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ORIGINAL PAPER

A general biorthogonal wavelet based on Karhunen–Loéve transform approximation

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Abstract The lifting style biorthogonal wavelet implementation has a nice property of enabling flexible design; it is immediately reversible and has a simple relation to subband filters. In this work, we present a general wavelet prediction (P) and update (U) filters of two-channel lifting structures. A previous research of the author introduced signal-specific wavelets, which minimizes the difference between block wavelet transform matrix of the wavelet and the Karhunen–Loéve transform of a stochastic process with certain autocorrelation sequence. This research introduces a general wavelet, which can work on any stochastic process. Numerical results are provided in terms of the filter coefficients and experimental performances on 16 test images.

Keywords Biorthogonal wavelets · Lifting scheme · Block wavelet transform · Filter design

1 Introduction

Despite their vast variation of applications and design bases, wavelets have a common property of de-correlation. The typical implementation of the discrete wavelet transform (DWT) is subband filtering. Due to its design flexibility and immediate relation to classical subband filtering, the lifting style decomposition is popular [7,8,11,12]. Besides, the synthesis operation is completely symmetric and the de-correlation concept is automatically implemented in the "prediction" scheme (P) of the lifting decomposition. Obviously, other types of transforms, including the classical matrix-based transforms, also aim for the de-correlation of the input random vector. The relation between a (multi-level) subband decomposition and a transform matrix was explained in [2], where a technique to define a wavelet decomposition structure in terms of a block transform matrix (hence the name; block wavelet transform—BWT) was devised. In that work, the generation method of the corresponding BWT from *any* DWT filter bank was explained, and various numerical examples of such transforms were provided [2]. The original methodology of BWT generation was feeding the balanced subband tree with shifted unit sample trains, each with a period of $L = 2^N$, where N is the number of decomposition levels. The outputs of each branch in the subband tree naturally become constant, rendering a column of the BWT.

In the previous research by the authors, the iterative determination of a $2^N \times 2^N$ BWT matrix generating algorithm (from smaller BWT counterparts) was developed, a filter design technique using the orthogonality constraints of BWT matrices for the lifting scheme (envisaged by Sweldens [11,12]) was proposed together with another filter design technique which yield BWT matrices that would mimic statistically optimal Karhunen–Loéve transform (KLT) matrices, which are, by construction, best de-correlating matrices [9,10]. In essence, the previous research is the dual problem of the method in [2], instead of generating a BWT from DWT, to try to determine filter coefficients of DWT, which is expected to produce a particular BWT (that is close to KLT in a Frobenius norm). One attempting method was investigated by Dogan and Gerek [4–6] for orthogonal wavelets using orthogonal QMF subband structures. However, QMF subband filters are too restrictive, and the distance between a KLT matrix and the BWT of a QMF DWT cannot be made arbitrarily close.

The drawback of author' s previous research is the wavelets' dependence to the test images, i.e., wavelets varied

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for each test image [9,10]. Questions are, would one wavelet of a certain test image work on the other test images or is there a possibility of the existence of a general wavelet? This research deals with the second question. Signal-specific wavelets bring problems with them, and a general solution must be found. This paper concludes the previous research [9] by generalizing the signal-specific wavelets described in [9]. Thus, the aim of this paper is to achieve a general solution by designing the general wavelet. Hence, in this work, the more flexible case of biorthogonal lifting style wavelet parametrization is not considered through the same KLT approximation idea as in the previous research [9,10], but a general wavelet is introduced which is only based on the KLT approximation idea.

The paper starts by briefly explaining the lifting scheme in Sect. 2. In the lifting scheme, the author describes the motivation behind. Next, the general wavelet is introduced in Sect. 3, and finally in Sect. 4, conclusions are provided.

2 Lifting scheme

Lifting is a smart method to implement a 2-channel decomposition in the polyphase domain (Fig. 1). For the 1-level lifting decomposition structure, the polyphase matrix is defined as

$$\mathbf{H}_p = \begin{bmatrix} H_{0,ev}(z) & H_{0,od}(z) \\ H_{1,ev}(z) & H_{1,od}(z) \end{bmatrix}$$

where

$$H_{0,ev}(z) = 1 - P(z)U(z)$$
$$H_{0,od}(z) = U(z)$$
$$H_{1,ev}(z) = -P(z)$$
$$H_{1,od}(z) = 1$$

and the subband decomposition filters are

$$H_0(z) = H_{0,ev}(z^2) + zH_{0,od}(z^2)$$

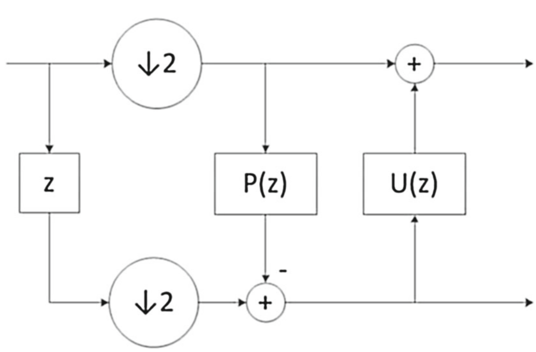

Fig. 1 1-level lifting structure

$$H_1(z) = H_{1,ev}(z^2) + zH_{1,od}(z^2)$$

In order to design a regularized wavelet having at least one vanishing moment so that its corresponding scaling and wavelet dilation equations converge iteratively, $H_0(z = -1)$ and $H_1(z = 1)$ must equal to 0. This condition automatically imposes that, when the prediction filter is in the form: $P(z) = \sum_i a_{-i}z^i$ and the update filter is in the form: $U(z) = \sum_i b_i z^{-i}$, the coefficients must obey the following conditions:

$$\begin{aligned} \alpha &= \sum_i a_{-i} = 1 \\ \beta &= \sum_i b_i = 0.5 \end{aligned} \qquad (1)$$

These filter coefficient constraints are, therefore, adopted as an initial condition for the BWT construction algorithms, while the minimization of the difference between the BWT and KLT matrices remains the optimization criterion.

Daubechies 5-tap/3-tap (Daub 5/3) wavelet has the following prediction ($P(z)$) and update ($U(z)$) filters

$$\begin{aligned} P(z) &= \frac{1}{2} + \frac{1}{2}z \\ U(z) &= \frac{1}{4} + \frac{1}{4}z^-1 \end{aligned} \qquad (2)$$

where it must be underlined that the Daub 5/3 wavelet has a major advantage. Its implementation consists of bitwise shifts and additions [1,3]. Hence, it would be better if the general wavelet to be proposed possesses such properties.

The basic methodology starts with the observation of 32 wavelet samples contained from 16 test images in the author's previous study [9]. Then, a general wavelet is proposed, which possesses the shifting and addition operations of Daubechies 5/3 wavelet. In the author's previous research, it has been described in detail that 8×8 design works worse than the 4×4 case [9]. So it must be noted that the author's operations are on 4×4 BWTs, i.e., 5-tap/3-tap biorthogonal wavelets.

3 Results and discussion

When the 32 wavelets of the previous research are investigated [9], it is seen that the update filter approximated for Barbara's column KLT matrix is of length 1, i.e., $U(z) = 0.5$. This is due to the fact that Barbara's column KLT matrix has second and third rows where the magnitudes of the elements are close to each other forcing the differences and summations of filter the coefficients to be equal. When the summation and difference of two variables are close to

Table 1 Variances of the wavelet tree images for the Daubechies 5/3 and the general wavelet

Test image	LLLL	LLLH	LLHL	LLHH	LLH	LHL	LHH	LH	HL	HH
Aerial, Daub 5/3	1474.75	927.91	1316.4	1730.7	489.75	812.23	575.57	110.45	192.70	60.402
Aerial, general	1147.75	595.03	803.03	1226.74	441.27	693.76	527.72	147.36	248.71	92.33
1st Plane, Daub 5/3	2322.74	747.636	603.588	532.46	186.485	259.872	116.906	48.5499	85.0399	10.9563
1st Plane, general	2002.32	437.057	419.323	380.71	181.419	264.845	117.483	82.6466	106.396	17.6138
2nd Plane, Daub 5/3	580.353	321.072	310.957	242.338	134.552	114.132	79.011	19.8558	23.0125	20.4354
2nd Plane, general	475.771	132.006	113.617	165.385	123.8	111.163	66.9966	23.629	29.3979	28.3982
Barbara, Daub 5/3	2171.94	369.743	259.986	277.296	536.058	148.513	623.975	472.443	34.799	75.5528
Barbara, general	1986.21	260.59	178.464	231.889	423.452	140.82	518.516	472.849	51.3802	131.041
Bus*, Daub 5/3	399.856	1216.95	332.905	1145.89	1880.08	563.4	1688.61	2412.79	221.111	453.78
Bus*, general	236.097	933.059	229.243	705.079	1514.98	434.68	1380.21	2461.03	258.386	649.34
Elaine, Daub 5/3	2329.48	286.069	268.352	210.787	109.29	121.275	84.1716	26.867	47.5747	110.481
Elaine, general	2115.84	209.891	155.34	170.111	112.564	122.161	73.3208	35.4041	53.7308	125.469
Foreman*, Daub 5/3	3.9733	20.391	8.8136	45.206	41.385	16.918	50.441	58.892	9.7729	21.404
Foreman*, general	2.6965	12.511	8.8461	37.203	32.544	12.870	40.667	58.312	10.686	28.093
House, Daub 5/3	2348.49	557.813	565.01	478.884	296.748	337.392	191.885	96.7139	127.167	19.0444
House, general	2019.58	532.601	390.338	357.519	261.488	316.891	173.928	131.831	149.172	30.8287
Lena, Daub 5/3	2464.13	272.89	518.885	490.206	122.431	212.094	138.107	21.9546	46.9018	21.1248
Lena, general	2253.33	157.469	362.422	330.592	120.968	209.867	119.307	31.2051	64.3419	29.632
Lena*, Daub 5/3	0.01251	0.02472	0.01225	0.05748	0.06479	0.02912	0.10691	0.07247	0.03463	0.10023
Lena*, general	0.01033	0.01731	0.00819	0.03913	0.04807	0.02109	0.08784	0.07151	0.03423	0.11957
Mandrill, Daub 5/3	1408.32	544.516	610.79	1253.65	459.305	850.513	950.538	187.441	572.74	156.942
Mandrill, general	1248.15	345.986	400.348	825.676	379.435	678.221	784.919	218.704	616.323	202.834
Mandrill*, Daub 5/3	0.01889	0.02714	0.03419	0.09875	0.07832	0.10237	0.23089	0.09852	0.14420	0.21238
Mandrill*, general	0.01572	0.01949	0.02668	0.06299	0.05796	0.07552	0.18036	0.09832	0.14050	0.25232
Peppers, Daub 5/3	3090.76	413.967	403.286	254.218	141.06	162.715	82.4316	26.875	31.1291	20.0732
Peppers, general	2833.27	309.113	224.123	198.988	147.07	162.443	76.6928	39.4183	43.0221	23.4969
Ruler, Daub 5/3	496.437	649.295	647.757	670.04	7368.59	7188.88	2342.07	2826.18	2771.34	527.882
Ruler, general	399.439	867.421	915.513	308.622	5431.19	5262.7	1970.89	3608.94	3557.94	651.943
Sailboat, Daub 5/3	4492.7	701.039	741.515	790.858	230.075	276.786	205.657	69.6518	91.1597	83.4426
Sailboat, general	4126.54	520.709	519.342	560.864	236.577	264.346	193.951	101.328	129.34	101.095
Tank, Daub 5/3	758.424	245.605	269.936	177.249	124.279	146.336	104.447	36.6123	52.689	36.8455
Tank, general	685.756	105.309	124.1	127.275	111.903	129.876	86.7573	40.9287	59.5856	49.0407

