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INVITED PAPER

Recent developments in the queueing problem

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Abstract
A group of agents must be served in a facility. The facility can serve only one agent at a time and agents incur waiting costs. The queueing problems is concerned with finding the order to serve agents and the monetary transfers. It can be solved by taking various approaches: the cooperative game theoretic approach, the normative approach, the strategic approach, the bargaining approach, and the combination of these approaches. In this paper, we provide a survey on the recent developments in the queueing problem.

Keywords Queueing problem · Cooperative game theoretic approach · Normative approach · Strategic approach · Bargaining approach

JEL Classification C72 · D63 · D82

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1 Introduction

Suppose there is a group of agents who must be served in a facility which can process only one agent at a time. Agents incur waiting costs. An agent's waiting cost is assumed to be constant per unit of time, but differs across agents. For such a queueing problem, we are interested in finding the order to serve agents and the monetary transfers they should receive.

There are a number of real-life examples of the queueing problem. Queueing arises when agents cannot coordinate amongst themselves on the service time (long queues at the ticket office, department store, etc). Even if they can coordinate, queueing can still arise because agents may have identical preferences regarding the service time and technological constraints prevent all agents being served simultaneously. For example, all faculty members of some department want to move into a new building as soon as possible, consumers want to implement a new computer program as soon as they can and so on.

There are two main approaches to the queueing problem. One is the *strategic approach* which focuses on the fact that the waiting cost of an agent is typically known to the agent only. Hence, incentives have to be provided to get agents to reveal their private information truthfully. This literature examines the tradeoffs involved in providing such incentives. The other approach assumes away the private information aspect and focuses instead on the *fairness aspect* of the problem. Since all agents cannot be served simultaneously in a queueing situation, some agents have to wait. Fairness demands that agents served earlier compensate those served later. What is a fair way of doing this? This question was raised by Maniquet (2003) and a large literature has developed in its wake. A more recent literature combines the strategic and normative approaches. In this paper, we provide a survey on the recent developments in the area of queueing problems.

After laying down the basic concepts in Sect. 2, Sect. 3 begins with the cooperative game theoretic approach, which applies well-known solution concepts of cooperative game theory to the queueing problem. In Sect. 4, we discuss the normative approach that proposes a set of axioms which a desirable rule should satisfy and then characterizes all rules satisfying those axioms. In Sect. 5, we investigate the existence of rules satisfying *strategy-proofness,* which requires that an agent should not have an incentive to misrepresent her waiting cost. In Sect. 6, we investigate the implications of normative requirements such as *egalitarian equivalence* (Pazner and Schmeidler 1978) and the *identical preferences lower bound* together with *queue-efficiency* and *strategy-proofness*. Section 7 tries to solve the queueing problem by adopting a bargaining approach which builds up a bargaining protocol such that players can negotiate among themselves to resolve the queueing conflicts. Finally, Section 8 discusses possible generalizations of the queueing problem for future research.

1.1 Brief history

An analysis on the queueing problem was initiated by Dolan (1978) who proposed a *strategy-proof,* but not *budget-balanced* rule. Suijs (1996) and Mitra (2001, 2002) provide a *strategyproof* and *budget-balanced* rule which has been later characterized by Kayi and Ramaekers (2015), Hashimoto and Saitoh (2012), and Chun, Mitra, and Mutuswami (in press). On the other hand, Maniquet (2003) focused on the fairness aspect of the problem which he addressed by applying the Shapley value, one of the most well-known solution concept of cooperative game theory. The resulting allocation rule is now known as the *minimal transfer rule*. This approach has also been adopted by Chun (2006a) who proposed a "pessimistic" definition for the worth of a coalition. Even though he applied the same Shapley value, he obtained a different rule, namely, the *maximal transfer rule*. Recent studies try to combine the strategic and the normative points of view. They characterize all rules satisfying *strategyproofness* together with some normative axioms (Kayi and Ramaekers 2010; Hashimoto and Saitoh 2012; Chun et al. 2014b, in press; Chun and Yengin 2017). Finally, Ju et al. (2014) adopted a bargaining approach to the queueing problem.[1]

