1}(\eta_1 \cdots \tau(\eta_{n-1})\eta_n$, by Lemma 3.1 we have

$$\begin{aligned} a_{\varepsilon_1,\cdots,\varepsilon_{n+1}} * v(\alpha) &= \sum_{\beta\in\Gamma}\sum_{\gamma\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\gamma)a_{\varepsilon_{n+1}}(\tau(\gamma)^{-1}\tau(\alpha_1)\eta_{n+1}\beta^{-1})v(\beta) \\ &= \sum_{\beta\in\Gamma}\sum_{\gamma\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha_1\gamma^{-1})a_{\varepsilon_{n+1}}(\tau(\gamma)\eta_{n+1}\beta^{-1})v(\beta) \\ &= \sum_{\gamma\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha_1\gamma^{-1})(A^{(\eta_{n+1})}_{\varepsilon_{n+1}}v)(\gamma) \\ &= a_{\varepsilon_1,\cdots,\varepsilon_n} * (A^{(\eta_{n+1})}_{\varepsilon_{n+1}}v)(\alpha_1). \end{aligned}$$

Then by induction hypothesis we have

$$a_{\varepsilon_1,\cdots,\varepsilon_n} * (A^{(\eta_{n+1})}_{\varepsilon_{n+1}}v)(\alpha) = A^{(\eta_1)}_{\varepsilon_1} \cdots A^{(\eta_{n+1})}_{\varepsilon_{n+1}} v(\gamma).$$

This completes the proof of Lemma 3.5.

Remark 3.6. When the set $\mathcal{M}$ has only one element and ψ is a scalar, this lemma has appeared in [10]. For the Euclidean case, this lemma can be found in [5].

Here, we point out, that for each $v \in (\ell_0(\Gamma))^r$, the minimal $\mathcal{M} := \{a_\varepsilon : \varepsilon \in D\}$-invariant subspace generated by v is finite dimensional (see [10,Lemma 4]).

§4 Mean size formula of subdivision tree on Heisenberg group

With the above preparations we state our main results.

Theorem 4.1. Let $\{\psi_{\varepsilon_1,\cdots,\varepsilon_n} : n \in \mathbf{N}, \varepsilon_1,\cdots,\varepsilon_n \in D\}$ be defined by (2.1), and $1 \le p \le \infty$. Suppose $\mathcal{M} := \{a_\varepsilon : \varepsilon \in D\}$ is a finite set of the compactly supported sequences of $r \times r$ matrices on Γ and the shifts of ψ is L_p-stable. Then, we have

$$\begin{aligned} M_p(\mathcal{M}, \psi) &:= \lim_{n\to\infty}\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in D} ||\psi_{\varepsilon_1,\cdots,\varepsilon_n}||_p^p\}^{\frac{1}{np}} \\ &= m^{-\frac{1}{p}} \max_{1\le j\le d} \rho_p(\{A^{(\eta)}_\varepsilon|_{V(\delta e_j)} : \varepsilon \in D, \eta \in E\}), \end{aligned}$$

where e_j is the jth column of $r \times r$ identity matrix, for $\alpha \in \Gamma$, δe_j is the sequence given by $\delta e_j(\alpha) = \delta(\alpha)e_j$, and here δ is a sequence defined on Γ given by $\delta(0) = 1$ and $\delta = 0$ otherwise, and $m = 2^Q$, where $Q = 2s + 2$ is the homogeneous dimension of Heisenberg group.

Proof. By Lemma 3.3 we have

$$||\psi_{\varepsilon_1,\cdots,\varepsilon_n}||_p = ||\sum_{\alpha\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(\alpha^{-1}\tau^n(q))||_p$$

$$= m^{-\frac{n}{p}}||\sum_{\alpha\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(\alpha^{-1}q)||_p.$$

Since the shifts of $\psi_1, \cdots, \psi_r$ are of L_p-stability, there exist two positive constants C_1 and C_2 such that

$$C_1||a_{\varepsilon_1,\cdots,\varepsilon_n}||_p \le ||\sum_{\alpha\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(\alpha^{-1}q)||_p \le C_2||a_{\varepsilon_1,\cdots,\varepsilon_n}||_p.$$

By Lemma 3.5,

$$||a_{\varepsilon_1,\cdots,\varepsilon_n}||_p^p = \sum_{j=1}^{d}\sum_{\eta_1,\cdots,\eta_n\in E}||A_{\varepsilon_1}^{(\eta_1)}\cdots A_{\varepsilon_n}^{(\eta_n)}(\delta e_j)||_p^p.$$

So we obtain for $\varepsilon_1, \cdots, \varepsilon_n \in D$ that

$$C_1 m^{-\frac{n}{p}}\{\sum_{j=1}^{d}\sum_{\eta_1,\cdots,\eta_n\in E}||A_{\varepsilon_1}^{(\eta_1)}\cdots A_{\varepsilon_n}^{(\eta_n)}(\delta e_j)||_p^p\}^{\frac{1}{p}}$$
$$\le ||\psi_{\varepsilon_1,\cdots,\varepsilon_n}||_p \le C_2 m^{-\frac{n}{p}}\{\sum_{j=1}^{d}\sum_{\eta_1,\cdots,\eta_n\in E}||A_{\varepsilon_1}^{(\eta_1)}\cdots A_{\varepsilon_n}^{(\eta_n)}(\delta e_j)||_p^p\}^{\frac{1}{p}}.$$

