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Convolution, correlation, and sampling theorems for the offset linear canonical transform

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Abstract The offset linear canonical transform (OLCT), which is a time-shifted and frequency-modulated version of the linear canonical transform, has been shown to be a powerful tool for signal processing and optics. However, some basic results for this transform, such as convolution and correlation theorems, remain unknown. In this paper, based on a new convolution operation, we formulate convolution and correlation theorems for the OLCT. Moreover, we use the convolution theorem to investigate the sampling theorem for the band-limited signal in the OLCT domain. The formulas of uniform sampling and low-pass reconstruction related to the OLCT are obtained. We also discuss the design method of the multiplicative filter in the OLCT domain. Based on the model of the multiplicative filter in the OLCT domain, a practical method to achieve multiplicative filtering through convolution in the time domain is proposed.

Keywords Offset linear canonical transform · Convolution theorem · Correlation theorem · Linear canonical transform · Sampling theorem · Multiplicative filtering

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1 Introduction

The OLCT [1–3], also called the special affine Fourier transformation [4] or the inhomogeneous canonical transform [1], is a six-parameter $(a, b, c, d, u_0, \omega_0)$ class of linear integral transform. It is a time-shifted and frequency-modulated version of the LCT [5–7]. The OLCT is more general and flexible than the original LCT for its two extra parameters u_0 and ω_0, which correspond to time shifting and frequency modulation, respectively. Many widely used linear transforms in engineering, such as the Fourier transform (FT), the offset FT [1,3], the fractional Fourier transform (FRFT) [6,8], the offset FRFT [1,3], the Fresnel transform (FRST) [9], the LCT, time shifting and scaling, frequency modulation, pulse chirping, and others, are special cases of the OLCT. Therefore, understanding the OLCT and developing relevant theorems for OLCT may help to gain more insights on its special cases and to carryover knowledge gained from one subject to others.

Conventional convolution and correlation operations for FT are fundamental in the theory of linear time-invariant (LTI) system [10]. The output of any continuous-time LTI system is found via the convolution of the input signal with the system impulse response. Correlation, which is similar to convolution, is another important operation in signal processing, as well as in optics, in pattern recognition, especially in detection applications [6,11,12]. During past years, many studies have shown that convolution and correlation operations could be extended to wider domains. For example, convolution and correlation operations for FRFT have been proposed some years ago [13,14]. Recently, with intensive research of the LCT, convolution and correlation operations have been further extended to LCT domain in the literature [15–17]. However, convolution and correlation operations for OLCT still yet remain unknown.

As a generalization of many other linear transforms, the OLCT has found wide applications in optics and signal processing [1–3]. It is theoretically interesting and practically useful to consider convolution and correlation theorems in the OLCT domain. Convolution theorem for a linear integral transform can be formulated in several ways. Based on the expression for the generalized translation in the LCT domain, the generalized convolution theorem has been derived in the LCT domain by Wei [16], which shows that the generalized convolution of two signals in time domain is equivalent to simple multiplication of their LCTs in the LCT domains. However, the generalized convolution structure in [16] is a triple integral. So, it is complicated to reduce the expression of the generalized convolution to a single integral form as in the ordinary convolution expression. In this paper, we propose a new convolution structure for the OLCT, which is different from the generalized convolution structure. It can be expressed by a simple one-dimensional integral and easy to implement in multiplicative filter design. In this work, based on a new convolution operation for OLCT, we will discuss convolution and correlation theorems in the OLCT domain. As an application, we also investigate the sampling theorem and multiplicative filtering for the band-limited signal in the OLCT domain by using convolution theorems proposed here.

The rest of the paper is organized as follows. Section 2 gives a brief description of the OLCT along with some properties. In Sect. 3, based on a new convolution operation, we derive convolution and correlation theorems for OLCT in detail. The sampling theorem for OLCT and discussions about multiplicative filter in the OLCT domain are presented in Sect. 4. Section 5 concludes the paper.

2 The OLCT

The OLCT with real parameters of $\mathbf{A} = (a, b, c, d, u_0, \omega_0)$ of a signal $f(t)$ is defined by [1,2]:

$$F_{\mathbf{A}}(u) = O_L^{\mathbf{A}}[f(t)](u) = \begin{cases} \int_{-\infty}^{+\infty} f(t)h_{\mathbf{A}}(t,u)\mathrm{d}t & b \neq 0 \\ \sqrt{d}\mathrm{e}^{j\frac{cd}{2}(u-u_0)^2+j\omega_0 u} f[d(u-u_0)] & b = 0 \end{cases}, \tag{1}$$

where

$$h_{\mathbf{A}}(t,u) = K_{\mathbf{A}}\mathrm{e}^{\frac{j}{2b}[at^2+2t(u_0-u)-2u(du_0-b\omega_0)+du^2]},$$
$$K_{\mathbf{A}} = \sqrt{\frac{1}{j2\pi b}}\mathrm{e}^{j\frac{d}{2b}u_0^2}, \tag{2}$$

and $ad - bc = 1$. The definition for case $b = 0$ is the limit of the integral in (1) for the case $b \neq 0$ as $|b| \to 0$. Therefore, from now on we shall confine our attention to OLCT for $b \neq 0$. And without loss of generality, we assume $b > 0$ in the following sections. The inverse of an OLCT with parameters $\mathbf{A} = (a, b, c, d, u_0, \omega_0)$ is given by an OLCT with parameters $\mathbf{A^{-1}} = (d, -b, -c, a, b\omega_0 - du_0, cu_0 - a\omega_0)$. The exact inverse OLCT expression is given by [1,2]:

$$f(t) = O_L^{\mathbf{A}^{-1}}[F_{\mathbf{A}}(u)](t) = C\int_{-\infty}^{\infty} F_{\mathbf{A}}(u)h_{\mathbf{A}^{-1}}(u,t)du, \tag{3}$$

where

$$C = \mathrm{e}^{j\frac{1}{2}(cdu_0^2-2adu_0\omega_0+ab\omega_0^2)}.$$

This can be verified by using the definition (1). Some of the special cases of the OLCT are listed in Table 1. Those relations can be easily verified by substituting the specific parameters $\mathbf{A}$ in Eq. (2).

A signal $f(t)$ is band-limited to $U_{\mathbf{A}}$ in the OLCT domain, which means that:

$$F_{\mathbf{A}}(u) = 0 \quad \text{for} \quad |u| > U_{\mathbf{A}},$$

where $U_{\mathbf{A}}$ is called the bandwidth of the signal $f(t)$ in the OLCT domain with parameters $\mathbf{A}$.

