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LIE SYMMETRY ANALYSIS AND PAINLEVÉ ANALYSIS OF THE NEW (2+1)-DIMENSIONAL KdV EQUATION

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Abstract. Lie point symmetries associated with the new (2+1)-dimensional KdV equation $u_t + 3u_x u_y + u_{xxy} = 0$ are investigated. Some similarity reductions are derived by solving the corresponding characteristic equations. Painlevé analysis for this equation is also presented and the soliton solution is obtained directly from the Bäcklund transformation.

§1 Introduction

In the analysis of nonlinear evolution equations(NLEEs), a lot of powerful methods such as inverse scattering transformation, Bäcklund transformation, Hirota bilinear approach, Lie symmetry analysis, Painlevé analysis have been established. Lie point symmetry method (group method) was originally developed by Sophus Lie and is a highly algorithmic method. There have been considerable important developments which include Lie-Bäcklund symmetry, potential symmetry and so on in Lie symmetry methods[1−2] in recent years. Usually, by means of the Lie symmetry method, we can study the invariance, symmetry properties and similarity reductions of NLEEs. In [3], Weiss, Tabor and Carnevale (WTC) presented the Painlevé test for NLEEs directly. According to WTC method, a NLEE has Painlevé property if it's solutions are single-valued about a movable singularity manifold. Up to the present time, the developments about this method include Kruskal's simplified method[4], Conte's invariant method[5] and Lou's extended method[6]. To a NLEE, no matter whether it possesses Painlevé property, certain physically interesting solutions can be derived by using the truncated Painlevé expansion.

In this paper, we consider the following new (2+1)-dimensional KdV equation[7]

$$\Delta = u_t + 3u_x u_y + u_{xxy} = 0. \tag{1.1}$$

In §2, we discuss Lie symmetries, Lie algebra of symmetry vector fields and similarity reductions. In §3, we investigate the above equation that is not Painlevé integrable by means of the WTC-Painlevé analysis method and obtain a soliton solution from Bäcklund transformation. The last section is a short summary and discussion.

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§2 Lie symmetries and similarity reductions

Setting the one-parameter Lie group of infinitesimal transformations

$$\begin{aligned} x &\to X = x + \varepsilon X(x, y, t, u), \\ y &\to Y = y + \varepsilon Y(x, y, t, u), \\ t &\to T = t + \varepsilon T(x, y, t, u), \\ u &\to U = u + \varepsilon U(x, y, t, u) \end{aligned} \tag{2.1}$$

with a small parameter $\varepsilon \ll 1$ and the vector field associated with the above group of transformations

$$V = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u}, \tag{2.2}$$

we can obtain the corresponding 3-order prolongation vector field as follows

$$\mathrm{pr}^{(3)}V = V + \sum_{1 \le i+j+l \le 3} U^{x^i y^j t^l} \frac{\partial}{\partial_{u_{x^i y^j t^l}}}, \tag{2.3}$$

where $U^{x^i y^j t^l}$ are defined by

$$U^{x^i y^j t^l} = D_x U^{x^{i-1} y^j t^l} - (D_x X) u_{x^i y^j t^l} - (D_x Y) u_{x^{i-1} y^{j+1} t^l} - (D_x T) u_{x^{i-1} y^j t^{l+1}}, \tag{2.4}$$

$$= D_y U^{x^i y^{j-1} t^l} - (D_y X) u_{x^{i+1} y^{j-1} t^l} - (D_y Y) u_{x^i y^j t^l} - (D_y T) u_{x^i y^{j-1} t^{l+1}}, \tag{2.5}$$

$$= D_t U^{x^i y^j t^{l-1}} - (D_t X) u_{x^{i+1} y^j t^{l-1}} - (D_t Y) u_{x^i y^{j+1} t^{l-1}} - (D_t T) u_{x^i y^j t^l}. \tag{2.6}$$

The invariance of Eq.(1.1) under the infinitesimal transformations (2.1) needs

$$\mathrm{pr}^{(3)}V[\Delta]\Big|_{\Delta=0} = 0. \tag{2.7}$$

Substituting Eqs.(2.4)-(2.6) into the above equation and using Maple programm, the expressions for X, Y, T, U can be derived as

$$X = \frac{c_1 - c_3}{2}x + c_6 t + c_7, \tag{2.8}$$

$$Y = c_3 y + c_5 t + c_4, \tag{2.9}$$

$$T = c_1 t + c_2, \tag{2.10}$$

$$U = \frac{c_3 - c_1}{2}u + \frac{c_5}{3}x + \frac{c_6}{3}y + c_8. \tag{2.11}$$

Here we omit the redundant computational process for simplification. Associated with this Lie group, we have an 8-dimensional Lie algebra that can be represented by the generators

