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EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS FOR NONLINEAR EVOLUTION EQUATION OF FOURTH ORDER

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Abstract. The existence and uniqueness of classical global solutions and the nonexistence of global solutions to the first boundary value problem and the second boundary value problem for the equation $u_{tt}-a_1u_{xx}-a_2u_{xxt}-a_3u_{xxtt}=\varphi(u_x)_x$ are proved.

§ 1 Introduction

In the study of the nonlinear waves in elastic rods, there arises the nonlinear hyperbolic equation of fourth order

$$u_{tt}-C_0^2[1+na_nu_x^{n-1}]u_{xx}-\beta u_{xxtt}=\gamma u_{xxt}, \tag{1.1}$$

where $C_0^2,\gamma>0,\beta>0$ and $a_n\neq 0$ are constants, and n is a natural number (see[1]). In [1] the author simplifies Eq. (1. 1) into the generalized Korteweg-de Vries-Burgers equation and the existence of the soliton wave to it is considered, but about the Eq. (1. 1) there hasn't been any discussion. In [2] the authors have proved the existence and uniqueness of the local classical solutions for the initial value problem and the first boundary value problem of Eq(1. 1).

In this paper, we are going to consider the following first boundary value problem

$$u_{tt}-a_1u_{xx}-a_2u_{xxt}-a_3u_{xxtt}=\varphi(u_x)_x, \tag{1.2}$$

$$u(0,t)=u(1,t)=0,\quad t\geqslant 0, \tag{1.3}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in[0,1], \tag{1.4}$$

or the second boundary value problem for Eq. (1. 2)

$$u_x(0,t)=u_x(1,t)=0,\qquad t\geqslant 0, \tag{1.5}$$

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in[0,1], \tag{1.6}$$

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where $u(x,t)$ is an unknown function, $a_i>0(i=1,2,3)$ are constants, $\varphi(s)$ is a given nonlinear function, $u_0(x)$ and $u_1(x)$ are given initial functions. The existence and uniqueness of classical global solutions and nonexistence of global solutions to the first boundary value problem (1.2)—(1.4) and the second boundary value problem (1.2), (1.5), (1.6) are proved.

§ 2 Existence and Uniqueness of Local Solutions for the Problem (1.2)—(1.4)

Let $K(x,\xi)$ be the Green's function[3] of the boundary value problem for the ordinary differential equation

$$y - a_3 y'' = 0, \qquad y(0) = y(1) = 0,$$

where $a_3>0$ is a real unmber, i.e.

$$K(x,\xi) = \frac{1}{\sqrt{a_3}\sinh\frac{1}{\sqrt{a_3}}}\begin{cases}\sinh\left[\frac{x}{\sqrt{a_3}}\right]\sinh\left[\frac{1-\xi}{\sqrt{a_3}}\right], & 0\leqslant x\leqslant \xi,\\ \sinh\left[\frac{1-x}{\sqrt{a_3}}\right]\sinh\left[\frac{\xi}{\sqrt{a_3}}\right], & \xi\leqslant x\leqslant 1.\end{cases} \tag{2.1}$$

Suppose that $u_0(x)$ and $u_1(x)$ are appropriately smooth and satisfy the boundary condition (1.3), $u(x,t)$ is the classical solution of the problem (1.2)—(1.4), then the solution $u(x,t)$ of Eq. (1.2) satisfying the condition (1.3) satisfies the integral equation

$$\begin{aligned}u_{tt}(x,t) = & a_1\int_0^1 K(x,\xi)u_{\xi\xi}(\xi,t)d\xi + a_2\int_0^1 K(x,\xi)u_{\xi\xi t}(\xi,t)d\xi + \\ & \int_0^1 K(x,\xi)\varphi(u_\xi(\xi,t))_\xi d\xi.\end{aligned} \tag{2.2}$$

Hence the classical solution of the problem (1.2)—(1.4) should satisfy the integral equation

$$\begin{aligned}u(x,t) = & u_0(x) + u_1(x)t - a_2\Big[\int_0^1 K(x,\xi)u_{0\xi\xi}(\xi)d\xi\Big]t + \\ & a_1\int_0^t\int_0^1 (t-\tau)K(x,\xi)u_{\xi\xi}(\xi,\tau)d\xi d\tau + a_2\int_0^t\int_0^1 K(x,\xi)u_{\xi\xi}(\xi,\tau)d\xi d\tau + \\ & \int_0^t\int_0^1 (t-\tau)K(x,\xi)\varphi(u_\xi(\xi,\tau))_\xi d\xi d\tau.\end{aligned} \tag{2.3}$$

By use of the properties of Green's funtion $K(x,\xi)$, it is easy to prove the following lemma.