The OLCT has many properties [1–7,17]. We present the following important space shift and phase shift properties of OLCT [7,17], which can also be easily verified by using the definition (1).

Property 1: The space shift property [7,17]

$$O_L^{\mathbf{A}}[f(t-\tau)](u) = F_{\mathbf{A}}(u-a\tau)\mathrm{e}^{-j\frac{ac\tau^2}{2}+jc\tau(u-u_0)+ja\tau\omega_0}. \tag{4}$$

Property 2: The phase shift property [17]

$$O_L^{\mathbf{A}}[f(t)\mathrm{e}^{jvt}](u) = F_{\mathbf{A}}(u-bv)\mathrm{e}^{-j\frac{bdv^2}{2}+jdv(u-u_0)+jbv\omega_0}. \tag{5}$$

Table 1 Some of the specific cases of the OLCT

Transform	Parameters $\mathbf{A}$
Offset linear canonical transform (OLCT)	$\mathbf{A} = (a, b, c, d, u_0, \omega_0)$
linear canonical transform (LCT)	$\mathbf{A} = (a, b, c, d, 0, 0)$
Fourier transform (FT)	$\mathbf{A} = (0, 1, -1, 0, 0, 0)$
Fractional Fourier transform (FRFT)	$\mathbf{A} = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta, 0, 0)$
Offset fractional Fourier transform (OFRFT)	$\mathbf{A} = (\cos\theta, \sin\theta, -\sin\theta, \cos\theta, u_0, \omega_0)$
Fresnel transform (FRST)	$\mathbf{A} = (1, b, 0, 1, 0, 0)$
Frequency modulation	$\mathbf{A} = (1, 0, 0, 1, 0, \omega_0)$
Time scaling	$\mathbf{A} = (d^{-1}, 0, 0, d, 0, 0)$
Time shifting	$\mathbf{A} = (1, 0, 0, 1, u_0, 0)$

Property 3: The space shift and phase shift properties [17]

$$O_L^{\mathbf{A}}[f(t-\tau)\mathrm{e}^{jvt}](u) = F_{\mathbf{A}}(u-a\tau-bv)\mathrm{e}^{-j\frac{ac\tau^2+bdv^2}{2}} \times \mathrm{e}^{j(c\tau+dv)(u-u_0)}\mathrm{e}^{-jbc\tau v}\mathrm{e}^{j(a\tau+bv)\omega_0}. \tag{6}$$

3 Convolution and correlation theorems for OLCT

In this section, we derive convolution and correlation theorems for OLCT based on the definition and properties of the OLCT. Some of the well-known results about the convolution and correlation theorems in FT domain, FRFT domain, and LCT domain are shown to be special cases of our achieved results.

3.1 Convolution theorems for OLCT

In the general framework of convolution theory (see [18, Ch. 4]), it is known that to every integral transformation $\Re$, one can, at least theoretically, associate with it a convolution operation, Θ, such that

$$\Re(f\Theta g) = \Re(f)\cdot\Re(g).$$

Let be W that subspace of the space of all integrable functions with the property that $f(t)\in W$ if and only if the FT of $f(t)$ is also in W [13,14]. Let us consider two functions, $f, h \in W$, the conventional convolution operator for FT, is given by [10]

$$f(t)*h(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)\mathrm{d}\tau.$$

And the convolution theorems for the FT are as follows:

$$\psi_{\mathrm{FT}}(f(t)*h(t)) = F(u)H(u),$$

where $\psi_{\mathrm{FT}}(\cdot)$ denotes the FT operator and $F(u), H(u)$ denotes the FT of $f(t) and h(t)$, respectively. In this paper, the FT is defined as follows [10]

$$F(u) = \psi_{\mathrm{FT}}(f(t)) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-jut}\mathrm{d}t,$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\mathrm{e}^{jut}\mathrm{d}u.$$

To obtain the convolution theorem for OLCT, we first define a new convolution operation for OLCT.

Definition 1 For any function $f(t), h(t) \in W$, let us define a new convolution operation $\overset{\mathbf{A}}{\otimes}$ for OLCT as follows [17]:

$$z(t) = (f\overset{\mathbf{A}}{\otimes}h)(t) = K_{\mathbf{A}}\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\mathrm{e}^{-j\frac{a\tau}{b}(t-\tau)}\mathrm{d}\tau \tag{7}$$

Based on the Definition 1, the expressions for the OLCT of a new convolution of two functions can be derived.

Theorem 1 *Let* $z(t)=(f\overset{\mathbf{A}}{\otimes}h)(t)$ *and* $Z_{\mathbf{A}}(u), F_{\mathbf{A}}(u), H_{\mathbf{A}}(u)$ *denote the OLCT of* z, f *and* h, *respectively. Then*

$$O_L^{\mathbf{A}}[z(t)](u) = Z_{\mathbf{A}}(u) = F_{\mathbf{A}}(u)\cdot H_{\mathbf{A}}(u)e^{\frac{j}{2b}[-du^2+2u(du_0-b\omega_0)]} \tag{8}$$

Proof *From Eqs. (1) and (6), we see that both* $O_L^{\mathbf{A}}[f(t)](u)$ *and* $O_L^{\mathbf{A}}[f(t-\tau)e^{jvt}](u)$ *depend on the same parameter if we choose* τ *and* v *such that*

$$a\tau + bv = 0 \tag{9}$$

According to Eq. (9), we get

$$v = -\frac{a}{b}\tau \tag{10}$$

Substituting (10) into (6) and making use of $ad - bc = 1$, *Eq. (6) can be reduce to*

$$O_L^{\mathbf{A}}\left[f(t-\tau)e^{-j\frac{a\tau}{b}t}\right](u) = F_{\mathbf{A}}(u)e^{-j\frac{a\tau^2}{2b}-j\frac{\tau}{b}(u-u_0)}. \tag{11}$$

Then, using the definition of the OLCT, we have

$$F_{\mathbf{A}}(u)\cdot H_{\mathbf{A}}(u) = F_{\mathbf{A}}(u)\int_{-\infty}^{\infty} K_{\mathbf{A}}e^{\frac{j}{2b}[a\tau^2+2\tau(u_0-u)-2u(du_0-b\omega_0)+du^2]}h(\tau)d\tau. \tag{12}$$