This in connection with (3.2) and an elementary limit identity implies

$$\begin{aligned}
&\lim_{n\to\infty}\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in D}||\psi_{\varepsilon_1,\cdots,\varepsilon_n}||_p^p\}^{\frac{1}{np}}\\
= \;& m^{-\frac{1}{p}}\lim_{n\to\infty}\{\sum_{j=1}^{d}\sum_{\eta_1,\cdots,\eta_n\in E}\sum_{\varepsilon_1,\cdots,\varepsilon_n\in D}||A_{\varepsilon_1}^{(\eta_1)}\cdots A_{\varepsilon_n}^{(\eta_n)}(\delta e_j)||_p^p\}^{\frac{1}{np}}\\
= \;& m^{-\frac{1}{p}}\max_{1\le j\le d}\lim_{n\to\infty}\{\sum_{\eta_1,\cdots,\eta_n\in E}\sum_{\varepsilon_1,\cdots,\varepsilon_n\in D}||A_{\varepsilon_1}^{(\eta_1)}\cdots A_{\varepsilon_n}^{(\eta_n)}(\delta e_j)||_p^p\}^{\frac{1}{np}}\\
= \;& m^{-\frac{1}{p}}\max_{1\le j\le d}\rho_p(\{A_\varepsilon^{(\eta)}|_{V(\delta e_j)} : \varepsilon\in D, \eta\in E\}).
\end{aligned}$$

This verifies Theorem 4.1.

Remark 4.2. We extend the mean size formula of wavelet packets on Euclidean space in [5] to Heisenberg group. We also generalize the conclusion in [10] to the vector case.

In general, without assuming the L_p-stability of ψ, the above theorem fails to hold. However, if some weak assumption about ψ is made, a variant of Theorem 4.1 can still be retained. Specifically, the assumption is as follows: for $\psi = (\psi_1, \cdots, \psi_r)$, there exist $d \in \mathbf{N}$ and a vector of compactly supported functions $g = (g^1, \cdots, g^d)^T \in (L_p(H^s))^d$ such that the shifts of g are L_p-stable and

$$\psi = \sum_{\alpha\in\Gamma} b(\alpha)U_\alpha g, \tag{4.1}$$

where $b := (b(\alpha))_{\alpha\in\Gamma}$ is in $(\ell_0(\Gamma))^{r\times d}$, which is the linear space of $r \times d$ matrix sequences of compactly supported on Γ. Such a vector g is called a generator of the shift-invariant space

$$S(\psi) := \left\{\sum_{k=1}^{r}\sum_{\alpha\in\Gamma} a_k(\alpha)U_\alpha\psi_k : a_1, \cdots, a_r \in \ell(\Gamma)\right\}.$$

We denote by $G(\psi)$ the set of all generators of $S(\psi)$.

It should be pointed out that there is a lot of researches on generators in the wavelet analysis of Euclidean space. See [21] for detailed discussions.

Suppose that $g \in G(\psi)$ and (4.1) holds. Then

$$||\psi_{\varepsilon_1,\cdots,\varepsilon_n}||_p = ||\sum_{\alpha\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(\alpha^{-1}\tau^n(q))||_p = ||\sum_{\alpha\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n} * g(\alpha)(\alpha^{-1}\tau^n(q))||_p,$$

where $a_{\varepsilon_1,\cdots,\varepsilon_n} * g \in (\ell_0(\Gamma))^{r\times d}$ is the convolution given by

$$a_{\varepsilon_1,\cdots,\varepsilon_n} * g(\alpha) := \sum_{\beta\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\beta)b(\beta^{-1}\alpha), \alpha \in \Gamma.$$

It follows from the L_p- stability hypothesis of g that for all $n \in \mathbf{N}, \varepsilon_1, \cdots, \varepsilon_n \in D$,

$$C_1||a_{\varepsilon_1,\cdots,\varepsilon_n} * b||_p \le ||\sum_{\alpha\in\Gamma} a_{\varepsilon_1,\cdots,\varepsilon_n}(\alpha)\psi(\alpha^{-1}q)||_p \le C_2||a_{\varepsilon_1,\cdots,\varepsilon_n} * b||_p,$$

here C_1 and C_2 are positive constants.

Repeating the similar procedure of the proof of Theorem 4.1, we can get a variant of Theorem 4.1.

Theorem 4.3. Let $\{\psi_{\varepsilon_1,\cdots,\varepsilon_n} : n \in \mathbf{N}, \varepsilon_1, \cdots, \varepsilon_n \in D\}$ be defined by (2.1), and $1 \le p \le \infty$. Suppose $\mathcal{M} := \{a_\varepsilon : \varepsilon \in D\}$ is a finite set of the compactly supported sequences of $r \times r$ matrices on Γ and $g \in G(\psi)$. Then we have

$$\begin{aligned} M_p(\mathcal{M}, \psi) \quad &:= \quad \lim_{n\to\infty}\{\sum_{\varepsilon_1,\cdots,\varepsilon_n\in D} ||\psi_{\varepsilon_1,\cdots,\varepsilon_n}||_p^p\}^{\frac{1}{np}} \\ &= \quad m^{-\frac{1}{p}} \max_{1\le j\le d} \rho_p(\{A_\varepsilon^{(\eta)}|_{V(be_j)} : \varepsilon \in D, \eta \in E\}), \end{aligned}$$

where e_j is the jth column of $d \times d$ identity matrix, for $\alpha \in \Gamma$, be_j is the sequence given by $be_j(\alpha) = b(\alpha)e_j$, and b is an $r \times d$ matrix sequence given by (4.1) for $\psi = (\psi_1, \cdots, \psi_r)^T$, and $m = 2^Q$, where $Q = 2s + 2$ is the homogeneous dimension of Heisenberg group.

Remark 4.4. Nielsen and Zhou[5] gave a corresponding formula in the analysis of the subdivision tree on Euclidean space.

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