Lemma 2.1. Suppose that $u_0(x), u_1(x) \in C^2[0,1]$, $u_0(0)=u_0(1)=u_1(0)=u_1(1)=0$, $\varphi(s) \in C^1(\mathbf{R})$, and $u(x,t)\in C([0,T];C^2[0,1])$ is the solution of (2.3), then $u(x,t)$ must be the classical solution of problem (1.2)—(1.4).

In the following we are going to prove the existence of local classical solution for the integral equation (2.3) by the contraction mapping principle. For this purpose, we define the function space

$$X(T) = \{u(x,t)\,|\,u \in C([0,T];C^2[0,1]),\quad u(0,t) = u(1,t) = 0\},$$

equipped with the norm defined by

$$\| u \|_{X(T)} = \max_{0\leqslant t\leqslant T}\Big[\sum_{i=1}^{2} \max_{0\leqslant x\leqslant 1} |u_{x^i}(\cdot,t)|\Big] = \| u \|_{C([0,T];C^2[0,1])}, \qquad \forall\, u \in X(T).$$

It is easy to see that $X(T)$ is a Banach space.

Let $U = \| u_{0x} \|_{C^1[0,1]} + \| u_{1x} \|_{C^1[0,1]}$ and define the set

$$P(U,T) = \{u\,|\,u \in X(T),\ \| u \|_{X(T)} \leqslant 2U\}.$$

Obviously, $P(U,T)$ is the nonempty bounded closed convex set for each $U,T>0$. We define the map S as follows:

$$\begin{aligned} Sw = {} & u_0(x) + u_1(x)t - a_2\Big[\int_0^1 K(x,\xi)u_{0\xi\xi}(\xi)\mathrm{d}\xi\Big]t + \\ & a_1\int_0^t\int_0^1 (t-\tau)K(x,\xi)w_{\xi\xi}(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau + a_2\int_0^t\int_0^1 K(x,\xi)w_{\xi\xi}(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau + \\ & \int_0^t\int_0^1 (t-\tau)K(x,\xi)\varphi(w_\xi(\xi,\tau))_\xi\mathrm{d}\xi\mathrm{d}\tau, \qquad \forall\, w \in P(U,T). \end{aligned} \tag{2.4}$$

Obviously, S maps $X(T)$ into $X(T)$. It is easy to prove

Lemma 2. 2. Suppose that $u_0, u_1 \in C^2[0,1], u_0(0) = u_0(1) = u_1(0) = u_1(1) = 0$ and $\varphi(s) \in C^2(\mathbf{R})$, then S maps $P(U,T)$ into $P(U,T)$ and $S: P(U,T) \to P(U,T)$ is strictly contractive if T is appropriately small relative to U.

By using the contraction mapping principle and the extension of the solution (see [4]), it is easy to prove

Theorem 2. 1. Let the assumptions of Lemma 2. 2 hold, then the integral equation (2. 3) has a unique solution $u(x,t) \in C([0,T_0);C^2[0,1])$, where $[0,T_0)$ is a maximal time interval. Moreover, if

$$\sup_{t\in[0,T_0)} (\| u_x \|_{C^1[0,1]} + \| u_{xt} \|_{C^1[0,1]}) < \infty, \tag{2.5}$$

then $T_0 = \infty$.