According to Eq. (11), we get

$$F_{\mathbf{A}}(u)\mathrm{e}^{j\frac{\tau}{b}(u_0-u)} = \mathrm{e}^{j\frac{a\tau^2}{2b}}O_L^{\mathbf{A}}\left[f(t-\tau)\mathrm{e}^{-j\frac{a\tau}{b}t}\right](u). \tag{13}$$

Substituting (13) into (12), we can obtain

$$F_{\mathbf{A}}(u)\cdot H_{\mathbf{A}}(u) = \int_{-\infty}^{\infty} K_{\mathbf{A}}\mathrm{e}^{\frac{j}{2b}[a\tau^2-2u(du_0-b\omega_0)+du^2]}h(\tau) \times\left(\mathrm{e}^{j\frac{a\tau^2}{2b}}O_L^{\mathbf{A}}\left[f(t-\tau)\mathrm{e}^{-j\frac{a\tau}{b}t}\right](u)\right)\mathrm{d}\tau. \tag{14}$$

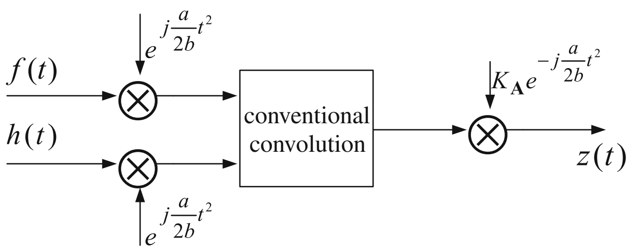

Fig. 1 A new convolution operation for OLCT

Using the definition of the OLCT, we rewrite (14) as

$$
\begin{aligned}
&F_{\mathbf{A}}(u)\cdot H_{\mathbf{A}}(u)\\
&\quad = K_{\mathbf{A}}^{2}\mathrm{e}^{\frac{j}{2b}[-2u(du_0-b\omega_0)+du^2]}\int_{-\infty}^{\infty}\mathrm{e}^{j\frac{a\tau^2}{b}}h(\tau)\\
&\quad\times\left(\int_{-\infty}^{\infty}\mathrm{e}^{\frac{j}{2b}[at^2+2t(u_0-u)-2u(du_0-b\omega_0)+du^2]}\right.\\
&\quad\left.\times f(t-\tau)\mathrm{e}^{-j\frac{a\tau}{b}t}\mathrm{d}t\right)\mathrm{d}\tau.
\end{aligned}\tag{15}
$$

According to (7), we get

$$
\begin{aligned}
z(t) &= K_{\mathbf{A}}\int_{-\infty}^{\infty} f(\tau)h(t-\tau)\mathrm{e}^{-j\frac{a\tau}{b}(t-\tau)}\mathrm{d}\tau\\
&= K_{\mathbf{A}}\int_{-\infty}^{\infty} h(\tau)f(t-\tau)\mathrm{e}^{-j\frac{a\tau}{b}(t-\tau)}\mathrm{d}\tau.
\end{aligned}\tag{16}
$$

Substituting (16) into (15), we can obtain

$$
\begin{aligned}
&F_{\mathbf{A}}(u)\cdot H_{\mathbf{A}}(u)\mathrm{e}^{\frac{j}{2b}[-du^2+2u(du_0-b\omega_0)]}\\
&\quad = O_L^{\mathbf{A}}[z(t)](u) = Z_{\mathbf{A}}(u).
\end{aligned}\tag{17}
$$

The proof of Theorem 1 is achieved.

From (16) and (17), we see that the function $z(t)$ could be a good candidate to be a new convolution of $f(t)$ and $h(t)$ for the OLCT. Further, based on the conventional convolution operator, we can express the new convolution operation (7) as

$$
\begin{aligned}
z(t) &= (f\overset{\mathbf{A}}{\otimes}h)(t)\\
&= K_{\mathbf{A}}\mathrm{e}^{-j\frac{at^2}{2b}}\left(f(t)\mathrm{e}^{j\frac{at^2}{2b}}\right)*\left(h(t)\mathrm{e}^{j\frac{at^2}{2b}}\right).
\end{aligned}\tag{18}
$$

See Fig. 1, a realization of the new convolution operation for OLCT. Equation (8) shows the OLCT of a new convolution of two functions is equivalent to simple multiplication of their OLCTs, multiplying by a quadratic phase function (linear chirp).This may be particularly useful in filter design and applications as will be shown later on.

Next, we derive the expressions for the OLCT of a product of two functions. This leads to the following convolution theorem.

Theorem 2 *For any function $f(t), h(t) \in W$, let $F_{\mathbf{A}}(u)$, $H_{\mathbf{A}}(u)$ denote the OLCT of f, h, then*

$$
\begin{aligned}
&O_L^{\mathbf{A}}[f(t)h(t)](u)\\
&\quad = \frac{1}{2\pi b}e^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]}\\
&\quad\times\left(\left(H_{\mathbf{A}}(u)e^{-j\frac{1}{2b}[du^2-2u(du_0-b\omega_0)]}\right)*F\left(\frac{u}{b}\right)\right).
\end{aligned}\tag{19}
$$

Proof Let $y(t) = f(t)h(t)$, then

$$
\begin{aligned}
Y_{\mathbf{A}}(u) &= O_L^{\mathbf{A}}[y(t)](u)\\
&= \int_{-\infty}^{\infty} K_{\mathbf{A}}\mathrm{e}^{\frac{j}{2b}[at^2+2t(u_0-u)-2u(du_0-b\omega_0)+du^2]}f(t)h(t)\mathrm{d}t.
\end{aligned}\tag{20}
$$