$$V_1 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}, \qquad V_2 = \frac{\partial}{\partial t}, \qquad V_3 = \frac{-x}{2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{u}{2}\frac{\partial}{\partial u},$$

$$V_4 = \frac{\partial}{\partial y}, \qquad V_5 = t\frac{\partial}{\partial y} + \frac{x}{3}\frac{\partial}{\partial u}, \qquad V_6 = t\frac{\partial}{\partial x} + \frac{y}{3}\frac{\partial}{\partial u}, \qquad V_7 = \frac{\partial}{\partial x},$$

$$V_8 = \frac{\partial}{\partial u}. \tag{2.12}$$

Thus, corresponding commutator table of $\{V_i, (i = 1, \cdots, 8)\}$ can be constructed (see Table 1).

Table 1 Commutator,$[V_i, V_j] = V_iV_j - V_jV_i$

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
V_1	0	$-V_2$	0	0	V_5	$\frac{V_6}{2}$	$\frac{-V_7}{2}$	$\frac{V_8}{2}$
V_2	V_2	0	0	0	V_4	V_7	0	0
V_3	0	0	0	$-V_4$	$-V_5$	$\frac{V_6}{2}$	$\frac{-V_7}{2}$	$\frac{-V_8}{2}$
V_4	0	0	V_4	0	0	$\frac{V_8}{3}$	0	0
V_5	$-V_5$	$-V_4$	V_5	0	0	0	$\frac{-V_8}{3}$	0
V_6	$\frac{-V_6}{2}$	$-V_7$	$\frac{-V_6}{2}$	$\frac{-V_8}{3}$	0	0	0	0
V_7	$\frac{V_7}{2}$	0	$\frac{V_7}{2}$	0	$\frac{V_8}{3}$	0	0	0
V_8	$\frac{-V_8}{2}$	0	$\frac{V_8}{2}$	0	0	0	0	0

In this paper, we don't pursue the further properties of this Lie algebra.

Theoretically, all of the similarity variables associated with Lie symmetries (2.8)-(2.11) can be derived by solving the following characteristic equation:

$$\frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}u}{U}. \tag{2.13}$$

Because of the complexity we only obtain certain special similarity reductions by selecting corresponding arbitrary constants. Some examples are listed as follows.

Case 1. $c_1 = c_3 = c_5 = c_6 = 0$

Similarity variables are

$$\xi = x - \frac{c_7}{c_2}t, \qquad \eta = y - \frac{c_4}{c_2}t, \qquad V(\xi,\eta) = u - \frac{c_8}{c_2}t. \tag{2.14}$$

Corresponding reduced equation is

$$\frac{-c_7}{c_2}V_\xi - \frac{c_4}{c_2}V_\eta + 3V_\xi V_\eta + V_{\xi\xi\eta} + \frac{c_8}{c_2} = 0. \tag{2.15}$$

Case 2. $c_1 \neq 0, c_4 \neq 0, c_3 = c_5 = c_6 = 0$

Similarity variables are

$$\xi = \left(\frac{c_1x}{2} + c_7\right)^2 (c_1t + c_7)^{-1}, \eta = \frac{c_1y}{c_4} - \ln|c_1t + c_2|, V(\xi,\eta) = \left(\frac{-c_1u}{2} + c_8\right)(c_1t + c_2)^{\frac{1}{2}}. \tag{2.16}$$

Corresponding reduced equation is

$$12\sqrt{\xi}V_\xi V_\eta + c_4V + \frac{c_4}{2}V_\eta + 2c_4\xi V_\xi - c_1^2V_{\xi\eta} - 2c_1^2\xi V_{\xi\xi\eta} = 0. \tag{2.17}$$

Case 3. $c_1 = c_3 \neq 0, c_6 \neq 0, c_5 = 0.$

Similarity variables are

$$\xi = x - \frac{c_6}{c_3}t - \frac{c_3c_7 - c_2c_6}{c_3^2}\ln|c_3t + c_2|, \qquad \eta = (c_3y + c_4)(c_3t + c_2)^{-1},$$

$$V(\xi,\eta)=u-\frac{c_6}{3c_3}\eta t-\frac{3c_8c_3-c_6c_4}{3c_3^2}\ln|c_3t+c_2|. \tag{2.18}$$

Corresponding reduced equation is

$$3c_8-\frac{c_4c_6}{c_3}+\frac{c_2c_6}{c_3}\eta-3c_7V_\xi-3c_3\eta V_\eta+9c_3V_\xi V_\eta+3c_3V_{\xi\xi\eta}=0. \tag{2.19}$$