§ 3 The Classical Global Solution of the Problem (1. 2)—(1. 4)

Lemma 3. 1. Suppose that $u_0(x), u_1(x) \in H^1[0,1]$ satisfy the boundary condition (1. 3), $\varphi(s) \in C^1(\mathbf{R})$, then the classical solution of the problem (1. 2)—(1. 4) satisfies the following identity:

$$\begin{aligned} E(t) = {} & \| u_t \|^2_{L_2[0,1]} + a_1 \| u_x \|^2_{L_2[0,1]} + a_3 \| u_{xt} \|^2_{L_2[0,1]} + \\ & 2a_2\int_0^t\int_0^1 u_{x\tau}^2\mathrm{d}x\mathrm{d}\tau + 2\int_0^1\int_0^{u_x}\varphi(s)\mathrm{d}s\mathrm{d}x = \\ & \| u_1 \|^2_{L_2[0,1]} + a_1 \| u_{0x} \|^2_{L_2[0,1]} + a_3 \| u_{1x} \|^2_{L_2[0,1]} + 2\int_0^1\int_0^{u_{0x}}\varphi(s)\mathrm{d}s\mathrm{d}x \equiv \\ & E(0), \qquad \forall\, t \in [0,T]. \end{aligned} \tag{3.1}$$

Proof. Multipling both sides of Eq. (1. 2) by u_t, integrating the product over $[0,1]\times[0,t]$

and integrating by parts with respect to x, we can obtain (3. 1). The lemma thus is proved.

Theorem 3. 1. Suppose that the conditions of Theorem 2. 1 and the following condition

$$|\varphi(s)| \leqslant A\int_0^s \varphi(y)\mathrm{d}y + B \tag{3.2}$$

hold, where A and B are positive constants, then the problem (1. 2)—(1. 4) has a unique classical global solution $u(x,t)$.

Remark 3. 1. The function $\varphi(s)$ satisfying (3. 2) exists. For example, $\varphi(s)=e^s$ satisfies Inequality (3. 2). When $r>0$ is a real number and n is an odd number, $\varphi(s)=rs^n$ which satisfies Inequality (3. 2) is the second example.

Proof of Theorem 3. 1. By virtue of Theorem 2. 1, we are only required to prove that (2. 5) holds. Let $T\in(0,T_0)$ be given. Integrating by parts in (2. 3), we obtain

$$\begin{aligned} u(x,t) =& u_0(x) + u_1(x)t - a_2\Big[\int_0^1 K(x,\xi)u_{0\xi\xi}(\xi)\mathrm{d}\xi\Big]t - \\ & a_1\int_0^t\int_0^1 (t-\tau)K_\xi(x,\xi)u_\xi(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau - a_2\int_0^t\int_0^1 K_\xi(x,\xi)u_\xi(\xi,\tau)\mathrm{d}\xi\mathrm{d}\tau - \\ & \int_0^t\int_0^1 (t-\tau)K_\xi(x,\xi)\varphi(u_\xi(\xi,\tau))\mathrm{d}\xi\mathrm{d}\tau, \qquad x\in[0,1], t\in[0,T]. \end{aligned} \tag{3.3}$$

It follows from (3. 3) that

$$\begin{aligned} u_{xtt}(x,t) =& -\frac{a_1}{a_3}u_x(x,t) - a_1\int_0^1 K_{\xi x}(x,\xi)u_\xi(\xi,t)\mathrm{d}\xi - \\ & \frac{a_2}{a_3}u_{xt}(x,t) - a_2\int_0^1 K_{\xi x}(x,\xi)u_{\xi,t}(\xi,t)\mathrm{d}\xi - \\ & \frac{1}{a_3}\varphi(u_x(x,t)) - \int_0^1 K_{\xi x}(x,\xi)\varphi(u_\xi(\xi,t))\mathrm{d}\xi, \qquad x\in[0,1], t\in[0,T]. \end{aligned} \tag{3.4}$$

Multipling both sides of (3. 4) by $2u_{xt}$, we get

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\Big[u_{xt}^2 + \frac{a_1}{a_3}u_x^2 + \frac{2a_2}{a_3}\int_0^t u_{x\tau}^2(x,\tau)\mathrm{d}\tau + \frac{2}{a_3}\int_0^{u_x}\varphi(s)\mathrm{d}s\Big] = \\ & 2\Big[-a_1\int_0^1 K_{\xi x}(x,\xi)u_\xi(\xi,t)\mathrm{d}\xi - a_2\int_0^1 K_{\xi x}(x,\xi)u_{\xi t}(\xi,t)\mathrm{d}\xi - \\ & \int_0^1 K_{\xi x}(x,\xi)\varphi(u_\xi(\xi,t))\mathrm{d}\xi\Big]u_{xt}. \qquad x\in[0,1], t\in[0,T]. \end{aligned} \tag{3.5}$$