According to the inverse OLCT expression (3), we get

$$
\begin{aligned}
h(t) &= C\int_{-\infty}^{\infty} H_{\mathbf{A}}(u)h_{\mathbf{A}^{-1}}(u,t)du\\
&= CK_{\mathbf{A}^{-1}}\int_{-\infty}^{\infty} H_{\mathbf{A}}(u)\mathrm{e}^{-j\frac{1}{2b}[du^2+2u(b\omega_0-du_0-t)]}\\
&\quad\times\mathrm{e}^{-j\frac{1}{2b}[-2t[a(b\omega_0-du_0)+b(cu_0-a\omega_0)]+at^2]}\mathrm{d}u\\
&= CK_{\mathbf{A}^{-1}}\int_{-\infty}^{\infty} H_{\mathbf{A}}(u)\mathrm{e}^{-j\frac{1}{2b}[du^2+2u(b\omega_0-du_0-t)]}\\
&\quad\times\mathrm{e}^{-j\frac{1}{2b}(2u_0t+at^2)}\mathrm{d}u,
\end{aligned}\tag{21}
$$

where

$$
\begin{aligned}
CK_{\mathbf{A}^{-1}} &= \mathrm{e}^{j\frac{1}{2}(cdu_0^2-2adu_0\omega_0+ab\omega_0^2)}\sqrt{\frac{1}{-j2\pi b}}\mathrm{e}^{-j\frac{a}{2b}(b\omega_0-du_0)^2}\\
&= \sqrt{\frac{1}{-j2\pi b}}\mathrm{e}^{-j\frac{d}{2b}u_0^2}.
\end{aligned}
$$

Substituting (21) into (20), we obtain

$$
\begin{aligned}
Y_{\mathbf{A}}(u) &= \int_{-\infty}^{\infty} K_{\mathbf{A}}\mathrm{e}^{\frac{j}{2b}[at^2+2t(u_0-u)-2u(du_0-b\omega_0)+du^2]}f(t)\\
&\quad\times CK_{\mathbf{A}^{-1}}\int_{-\infty}^{\infty} H_{\mathbf{A}}(v)\mathrm{e}^{-j\frac{1}{2b}[dv^2+2v(b\omega_0-du_0-t)]}\\
&\quad\times\mathrm{e}^{-j\frac{1}{2b}(2u_0t+at^2)}\mathrm{d}v\mathrm{d}t
\end{aligned}
$$

$$= K_{\mathbf{A}}\sqrt{\frac{1}{-j2\pi b}}\mathrm{e}^{-j\frac{d}{2b}u_0^2}\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]} \times \int_{-\infty}^{\infty} H_{\mathbf{A}}(v)\mathrm{e}^{-j\frac{1}{2b}[dv^2-2v(du_0-b\omega_0)]} \times \left(\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-j\frac{t}{b}(u-v)}\mathrm{d}t\right)\mathrm{d}v$$
$$= \frac{1}{2\pi b}\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]} \times \left(\left(H_{\mathbf{A}}(u)\mathrm{e}^{-j\frac{1}{2b}[du^2-2u(du_0-b\omega_0)]}\right) * F\left(\frac{u}{b}\right)\right),$$

where $F(\cdot)$ denotes the FT of $f(t)$. Thus, the proof of Theorem 2 is achieved. These results of the above-derived theorems are extensions of convolution theorems of the FT and LCT to OLCT domain.

Corollary 1 *If the parameter of OLCT changes to* $(a, b, c, d, u_0, \omega_0) = (0, 1, -1, 0, 0, 0)$, *then convolution formulas of Theorems 1 and 2 reduce to conventional convolution formulas in FT domain as follows:*

$$f(t) * h(t) \overset{\mathrm{FT}}{\longleftrightarrow} F(u)H(u),$$

and

$$f(t)h(t) \overset{\mathrm{FT}}{\longleftrightarrow} \frac{1}{2\pi}F(u) * H(u).$$

Replacing parameters $\mathbf{A} = (a, b, c, d, u_0, \omega_0)$ *with* $\mathbf{A} = (0, 1, -1, 0, 0, 0)$ *in Eqs. (7), (8), and (19), respectively, the proof of the Corollary 1 is achieved.*

Corollary 2 *If the parameter of OLCT changes to* $(a, b, c, d, u_0, \omega_0) = (a, b, c, d, 0, 0)$, *then convolution formulas of Theorems 1 and 2 reduce to convolution formulas in LCT domain [15] as follows:*

$$\sqrt{\frac{1}{j2\pi b}}e^{-j\frac{at^2}{2b}}\left(f(t)e^{j\frac{at^2}{2b}}\right) * \left(h(t)e^{j\frac{at^2}{2b}}\right) \overset{LCT}{\longleftrightarrow} F_{(a,b,c,d)}(u)\cdot H_{(a,b,c,d)}(u)e^{-j\frac{du^2}{2b}},$$

and

$$f(t)h(t) \overset{LCT}{\longleftrightarrow} \frac{1}{2\pi b}e^{j\frac{du^2}{2b}}\left(\left(H_{(a,b,c,d)}(u)e^{-j\frac{du^2}{2b}}\right) * F\left(\frac{u}{b}\right)\right),$$

where $F_{(a,b,c,d)}(u)$, $H_{(a,b,c,d)}(u)$ *denote the LCT of* $f(t)$, $h(t)$, *respectively. The proof of the Corollary 2 is similar to the proof of the Corollary 1 and is omitted.*

In addition to the FT and LCT, convolution theorems for the specific OLCT cases FRST and FRFT shown in Table 1 can also be obtained from the above-derived theorems [9,13,14].

3.2 Correlation theorem for the OLCT

Correlation, which is similar to convolution, is another important operation in signal processing. Since the correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed and complex conjugated, the property of the new convolution results in the property of the correlation. Making use of the new convolution structure (7) for OLCT, we can present a new correlation operation in the OLCT domain as following

Definition 2 For any function $f(t), h(t) \in W$, let us define a new correlation operation $\overset{\mathbf{A}}{\oplus}$ for OLCT as [17]

$$z(t) = (f \overset{\mathbf{A}}{\oplus} h)(t) = K_{\mathbf{A}}K_{\mathbf{A}}^{*}\int_{-\infty}^{\infty} f(\tau)h^{*}(\tau - t)\mathrm{e}^{j\frac{at}{b}(\tau-t)}\mathrm{d}\tau, \tag{22}$$

where $K_{\mathbf{A}}^{*} = \sqrt{\frac{1}{-j2\pi b}}\mathrm{e}^{-j\frac{d}{2b}u_0^2}$ and superscript "$*$" denotes complex conjugation, which is also used in the following content of this paper. Further, based on the conventional convolution operator, we can express the correlation operator (22) in the OLCT domain as follow:

$$\begin{aligned} z(t) &= (f \overset{\mathbf{A}}{\oplus} h)(t) \\ &= K_{\mathbf{A}}K_{\mathbf{A}}^{*}\int_{-\infty}^{\infty} f(\tau)h^{*}(\tau - t)\mathrm{e}^{j\frac{at}{b}(\tau-t)}\mathrm{d}\tau \\ &= K_{\mathbf{A}}K_{\mathbf{A}}^{*}\mathrm{e}^{-j\frac{at^2}{2b}}\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{j\frac{a\tau^2}{2b}}h^{*}(-(t-\tau))\mathrm{e}^{-j\frac{a}{2b}(t-\tau)^2}\mathrm{d}\tau \\ &= K_{\mathbf{A}}K_{\mathbf{A}}^{*}\mathrm{e}^{-j\frac{at^2}{2b}}\left(f(t)\mathrm{e}^{j\frac{at^2}{2b}}\right) * \left(h^{*}(-t)\mathrm{e}^{-j\frac{at^2}{2b}}\right) \end{aligned} \tag{23}$$