each other, one variable approaches to 0. In this case, Barbara' s column KLT approximation resulted an update filter $U(z) = 0.5000$, and the prediction filter was no different, which was $P(z) = 0.9932 + 0.0068z$.

However, for the rest of the 31 cases (including 8 cases given in Table 4 of [9]), a_0 varies around 0.75, whereas a_{-1} varies around 0.25. Similarly, b_0 varies around 0.4375, whereas a_{-1} varies around 0.0625 [9]. These picked pivot numbers are the closest coefficients that can be described by shifting and addition bitwise operations.

3.1 Definition of a general wavelet using the 4 × 4 KLT approximation wavelets

Using these filter coefficient results obtained in our previous research [9], we have devised general wavelet coefficients as shown in Eq. 3

$$\begin{aligned} P &= \frac{3}{4} + \frac{1}{4}z \\ U &= \frac{7}{16} + \frac{1}{16}z^{-1} \end{aligned} \tag{3}$$

The importance of these prediction and update filters is not only their character to be generalized definitions, but also they include bitwise additions and shifts as in Daubechies 5/3.

The delayer components (i.e., $z/4$ and $z^{-1}/16$) are plain bitwise shifting and addition operations. Compared to the Daubechies' delayer shiftings of 1 for P and 2 for U (i.e., $z/2$ and $z^{-1}/4$), the wavelet introduced in this paper has delayer shiftings of 2 for P and 4 for U.

Likewise, the wavelet introduced in this paper has scalar shiftings of 2 for P (1/4) and 4 for U (1/16) compared to Daubechies' scalar shiftings of 1 for P (1/2) and 2 for U (1/4). On the other hand, using parallel architecture, it is possible to achieve bitwise shifting and addition operations for scalar components, which are 3/4 and 7/16. Off course, this adds 3 addition operations for P and 7 addition operations to U realizations.

The lifting filters of Eq. 3 result analysis subband filters such as

$$\begin{aligned} H_0(z) &= \frac{1}{64}\left(-7z^2 + 28z + 42 + 4z^{-1} - 3z^{-2}\right) \\ H_1(z) &= \frac{1}{4}\left(-z^2 + 4z - 3\right) \end{aligned} \tag{4}$$

and the synthesis subband filters become

$$\begin{aligned} G_0(z) &= -z^{-1} H_1(-z) \\ &= \frac{\left(z + 4 + 3z^{-1}\right)}{4} \end{aligned} \tag{5}$$

$$\begin{aligned} G_1(z) &= z^{-1} H_0(-z) \\ &= \frac{\left(-7z - 28 + 42z^{-1} - 4z^{-2} - 3z^{-3}\right)}{64} \end{aligned} \tag{6}$$

The variances of the wavelet tree images are listed in Table 1. As can be seen, the wavelet introduced in this research gives better variance results with better coding gain and energy unbalance.

4 Conclusions

In this research, a general wavelet is presented based on a signal-specific methodology to design lifting wavelets at certain sizes (5/3 for this case). The described general wavelet is inspired by wavelets which construct a 4×4 orthogonal BWT matrix that mimic a 4×4 KLT matrix corresponding to a time series. Experimentally, the general wavelet was tested on the 16 different typical test images, and it was observed that the designed wavelet has good regularity properties and also provides plausible de-correlation performances as compared to the Daubechies 5/3 wavelet.

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