Let us denote $u_{1x}^2(x)+\frac{a_1}{a_3}u_{0x}^2+\frac{2}{a_3}\left|\int_0^{u_{0x}(x)}\varphi(s)\mathrm{d}s\right|$ by $E_1(x)$. Integrating both sides of (3. 5) with respect to t and making use of Conditions (3. 2) and (3. 1), we can obtain

$$\begin{aligned} & u_{xt}^2 + \frac{a_1}{a_3}u_x^2 + \frac{2a_2}{a_3}\int_0^t u_{x\tau}^2(x,\tau)\mathrm{d}\tau + \frac{2}{a_3}\int_0^{u_x}\varphi(s)\mathrm{d}s \leqslant \\ & E_1(x) + \int_0^t\{[\int_0^1 4a_1K_{\xi x}^2(x,\xi)\mathrm{d}\xi]^{\frac12}[\int_0^1 a_1u_\xi^2(\xi,\tau)\mathrm{d}\xi]^{\frac12} + \\ & [\int_0^1 \frac{4a_2^2}{a_3}K_{\xi x}^2(x,\xi)\mathrm{d}\xi]^{\frac12}[\int_0^1 a_3u_{\xi\tau}^2(\xi,\tau)\mathrm{d}\xi]^{\frac12} + C_1\int_0^1|\varphi(u_\xi(\xi,\tau))|\mathrm{d}\xi\}|u_{x\tau}|\mathrm{d}\tau \leqslant \end{aligned}$$

$$E_1(x) + \int_0^t \{C_2 + a_1\int_0^1 u_x^2(x,\tau)\mathrm{d}x + a_3\int_0^1 u_{x\tau}^2(x,\tau)\mathrm{d}x +$$

$$C_1\int_0^1 [A\int_0^{u_x(x,\tau)} \varphi(s)\mathrm{d}s + B]\mathrm{d}x\} |u_{x\tau}|\mathrm{d}\tau \leqslant$$

$$E_1(x) + \int_0^t \{C_3 + C_4E(0)\} |u_{x\tau}|\mathrm{d}\tau \leqslant$$

$$E_1(x) + \frac{1}{4}[C_3 + C_4E(0)]^2T + \int_0^t u_{x\tau}^2\mathrm{d}\tau, x \in [0,1], t \in [0,T]. \tag{3.6}$$

Multipling both sides of (3.6) by A, adding the product to $\frac{2B}{a_3}$ and using (3.2), we get

$$Au_{xt}^2 + \frac{Aa_1}{a_3}u_x^2 + \frac{2Aa_2}{a_3}\int_0^t u_{x\tau}^2\mathrm{d}\tau + \frac{2}{a_3}|\varphi(u_x)| \leqslant$$

$$M_1(T) + A\int_0^t u_{x\tau}^2\mathrm{d}\tau, \qquad x \in [0,1], t \in [0,T]. \tag{3.7}$$

It follows from (3.7) by Gronwall's inequality that

$$Au_{xt}^2 + \frac{Aa_1}{a_3}u_x^2 + \frac{2Aa_2}{a_3}\int_0^t u_{x\tau}^2\mathrm{d}\tau + \frac{2}{a_3}|\varphi(u_x)| \leqslant M_1(T)\mathrm{e}^{AT}, \quad x \in [0,1], t \in [0,T].$$

Therefore

$$\sup_{0\leqslant t\leqslant T} \| u_x \|_{C[0,1]}, \sup_{0\leqslant t\leqslant T} \| u_{xt} \|_{C[0,1]}, \quad \sup_{0\leqslant t\leqslant T} \| \varphi(u_x) \|_{C[0,1]}, \quad \int_0^t u_{xt}^2\mathrm{d}t \leqslant M_2(T), \tag{3.8}$$

where $M_2(T)$ is a constant dependent on T. Differentiating (3.3) with respect to x twice, we obtain the estimate

$$|u_{xx}(x,t)| \leqslant M_3(T) + M_4(T)\int_0^t |u_{xx}(x,\tau)|\mathrm{d}\tau, \quad x \in [0,1], t \in [0,T].$$