Theorem 3 *Let* $z(t) = (f \overset{\mathbf{A}}{\oplus} h)(t)$ *and* $Z_{\mathbf{A}}(u)$, $F_{\mathbf{A}}(u)$, $H_{\mathbf{A}}(u)$ *denote the OLCT of z, f and h, respectively. Then*

$$O_L^{\mathbf{A}}[z(t)](u) = Z_{\mathbf{A}}(u) = K_{\mathbf{A}}F_{\mathbf{A}}(u)\cdot H_{\mathbf{A}}^{*}(u)e^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]}. \tag{24}$$

Proof Let us rewrite (23) in the form

$$z(t) = (f \overset{\mathbf{A}}{\oplus} h)(t) = K_{\mathbf{A}}\mathrm{e}^{-j\frac{at^2}{2b}}\left(f(t)\mathrm{e}^{j\frac{at^2}{2b}}\right) * \left(K_{\mathbf{A}}^{*}h^{*}(-t)\mathrm{e}^{-j\frac{at^2}{b}}\mathrm{e}^{j\frac{at^2}{2b}}\right). \tag{25}$$

And let $g(t)$ be such that

$$g(t) = K_{\mathbf{A}}^{*}h^{*}(-t)\mathrm{e}^{-j\frac{at^2}{b}}. \tag{26}$$

Substituting (26) into (25), we obtain

$$z(t) = (f \overset{\mathbf{A}}{\oplus} h)(t) = K_{\mathbf{A}} \mathrm{e}^{-j\frac{at^2}{2b}} \left(f(t)\mathrm{e}^{j\frac{at^2}{2b}} \right) * \left(g(t)\mathrm{e}^{j\frac{at^2}{2b}} \right).$$

According to Theorem 1, we get

$$Z_{\mathbf{A}}(u) = F_{\mathbf{A}}(u) \cdot G_{\mathbf{A}}(u)\mathrm{e}^{\frac{j}{2b}[-du^2+2u(du_0-b\omega_0)]}, \tag{27}$$

where $G_{\mathbf{A}}(u)$ denote the OLCT of $g(t)$. Using the definition of the OLCT, we have

$$\begin{aligned} G_{\mathbf{A}}(u) &= \int_{-\infty}^{\infty} K_{\mathbf{A}} \mathrm{e}^{\frac{j}{2b}[at^2+2t(u_0-u)-2u(du_0-b\omega_0)+du^2]} \\ &\quad \times K_{\mathbf{A}}^* h^*(-t)\mathrm{e}^{-j\frac{at^2}{b}}\mathrm{d}t \\ &= K_{\mathbf{A}}\mathrm{e}^{-\frac{j}{b}[2u(du_0-b\omega_0)-du^2]} \\ &\quad \times \left(\int_{-\infty}^{\infty} K_{\mathbf{A}} \mathrm{e}^{\frac{j}{2b}[at^2+2t(u_0-u)]} \right. \\ &\quad \left. \cdot\, \mathrm{e}^{\frac{j}{2b}[-2u(du_0-b\omega_0)+du^2]} h(t)\mathrm{d}t \right)^* \\ &= K_{\mathbf{A}}\mathrm{e}^{\frac{j}{b}[du^2-2u(du_0-b\omega_0)]} H_{\mathbf{A}}^*(u). \end{aligned} \tag{28}$$

Substituting (28) into (27), we can obtain

$$Z_{\mathbf{A}}(u) = K_{\mathbf{A}} F_{\mathbf{A}}(u) \cdot H_{\mathbf{A}}^*(u)\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]}.$$

The proof of Theorem 3 is achieved. By substituting with the specific OLCT parameters of $\mathbf{A}$ in Eqs. (23) and (24), correlation theorems for the special OLCT cases FT, FRST, FRFT, and LCT [9–17] shown in Table 1 can also be obtained from the above-derived Theorem 3.

4 Sampling and filtering for the OLCT

Sampling and filtering are two basic theoretical problems in signal processing. In this section, the above-mentioned convolution theorems will be utilized to resolve these problems associated with the OLCT.

4.1 Sampling theorem for the OLCT

The sampling process is central in almost any domain, and it explains how to sample continuous signals without aliasing. The sampling theorem expansions for the OLCT have been derived in [2], which provide the link between the continuous signals and the discrete signals, and can be used to reconstruct the original signal from their samples satisfying the Nyquist rate of that domain. Here, utilizing the convolution theorem, sampling of band-limited signals in the OLCT domain is further investigated. In particular, the formulas of uniform sampling and low-pass reconstruction are obtained.

Firstly, we define the uniform sampled signal as follows:

$$\hat{f}(t) = f(t)s_\delta(t) = f(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT)$$

where T is the sampling period and $s_\delta(t)$ is the uniform impulse train. Using the definition of the OLCT, we have

$$\begin{aligned} \hat{F}_{\mathbf{A}}(u) &= O_L^{\mathbf{A}}[\hat{f}(t)](u) \\ &= \int_{-\infty}^{\infty} K_{\mathbf{A}} \mathrm{e}^{\frac{j}{2b}[at^2+2t(u_0-u)-2u(du_0-b\omega_0)+du^2]} \\ &\quad \times \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT)\mathrm{d}t \\ &= K_{\mathbf{A}}\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]} \\ &\quad \times \sum_{n=-\infty}^{\infty} f(nT)\mathrm{e}^{\frac{j}{2b}[a(nT)^2+2nT(u_0-u)]}. \end{aligned} \tag{29}$$

Equation (29) shows how to obtain the OLCT of a discrete time signal $f(nT)$. We refer to it as the discrete-time OLCT.