Case 4. $c_1\neq c_3, c_1\neq 0, c_3\neq 0, c_5=c_6=0$

Similarity variables are

$$\xi=\left(\frac{c_1-c_3}{2}x+c_7\right)(c_1t+c_2)^{\frac{c_3-c_1}{2c_1}},\qquad \eta=(c_3y+c_4)(c_1t+c_2)^{\frac{-c_3}{c_1}},$$

$$V(\xi,\eta)=\left(\frac{c_3-c_1}{2}u+c_8\right)(c_1t+c_2)^{\frac{c_1-c_3}{2c_1}}. \tag{2.20}$$

Corresponding reduced equation is

$$2(c_1-c_3)V+4c_3\eta V_\eta+2(c_1-c_3)\xi V_\xi+12c_3V_\xi V_\eta-(c_1-c_3)^2c_3V_{\xi\xi\eta}=0. \tag{2.21}$$

§3 Painlevé analysis and soltion solution

For a NLEE, its Painlevé test has been performed by Weiss, Tabor and Carnevale[3]. According to this theory, we may expand the field u about a singularity manifold $\phi(x,y,t)=0$ as

$$u=\sum_{i=0}^{\infty}u_i(x,y,t)\phi^{i+\alpha}(x,y,t), \tag{3.1}$$

where the constant α should be a negative integer and there should be N arbitrary functions in $u_i(x,y,t)$ and $\phi(x,y,t)$ of every branch if it is a N-order NLEE. Inserting $u\propto u_0(x,y,t)\phi^\alpha(x,y,t)$ we can find that $\alpha=-1$, $u_0(x,y,t)=2\phi_x(x,y,t)$ and Eq.(3.1) is modified to

$$u=\frac{2\phi_x(x,y,t)}{\phi(x,y,t)}+\sum_{i=1}^{\infty}u_i(x,y,t)\phi^{i-1}(x,y,t). \tag{3.2}$$

Then, substituting the above equation into KdV equaiton (1.1), we have

$$(i+1)(i-1)(i-6)\phi_y\phi_x^2u_i=F(u_{i-1},\cdots,u_0,\phi_t,\phi_x,\phi_{xx},\cdots), \tag{3.3}$$

where we set $u_i(x,y,t)\equiv u_i, \phi(x,y,t)=\phi$ for simplification. This means that the resonances are given by $-1,1$ and 6. From Eq.(3.3), the following equations can be found.

$i=0,\qquad u_0=2\phi_x, \qquad (3.4)$

$i=1,\qquad$ the compatibility condition is satisfied identically,
this means u_1 is an arbitrary function

$i=2,\qquad \phi_t\phi_x+6u_2\phi_y\phi_x^2+3u_{1y}\phi_x^2+3\phi_y\phi_xu_{1x}-3\phi_{xy}\phi_{xx}+3\phi_x\phi_{xxy}+\phi_y\phi_{xxx}=0, \quad (3.5)$

$$
\begin{aligned}
i=3, \qquad & -12u_3\phi_y\phi_x^2-3u_{2y}\phi_x^2-3\phi_y\phi_xu_{2x}+\phi_{xt}+3u_2\phi_x\phi_{xy}+3u_{1x}\phi_{xy}+3u_2\phi_y\phi_{xx}\\
& +3u_{1y}\phi_{xx}+\phi_{xxxy}=0, \qquad (3.6)\\
i=4, \qquad & u_{1t}+3u_2^2\phi_y\phi_x-30u_4\phi_y\phi_x^2-4u_{3y}\phi_x^2+3u_{1y}u_{1x}-2\phi_y\phi_xu_{3x}+16u_3\phi_x\phi_{xy}\\
& +8u_{2x}\phi_{xy}+2\phi_xu_{2xy}+14u_3\phi_y\phi_{xx}+7u_{2y}\phi_{xx}+\phi_yu_{2xx}+u_2\phi_t+3u_2u_{1y}\phi_x\\
& +3u_2\phi_yu_{1x}+u_2\phi_{xxy}+u_{1xxy}=0, \qquad (3.7)\\
i=5, \qquad & u_{2t}+3u_2u_{2y}\phi_x-24u_5\phi_y\phi_x^2+3u_{2y}u_{1x}+3u_2\phi_yu_{2x}+3u_{1y}u_{2x}+6\phi_y\phi_xu_{4x}\\
& +30u_4\phi_x\phi_{xy}+10u_{3x}\phi_{xy}+4\phi_xu_{3xy}+24u_4\phi_y\phi_{xx}+8u_{3y}\phi_{xx}+2\phi_yu_{3xx}\\
& +2u_3\phi_t+12u_3u_2\phi_y\phi_x+6u_3u_{1y}\phi_x+6u_3\phi_yu_{1x}+2u_3\phi_{xxy}+u_{2xxy}=0, \qquad (3.8)\\
i=6, \qquad & u_{3t}+12u_3^2\phi_y\phi_x+6u_3u_{2y}\phi_x+3u_2u_{3y}\phi_x+6u_{5y}\phi_x^2+3u_{3y}u_{1x}+6u_3\phi_yu_{2x}\\
& +3u_{2y}u_{2x}+3u_2\phi_yu_{3x}+3u_{1y}u_{3x}+18\phi_y\phi_xu_{5x}+48u_5\phi_x\phi_{xy}+12u_{4x}\phi_{xy}\\
& +6\phi_xu_{4xy}+36u_5\phi_y\phi_{xx}+9u_{4y}\phi_{xx}+3\phi_yu_{4xx}+3u_4\phi_t+18u_4u_2\phi_y\phi_x\\
& +9u_4u_{1y}\phi_x+9u_4\phi_yu_{1x}+3u_4\phi_{xxy}+u_{3xxy}=0. \qquad (3.9)
\end{aligned}
$$