Making use of Gronwall's inequality, we have

$$\sup_{0\leqslant t\leqslant T} \| u_{xx}(\cdot,t) \|_{C[0,1]} \leqslant M_5(T), \tag{3.9}$$

where $M_5(T)$ is a constant dependent on T. Similarly, we can obtain

$$\sup_{0\leqslant t\leqslant T} \| u_{xxt}(\cdot,t) \|_{C[0,1]} \leqslant M_6(T), \tag{3.10}$$

where $M_6(T)$ is a constant dependent on T. From (3.8)—(3.10) it follows that

$$\sup_{t\in[0,T_0)} (\| u_x \|_{C^1[0,1]} + \| u_{xt} \|_{C^1[0,1]}) < \infty.$$

By virtue of Theorem 2.1 and Lemma 2.1 we know that the problem (1.2)—(1.4) has a unique classical global solution $u(x,t)$. The theorem is proved.

§ 4 Nonexistence of Global Solution of the Problem (1.2)—(1.4)

Theorem 4.1. Suppose that the following conditions hold:

(1) $E(0)\leqslant 0$,

(2) $\varphi(s)\in C^1(\mathbf{R}), \varphi(s)s\leqslant 2(2\sigma+1)\int_0^s \varphi(y)\mathrm{d}y+2\sigma a_1 s^2$,

(3) $2\sigma\int_0^1 (u_0u_1+a_3u_{0x}u_{1x})\mathrm{d}x-a_2\int_0^1 u_{0x}^2\mathrm{d}x>0$,

where $\sigma>0$ is a constant, then the classical solutions of the problem (1.2)—(1.4) will blow up in finite time.

Proof. The proof is made by use of so called "concavity" arguments. Assume that $u(x,t)$ is the classical solution of the problem (1.2)—(1.4) on $[0,1]\times[0,T]$. Set

$$F(t)=\int_0^1(u^2+a_3u_x^2)\mathrm{d}x+\int_0^1\int_0^t a_2u_x^2\mathrm{d}x\mathrm{d}t+(T-t)\int_0^1 a_2u_{0x}^2\mathrm{d}x. \tag{4.1}$$

Using Cauchy's inequality and Eq. (1.2), it follows from (4.1) that

$$FF''-(1+\sigma)(F')^2\geqslant F[-2(2\sigma+1)\int_0^1(u_t^2+a_3u_{xt}^2+2a_2\int_0^t u_{xt}^2\mathrm{d}t)\mathrm{d}x-$$
$$2\int_0^1\varphi(u_x)u_x\mathrm{d}x-2a_1\int_0^1 u_x^2\mathrm{d}x+4\sigma a_2\int_0^1\int_0^t u_{xt}^2\mathrm{d}x\mathrm{d}t]. \tag{4.2}$$

From (4.2), (3.1) and assumptions (1), (2) it follows that

$$FF''-(1+\sigma)(F')^2\geqslant F\{-2(2\sigma+1)[E(0)-2\int_0^1\int_0^{u_x}\varphi(s)\mathrm{d}s-a_1\int_0^1 u_x^2\mathrm{d}x]-$$
$$2\int_0^1\varphi(u_x)u_x\mathrm{d}x-2a_1\int_0^1 u_x^2\mathrm{d}x+4\sigma a_2\int_0^1\int_0^t u_{xt}^2\mathrm{d}x\mathrm{d}t\geqslant$$
$$-2(2\sigma+1)FE(0)\geqslant 0,\qquad t\in[0,T].$$

We see that $F(t)>0$ for all $t\in[0,T]$ and that from Assumption (3) above $F'(0)>0$. From "concavity" arguments (see [5,6]) we know that there exists a constant t_0, such that

$$\lim_{t\to t_0^-}(\|u\|^2_{L_2[0,1]}+a_3\|u_x\|^2_{L_2[0,1]})=+\infty$$

and

$$T<t_0=\left\{\int_0^1(u_0^2+a_3u_{0x}^2)\mathrm{d}x\right\}\Big/\left\{2\sigma\int_0^1(u_0u_1+a_3u_{0x}u_{1x})\mathrm{d}x-a_2\int_0^1 u_{0x}^2\mathrm{d}x\right\}.$$

Theorem 4.1 is proved.