Using the definition of the FT, we have

$$S_\delta(u) = \psi_{\mathrm{FT}}(s_\delta(t)) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(u-n\frac{2\pi}{T})$$

According to Theorem 2, we can obtain

$$\begin{aligned} \hat{F}_{\mathbf{A}}(u) &= O_L^{\mathbf{A}}[\hat{f}(t)](u) \\ &= \frac{1}{2\pi b}\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]} \\ &\quad \times \left((F_{\mathbf{A}}(u)\mathrm{e}^{-j\frac{1}{2b}[du^2-2u(du_0-b\omega_0)]}) * S_\delta\left(\frac{u}{b}\right) \right) \\ &= \frac{1}{2\pi b}\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]} \left(\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\frac{u}{b}-n\frac{2\pi}{T}\right) \right. \\ &\quad \left. * \left(F_{\mathbf{A}}(u)\mathrm{e}^{-j\frac{1}{2b}[du^2-2u(du_0-b\omega_0)]}\right) \right) \\ &= \frac{1}{T}\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]} \left(\sum_{n=-\infty}^{\infty}\delta\left(u-n\frac{2\pi b}{T}\right) \right. \\ &\quad \left. * \left(F_{\mathbf{A}}(u)\mathrm{e}^{-j\frac{1}{2b}[du^2-2u(du_0-b\omega_0)]}\right) \right). \end{aligned} \tag{30}$$

From (30) we find that $F_{\mathbf{A}}(u)$ replicates with a period of $2\pi b/T$, along with linear phase modulation depending on the harmonic order n. When $n = 0$, we obtain

$$\begin{aligned}\hat{F}_{\mathbf{A}}(u) &= O_L^{\mathbf{A}}[\hat{f}(t)](u)\\ &= \frac{1}{T}\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]}F_{\mathbf{A}}(u)\mathrm{e}^{-j\frac{1}{2b}[du^2-2u(du_0-b\omega_0)]}\\ &= \frac{1}{T}F_{\mathbf{A}}(u).\end{aligned}$$

The expression above shows that the part of $\hat{F}_{\mathbf{A}}(u)$ with $n = 0$ modulates the amplitude of $F_{\mathbf{A}}(u)$ by $1/T$, but it does not modulate the phase. If $f(t)$ is a band-limited signal in the OLCT domain, in other words, the support interval of $F_{\mathbf{A}}(u)$ is $[-U_{\mathbf{A}}, U_{\mathbf{A}}]$, then no overlapping occurs in $\hat{F}_{\mathbf{A}}(u)$ after sampling only if

$$\frac{2\pi b}{T} \geq 2U_{\mathbf{A}},$$

or the sampling frequency

$$\omega \geq \frac{2U_{\mathbf{A}}}{b}.$$

Thus, $F_{\mathbf{A}}(u)$ can be recovered, and the other replicated spectrums are filtered out through a low-pass filter $\tilde{H}_{\mathbf{A}}(u)$ having the gain T and the cutoff frequency ω_c in the OLCT domain, whose transfer function is shown by (31). The original signal $f(t)$ can be reconstructed without any distortion by inverse OLCT of the recovered $F_{\mathbf{A}}(u)$.

$$\tilde{H}_{\mathbf{A}}(u) = \begin{cases} T, & |u| < \omega_c \\ 0, & |u| \geq \omega_c \end{cases}, \ \omega_c \in [U_{\mathbf{A}}, b\omega - U_{\mathbf{A}}]. \quad (31)$$

The expression of the above-mentioned reconstruction signal can be obtained by Theorem 1. Assume

$$\tilde{H}_{\mathbf{A}}(u) = H_{\mathbf{A}}(u)\mathrm{e}^{\frac{j}{2b}[-du^2+2u(du_0-b\omega_0)]} = \begin{cases} T, & |u| < \omega_c \\ 0, & |u| \geq \omega_c \end{cases},$$

then

$$\begin{aligned}h(t) &= O_L^{\mathbf{A}^{-1}}[H_{\mathbf{A}}(u)](t)\\ &= O_L^{\mathbf{A}^{-1}}[\tilde{H}_{\mathbf{A}}(u)\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]}](t)\\ &= CK_{\mathbf{A}^{-1}}\int_{-\omega_c}^{\omega_c}\mathrm{e}^{-j\frac{1}{2b}[du^2+2u(b\omega_0-du_0-t)+2u_0t+at^2]}\\ &\quad\times T\mathrm{e}^{\frac{j}{2b}[du^2-2u(du_0-b\omega_0)]}\mathrm{d}u\\ &= TCK_{\mathbf{A}^{-1}}\int_{-\omega_c}^{\omega_c}\mathrm{e}^{-j\frac{1}{2b}[2u_0t+at^2]}\mathrm{e}^{j\frac{t}{b}u}\mathrm{d}u\\ &= TCK_{\mathbf{A}^{-1}}\mathrm{e}^{-j\frac{1}{2b}[2u_0t+at^2]}\frac{2b\sin\frac{\omega_c}{b}t}{t}.\end{aligned}$$

According to Theorem 1, we have

$$\hat{F}_{\mathbf{A}}(u)\tilde{H}_{\mathbf{A}}(u) = O_L^{\mathbf{A}}[(\hat{f}\overset{\mathbf{A}}{\otimes}h)(t)](u).$$

So the output of the low-pass filter is as follows:

$$\begin{aligned}f(t) &= (\hat{f}\overset{\mathbf{A}}{\otimes}h)(t)\\ &= K_{\mathbf{A}}\mathrm{e}^{-j\frac{at^2}{2b}}(\hat{f}(t)\mathrm{e}^{j\frac{at^2}{2b}}) * (h(t)\mathrm{e}^{j\frac{at^2}{2b}})\\ &= TCK_{\mathbf{A}^{-1}}K_{\mathbf{A}}\mathrm{e}^{-j\frac{at^2}{2b}}\int_{-\infty}^{\infty}\hat{f}(\tau)\mathrm{e}^{j\frac{a\tau^2}{2b}}\\ &\quad\times\mathrm{e}^{-j\frac{1}{b}u_0(t-\tau)}\frac{2b\sin\frac{\omega_c}{b}(t-\tau)}{t-\tau}\mathrm{d}\tau\\ &= \frac{T}{\pi}\mathrm{e}^{-j\frac{at^2}{2b}}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}f(\tau)\delta(\tau-nT)\mathrm{e}^{j\frac{a\tau^2}{2b}}\\ &\quad\times\mathrm{e}^{-j\frac{1}{b}u_0(t-\tau)}\frac{\sin\frac{\omega_c}{b}(t-\tau)}{t-\tau}\mathrm{d}\tau\\ &= \mathrm{e}^{-j\frac{at^2}{2b}}\sum_{n=-\infty}^{\infty}f(nT)\mathrm{e}^{j\frac{a(nT)^2}{2b}}\\ &\quad\times\mathrm{e}^{-j\frac{1}{b}u_0(t-nT)}\frac{T\sin\frac{\omega_c}{b}(t-nT)}{\pi(t-nT)}.\end{aligned} \quad (32)$$