By using Maple programm and Kruskal's simplified ansatz[4] $\phi=x+\psi(y,t)$, we can obtain $u_i,(i=2,3,4,5)$ in turn from Eqs.(3.5)-(3.8) and substituting them into Eq.(3.9) yield

$$
\begin{aligned}
& \psi_{yt}^2-\psi_{tt}\psi_{yy}-3\psi_{yy}u_{1yt}+3\psi_{yt}u_{1yy}+3\psi_y\psi_{yy}u_{1xt}-6\psi_y\psi_{yt}u_{1xy}\\
& +3\psi_t\psi_{yy}u_{1xy}+3\psi_y^2\psi_{yt}u_{1xy}-3\psi_t\psi_y\psi_{yy}u_{1xx}=0. \qquad (3.10)
\end{aligned}
$$

It means that ψ, u_1 are not arbitrary, namely, this new (2+1)-dimensional KdV equation (1.1) has not Painlevé property.

Despite this equation does't possess Painlevé property, certain physically interesting solutions can be derived from corresponding Bäcklund transformation,

$$u=2\big(\ln\phi\big)_x+u_1, \qquad (3.11)$$

$$u_{1t}+3u_{1x}u_{1y}+u_{1xxy}=0, \qquad (3.12)$$

$$\phi_t\phi_x+3u_{1y}\phi_x^2+3\phi_y\phi_xu_{1x}-3\phi_{xy}\phi_{xx}+3\phi_x\phi_{xxy}+\phi_y\phi_{xxx}=0, \qquad (3.13)$$

$$\phi_{xt}+3u_{1x}\phi_{xy}+3u_{1y}\phi_{xx}+\phi_{xxxy}=0, \qquad (3.14)$$

which is obtained by using the truncated Painlevé expansion (3.2) at the constant level term. If setting $u_1=a$ and $\phi=1+\mathrm{e}^{kx+ly-k^2lt}$, the following soliton solution

$$u=\frac{2k\mathrm{e}^{kx+ly-k^2lt}}{1+\mathrm{e}^{kx+ly-k^2lt}}+a=a+k-k\tanh\left[\frac{-kx-ly+k^2lt}{2}\right] \qquad (3.15)$$

can be obtained from Eq.(3.11).

§4 Summary and discussion

In this paper we obtain the following two theorems.

Theorem 1.

The new (2+1)-dimensional KdV equation(1.1) has Lie point symmetries (2.8)-(2.11) and the generators construct an 8-dimensional Lie algebra.

Theorem 2.

The new (2+1)-dimensional KdV equation(1.1) has no Painlevé property in WTC meaning. An auto-Bäcklund transformation (3.11)-(3.14) is obtained by using the truncated Painlevé expansion (3.2) at the constant level term.

In addition, similarity reductions and those embedded properties which may be used in other methods [8−9] are worthy of studying further. For instance, if setting similarity variables as $\xi = ct, \eta = -x + cy, V(\xi, \eta) = u$, we have a similarity reduction equation

$$V_\xi - 3V_\eta^2 + V_{\eta\eta\eta} = 0. \tag{4.1}$$

Thus, from its nonlinear superposition principle, another new solution

$$u = \frac{6}{2\coth[2(-x+cy) - 32ct] - \tanh[-x+cy-4ct]} \tag{4.2}$$

can be obtained.

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