Theorem 4.2. Suppose that $u(x,t)$ is a classical solution of the problem (1.2)—(1.4) and the conditions $u_0(x),u_1(x)\in C^1[0,1]$, $\varphi\in C^1(\mathbf{R})$, $2\alpha\int_0^s\varphi(z)\mathrm{d}z+\varphi(s)s+(\alpha+1)a_1s^2\geqslant 0$, $\alpha\geqslant 1$, are satisfied. Then the solution $u(x,t)$ cannot exist for all time even provided the initial data satisfies one of the following conditions:

(1) $E(0)<0$,

(2) $E(0)=0$, $\int_0^1(u_0u_1+\frac{a_2u_{0x}^2}{2}+a^3u_{0x}u_{1x})\mathrm{d}x<0$,

(3) $E(0)>0$, $\int_0^1(u_0u_1+\frac{a_2u_{0x}^2}{2}+a_3u_{0x}u_{1x})\mathrm{d}x<0$,

and $[\int_0^1(u_0u_1+\frac{a_2u_{0x}^2}{2}+a_3u_{0x}u_{1x})\mathrm{d}x]^2\geqslant 8[\int_0^1(u_0^2+a_3u_{1x}^2)\mathrm{d}x]E(0)$.

Proof. We define $F(t)=\int_0^1(u^2+a_2\int_0^t u_x^2\mathrm{d}\tau+a_3u_x^2)\mathrm{d}x$, thus we have

$$F'(t)=2\int_0^1(uu_t+\frac{a_2u_x^2}{2}+a_3u_xu_{xt})\mathrm{d}x.$$

Let $f(t)$ be a positive real valued twice continuously differentiable function defined on

$[0,T)$. A further restriction will be imposed later on f. We get

$$f^2(t)F'(t) - f^2(0)F'(0) = 2\int_0^1\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\left\{[uu_s + \frac{a_2u_x^2}{2} + a_3u_xu_{xs}]f^2(s)\right\}\mathrm{d}s\mathrm{d}x =$$

$$2\int_0^1\int_0^t[2f(s)f'(s)(uu_s + a_3u_xu_{xs}) + f^2(s)(u_s^2 + a_3u_{xs}^2)]\mathrm{d}s\mathrm{d}x +$$

$$2a_2\int_0^1\int_0^t[f(s)f'(s)u_x^2\mathrm{d}s\mathrm{d}x + 2\int_0^1\int_0^t f^2(s)[uu_{ss} + a_2u_xu_{xs} + a_3u_xu_{xss}]\mathrm{d}s\mathrm{d}x. \tag{4.3}$$

For $\alpha\geqslant 1$ we have

$$2\int_0^1\int_0^t[2f(s)f'(s)(uu_s + a_3u_xu_{xs}) + f^2(s)(u_s^2 + a_3u_{xs}^2)]\mathrm{d}s\mathrm{d}x =$$

$$2(1-\alpha)\int_0^1\int_0^t\{[(f(s)u)_s]^2 + a_3[(f(s)u_x)_s]^2 - [f'(s)]^2(u^2 + a_3u_x^2)\}\mathrm{d}s\mathrm{d}x +$$

$$2\alpha\int_0^1\int_0^t[2f(s)f'(s)(uu_s + a_3u_xu_{xs}) + f^2(s)(u_s^2 + a_3u_{xs}^2)]\mathrm{d}s\mathrm{d}x \leqslant$$

$$2(\alpha-1)\int_0^1\int_0^t[f'(s)]^2[u^2 + a_3u_x^2]\mathrm{d}s\mathrm{d}x + 2\alpha\int_0^1\int_0^t\frac{\mathrm{d}}{\mathrm{d}s}[f(s)f'(s)(u^2 + a_3u_x^2)]\mathrm{d}s\mathrm{d}x -$$

$$2\alpha\int_0^1\int_0^t[f'(s)]^2(u^2 + a_3u_x^2)\mathrm{d}s\mathrm{d}x - 2\alpha\int_0^1\int_0^t f'(s)f''(s)(u^2 + a_3u_x^2)\mathrm{d}s\mathrm{d}x +$$

$$2\alpha\int_0^1\int_0^t f^2(s)(u_s^2 + a_3u_{xs}^2)\mathrm{d}s\mathrm{d}x. \tag{4.4}$$