Equation (32) is the reconstruction formula of band-limited signal in the OLCT domain. Let $\omega_c = U_{\mathbf{A}}$ in (31), the interpolation formula (32) can be presented as follows:

$$\begin{aligned}f(t) &= \mathrm{e}^{-j\frac{at^2}{2b}}\sum_{n=-\infty}^{\infty}f(nT)\mathrm{e}^{j\frac{a(nT)^2}{2b}}\\ &\quad\mathrm{e}^{-j\frac{1}{b}u_0(t-nT)}\frac{T\sin\frac{U_{\mathbf{A}}}{b}(t-nT)}{\pi(t-nT)}\end{aligned} \quad (33)$$

From the above-derived results, we can conclude uniform sampling theorem for band-limited signal in the OLCT domain as following

Theorem 4 *Let signal $f(t) \in W$ is band-limited to $U_{\mathbf{A}}$ in the OLCT domain with parameters $\mathbf{A} = (a, b, c, d, u_0, \omega_0)$ and $b > 0$.Then, the signal $f(t)$ can be exactly reconstructed from its sampled version $f(nT), n \in Z$, providing that the sampling interval satisfies*

$$T = \frac{\pi b}{U_{\mathbf{A}}}. \quad (34)$$

The reconstruction formula is given by (33).

By exchanging the role of the signal and its OLCT, the sampling theorem of time-limited signal in OLCT domain derived in [2] can be easily obtained from Theorem 4. Equations (33) and (34) are the sampling theorem expansions of the classical results in OLCT domain. Previously developed sampling theorems for FT, FRFT, and LCT [19–22] can also be obtained by substituting with the specific OLCT parameters of $\mathbf{A}$ in Eqs. (33) and (34).

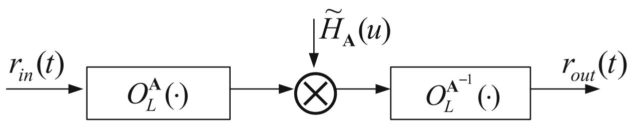

Fig. 2 The multiplicative filter in the OLCT domain

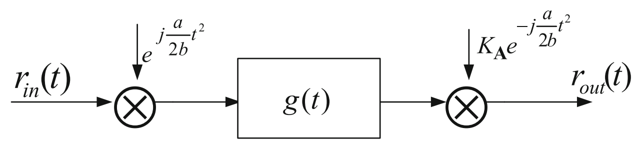

Fig. 3 The method of realizing the multiplicative filter in the OLCT domain by convolution in the time domain

4.2 The multiplicative filter in the OLCT domain

Many papers [6,8,15,16] discuss the use of the multiplicative filter designed by the FRFT or LCT to remove noise or distortion. Here, based on Theorem 1, we can discuss the design methods and the performance of the multiplicative filter designed by the OLCT.

The model of the multiplicative filter in the OLCT domain is shown in Fig. 2. The effect of the multiplicative filter in Fig. 2 can be written in the following equation:

$$r_{\text{out}}(t) = O_L^{\mathbf{A}^{-1}}\left[R_{\mathbf{A}}(u)\cdot\tilde{H}_{\mathbf{A}}(u)\right](t), \tag{35}$$

where $R_{\mathbf{A}}(u)$ denotes the OLCT of received signal $r_{\text{in}}(t)$. There are many possible types of multiplicative filter by designing different transform function $\tilde{H}_{\mathbf{A}}(u)$, such as low pass, high pass, band pass, band stop, and so on. In practical application, if the received signal $r_{\text{in}}(t)$ comprises two parts of the designed signal $f(t)$ and noise $n(t)$, then the desired signal can be reserved and the noise can be discarded to a large extent for increasing signal-to-noise ratio (SNR) through a multiplicative filter in the OLCT domain. For example, we are interested only in the frequency spectrum of the OLCT in the region $[u_l, u_h]$ of the designed signal $f(t)$. We also suppose that the OLCT parts of the received signal $r_{\text{in}}(t)$, $F_{\mathbf{A}}(u)$ and $N_{\mathbf{A}}(u)$, have no overlapping or minimal overlapping. According to Theorem 1, we choose function $H_{\mathbf{A}}(u)\mathrm{e}^{\frac{j}{2b}[-du^2+2u(du_0-b\omega_0)]}$ in (8) as the transfer function of the multiplicative filter. So, $\tilde{H}_{\mathbf{A}}(u)$ in Fig. 2 can be expressed as

$$\tilde{H}_{\mathbf{A}}(u) = H_{\mathbf{A}}(u)\mathrm{e}^{\frac{j}{2b}[-du^2+2u(du_0-b\omega_0)]}. \tag{36}$$

We can design the transfer function, $\tilde{H}_{\mathbf{A}}(u)$, so that it is constant over $[u_l, u_h]$ and zero or of rapid decay outside that region. Passing the output of the filter yields that part of the spectrum of $f(t)$ over $[u_l, u_h]$. By inverse OLCT, the designed signal $f(t)$ can be obtained.

The above-mentioned multiplicative filter in the OLCT domain can also be achieved through convolution in the time domain. Figure 3 shows a method of realizing the multiplicative filter in the OLCT domain by convolution in the time domain. From Fig. 3, the output of the filter is expressed as

$$r_{\text{out}}(t) = K_{\mathbf{A}}\mathrm{e}^{-j\frac{at^2}{2b}}[(r_{\text{in}}(t)\mathrm{e}^{j\frac{at^2}{2b}}) * g(t)]. \tag{37}$$

Making use of (18), the convolution function $g(t)$ in Fig. 3 can be designed as

$$g(t) = h(t)\mathrm{e}^{j\frac{at^2}{2b}}. \tag{38}$$

According to the inverse OLCT expression (3), we get

$$\begin{aligned} h(t) &= O_L^{\mathbf{A}^{-1}}[H_{\mathbf{A}}(u)](t) \\ &= CK_{\mathbf{A}^{-1}}\int_{-\infty}^{\infty} H_{\mathbf{A}}(u)\mathrm{e}^{j\frac{1}{2b}[-du^2+2u(du_0-b\omega_0)]}\mathrm{e}^{j\frac{u}{b}t} \\ &\quad\times\mathrm{e}^{-j\frac{1}{2b}(2u_0t+at^2)}\mathrm{d}u \\ &= \sqrt{\frac{1}{-j2\pi b}}\mathrm{e}^{-j\frac{d}{2b}u_0^2} \\ &\quad\times\int_{-\infty}^{\infty} H_{\mathbf{A}}(u)\mathrm{e}^{j\frac{1}{2b}[-du^2+2u(du_0-b\omega_0)]}\mathrm{e}^{j\frac{u}{b}t} \\ &\quad\times\mathrm{e}^{-j\frac{1}{2b}(2u_0t+at^2)}\mathrm{d}u. \end{aligned} \tag{39}$$

Substituting (36) into (39), we can obtain

$$\begin{aligned} h(t) &= \sqrt{\frac{1}{-j2\pi b}}\mathrm{e}^{-j\frac{d}{2b}u_0^2}\mathrm{e}^{-j\frac{1}{2b}(2u_0t+at^2)}\int_{-\infty}^{\infty}\tilde{H}_{\mathbf{A}}(u)\mathrm{e}^{j\frac{u}{b}t}\mathrm{d}u \\ &= \sqrt{\frac{j2\pi}{b}}\mathrm{e}^{-j\frac{d}{2b}u_0^2}\mathrm{e}^{-j\frac{1}{2b}(2u_0t+at^2)}\widehat{h}\left(\frac{t}{b}\right). \end{aligned} \tag{40}$$

From Eqs. (38) and (40), we get

$$g(t) = \sqrt{\frac{j2\pi}{b}}\mathrm{e}^{-j\frac{d}{2b}u_0^2}\mathrm{e}^{-j\frac{1}{b}u_0t}\widehat{h}\left(\frac{t}{b}\right), \tag{41}$$

where $\widehat{h}(t)$ is the inverse FT of $\tilde{H}_{\mathbf{A}}(u)$. According to the requirement of filter or the time-frequency distribution [2,6,15] of received signal $r_{\text{in}}(t)$, the transfer function $\tilde{H}_{\mathbf{A}}(u)$ of the multiplicative filter should be designed in the OLCT domain first. Substituting (41) into (37), we can obtain

$$r_{\text{out}}(t) = \frac{1}{b}\mathrm{e}^{-j\frac{at^2}{2b}}\left(r_{\text{in}}(t)\mathrm{e}^{j\frac{at^2}{2b}}\right) * \left(\mathrm{e}^{-j\frac{1}{b}u_0t}\widehat{h}\left(\frac{t}{b}\right)\right). \tag{42}$$

According to Theorem 1, it will be easy to prove that the output of filter shown in Fig. 3 is same as that in Fig. 2. In an actual application, the method shown in Fig. 3 has less computational complexity than that in Fig. 2 for the reason that the major computation load of the former method is convolution, which can be done by the classical FFT, while the latter method needs to calculate OLCT twice, and there exists still no satisfactory fast digital algorithm for OLCT so far. In the following paragraphs, we will analyze the computation complexity of the multiplicative filter achieved in the time domain from (42) in detail.

According to the conventional convolution theorem, Eq. (42) can be written as

$$\begin{aligned} r_{\text{out}}(t) &= \frac{1}{b} e^{-j\frac{at^2}{2b}} \left(r_{\text{in}}(t) e^{j\frac{at^2}{2b}} \right) * \left(e^{-j\frac{1}{b}u_0 t} \widehat{h}\left(\frac{t}{b}\right) \right) \\ &= \frac{1}{b} e^{-j\frac{at^2}{2b}} F^{-1}\left[F[\varphi_1](u) F[\varphi_2](u) \right](t), \end{aligned} \tag{43}$$

where $\varphi_1(t) = r_{\text{in}}(t) e^{j\frac{at^2}{2b}}$, $\varphi_2(t) = e^{-j\frac{1}{b}u_0 t}\widehat{h}(\frac{t}{b})$, $F^{-1}(\cdot)$ denote the inverse FT operator. Since $\widehat{h}(t)$ is the inverse FT of $\tilde{H}_{\mathbf{A}}(u)$, Eq. (43) can be expressed as

$$\begin{aligned} r_{\text{out}}(t) &= \frac{1}{b} e^{-j\frac{at^2}{2b}} F^{-1}\left[F[\varphi_1](u) b \tilde{H}_{\mathbf{A}}(bu + u_0) \right](t) \\ &= e^{-j\frac{at^2}{2b}} F^{-1}[\varphi_3](t), \end{aligned} \tag{44}$$

where $\varphi_3(u) = F[\varphi_1](u)\tilde{H}_{\mathbf{A}}(bu + u_0)$. According to (44), the FFT can be used to reduce the computation complexity of this filtering method. Since the computation complexity of FFT is $o(N \log_2^N)$, then we can draw the conclusion that the computation complexity of the multiplicative filter achieved in the time domain can be reduced to $o(N \log_2^N)$ for N point of samples.

5 Conclusion

In this paper, we first introduced the definition and some properties of OLCT. Then, convolution theorems for OLCT, that is, Theorems 1 and 2 have been derived in detail, presenting the OLCT domain's behavior of a new convolution as well as a product between two functions in the time domain. We also obtain the expressions for OLCT of a new correlation operation of two functions based on Theorem 1. Since the correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed and complex-conjugated, the property of the new convolution structure for OLCT results in the property of the correlation for OLCT. All these results developed for OLCT can be applied to a large class of signals and system outputs and can be regarded as unified and extended versions of previously developed convolution and correlation theorems for special cases of the OLCT. Finally, as an application, utilizing convolution theorems for OLCT derived in this paper, sampling of band-limited signals in the OLCT domain has been further investigated. The formulas of uniform sampling and low-pass reconstruction are obtained. Moreover, the multiplicative filter in the OLCT domain has also been discussed in this paper. A practical method to achieve the multiplicative filter through convolution in the time domain is proposed, which can be realized by classical FFT and has the same capability, but less computational complexity compared with the method achieved in the OLCT domain.

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