Integrating by parts with respect to x in last term on the right of (4.3), using Eq. (1.2) and substituting (4.4) into (4.3), we obtain

$$f^2(t)F'(t) - f^2(0)F'(0) \leqslant 2(\alpha-1)\int_0^1\int_0^t[f'(s)]^2(u^2 + a_3u_x^2)\mathrm{d}s\mathrm{d}x +$$

$$2\alpha\int_0^1 f(t)f'(t)(u^2 + a_3u_x^2)\mathrm{d}s\mathrm{d}x - 2\alpha f'(0)f(0)\int_0^1(u_0^2 + a_3u_{0x}^2)\mathrm{d}x -$$

$$2\alpha\int_0^1\int_0^t[f'(s)]^2(u^2 + a_3u_x^2)\mathrm{d}s\mathrm{d}x - 2\alpha\int_0^1\int_0^t f'(s)f''(s)(u^2 + a_3u_x^2)\mathrm{d}s\mathrm{d}x +$$

$$2a_2\int_0^1\int_0^t f(s)f'(s)u_x^2\mathrm{d}s\mathrm{d}x + 2\int_0^1\int_0^t f^2(s)[\alpha(u_s^2 + a_3u_{xs}^2) - (a_1u_x^2 + \varphi(u_x)u_x)]\mathrm{d}x\mathrm{d}s \leqslant$$

$$2\alpha f(t)f'(t)F(t) - 2\alpha f'(0)f(0)F(0) + 2\alpha\int_0^t f^2(s)E(0)\mathrm{d}s -$$

$$2\int_0^1\int_0^t[(f'(s))^2 + \alpha f(s)f''(s)](u^2 + a_3u_x^2)\mathrm{d}s\mathrm{d}x +$$

$$2a_2(1-\alpha)\int_0^1\int_0^t f(s)f'(s)u_x^2\mathrm{d}s\mathrm{d}x - 4a_2\alpha\int_0^1\int_0^t f^2(s)u_{xs}^2\mathrm{d}s\mathrm{d}x -$$

$$2\int_0^t\int_0^1 f^2(s)[(\alpha+1)a_1u_x^2 + 2\alpha\int_0^{u_x}\varphi(s)\mathrm{d}s + \varphi(u_x)u_x]\mathrm{d}x\mathrm{d}s. \tag{4.5}$$

We now demand f to satisfy $f'(s)+\alpha ff''\geqslant 0$. For this purpose we take $f(t)=\mathrm{e}^{\delta t}, \delta>0$. Since $(\alpha+1)a_1s^2+2\alpha\int_0^s\varphi(z)\mathrm{d}z+\varphi(s)s\geqslant 0$, from (4.5) we have

$$f^2(t)F'(t) - f^2(0)F'(0) \leqslant 2\alpha f(t)f'(t)F(t) - 2\alpha f'(0)f(0)F(0) + 2\alpha\int_0^t f^2(s)E(0)\mathrm{d}s. \tag{4.6}$$

It follows from (4. 6) that

$$F'(t) \leqslant 2\alpha\delta F(t) + (F'(0) - 2\alpha\delta F(0) - \frac{\alpha E(0)}{\delta})e^{-\alpha\delta t} + \frac{\alpha E(0)}{\delta}.$$

Finally we obtain

$$F(t) \leqslant H_0 e^{2\alpha\delta t} + L_0 e^{-2\delta t} - \frac{E(0)}{2\delta^2}, \tag{4.7}$$

where

$$2(\alpha + 1)\delta^2 H_0 = \delta F'(0) + 2\delta^2 F(0) + E(0),$$
$$2(\alpha + 1)\delta^2 L_0 = 2\alpha\delta^2 F(0) - \delta F'(0) + \alpha E(0).$$

If $H_0 < 0$, then for sufficiently large t the left-hand side of (4. 7) is greater than or equal to zero, but the right-hand side of (4. 7) is less than zero. This is a contradiction. Thus we may conclude that for the problem (1. 2)—(1. 4) does not exist classical global solution. Conditions (1)—(3) are sufficient to ensure $H_0 < 0$ (for approximate choices of δ) and so the theorem is proved.

Remark 4. 1. By the same method as in Sections 1-4 we can obtain the similar results of the problem (1. 2), (1. 5), (1. 6). It is easy to use the results of the problem (1. 2)—(1. 4) to the problem (1. 1), (1. 3), (1. 4) and the results of the problem (1. 2), (1. 5), (1. 6) to the problem (1. 1), (1. 5), (1. 6).

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