2 Basic concepts

2.1 The model

Let $I \equiv \{1, 2, \ldots\}$ be an (infinite) universe of "potential" agents, and $\mathcal{N}$ be the family of non-empty finite subsets of I. Each agent $i \in I$ is characterized by her (unit) waiting cost, $\theta_i \geq 0$. Given $N \in \mathcal{N}$, each agent $i \in N$ is assigned a position $\sigma_i \in \{1, \ldots, |N|\}$ in a queue[2] and a monetary transfer $t_i \in \mathbb{R}$. Each agent has one job to process and the facility can serve only one agent at a time. Each agent needs the same amount of processing time which is normalized to one. If agent $i \in N$ is served in the σ_ith position, her total waiting cost is $(\sigma_i - 1)\theta_i$. Since each agent is assumed to have a quasi-linear utility function, the utility of agent i from the bundle (σ_i, t_i) is given by

$$u(\sigma_i, t_i; \theta_i) = -(\sigma_i - 1)\theta_i + t_i.$$

A *queueing problem* for $N \in \mathcal{N}$ is defined as the vector of waiting costs of all agents $\theta = (\theta_i)_{i \in N} \in \mathbb{R}^N_+$.[3] Let $\mathcal{Q}^N$ be the class of all problems for N and $\mathcal{Q} = \cup \mathcal{Q}^N$. An *allocation* for $\theta \in \mathcal{Q}^N$ is a pair (σ, t), where for each $i \in N$, σ_i denotes agent i's position in the queue and t_i the monetary transfer to her. An allocation is *feasible* if no two agents are assigned the same position. Thus, the set of feasible allocations $Z(\theta)$ consists of all pairs (σ, t) such that for all $i, j \in N$, $i \neq j$ implies $\sigma_i \neq \sigma_j$. For each profile θ and each $i \in N$, let $\theta_{N \setminus \{i\}}$ be the vector of waiting costs of all agents except i.

Given $\theta \in \mathcal{Q}^N$, an allocation $(\sigma, t) \in Z(\theta)$ is *queue-efficient* if it minimizes the aggregate waiting costs, that is, for all $(\sigma', t') \in Z(\theta)$, $\sum_{i \in N}(\sigma_i - 1)\theta_i \leq \sum_{i \in N}(\sigma'_i - 1)\theta_i$. It is straightforward to check that an efficient queue serves agents in the non-increasing order of their waiting costs and that any queue with this property is also

[1] See Chun (2016) for a survey of the literature on the queueing problem.

[2] For any set A, $|A|$ denotes the cardinality of A.

[3] $\mathbb{R}_+$ denotes the non-negative orthant of the real line.

efficient. The efficient queue of a problem does not depend on the transfers. Moreover, it is unique except for agents with identical waiting costs. These agents have to be served consecutively but in any order. The set of efficient queues for $\theta \in \mathcal{Q}^N$ is denoted by $E(\theta)$. An allocation $(\sigma, t) \in Z(\theta)$ is *budget-balanced* if $\sum_{i \in N} t_i = 0$. An allocation rule, or simply a *rule,* is a function φ defined on $\mathcal{Q}$ which associates with each $N \in \mathcal{N}$ and each $\theta \in \mathcal{Q}^N$ a tuple $\varphi(\theta) = (\sigma, t)$ of feasible allocations. The pair $\varphi_i(\theta) = (\sigma_i, t_i)$ is the assignment to agent i by φ. Given $\theta \in \mathcal{Q}^N$, $(\sigma, t) \in Z(\theta)$, and $i \in N$, let $P_i(\sigma) = \{j \in N \mid \sigma_j < \sigma_i\}$ be the set of agents preceding agent i in the queue σ and $F_i(\sigma) = \{j \in N \mid \sigma_j > \sigma_i\}$ the set of agents following her in the queue σ. The set of all possible queues for N is $\Sigma(N)$. Similarly, for all $S \subseteq N$, the set of all possible queues for S is $\Sigma(S)$.

Remark 2.1 The efficient queue is unique at all profiles where no two agents have identical waiting costs. However, if there are some agents with identical waiting costs, then the efficient queue is not unique. Since *queue-efficiency* is the only axiom that we impose on the queue, it is not clear which queue we should choose for the profile. In this paper, we implicitly assume the existence of a tie-breaking rule, which selects an efficient queue whenever there is more than one such queue. The same rule is used to break ties when a queue involving subsets of agents has to be selected. Let $\mathcal{T}$ be the set of all possible tie-breaking rules for N and τ be a typical element of $\mathcal{T}$. We note that our result applies for any choice of tie-breaking rule.

Remark 2.2 A queueing problem can be generalized to a *sequencing problem,* which is a list (r, θ), where $r \equiv (r_i)_{i \in N}$ is the vector representing the processing time of agents and $\theta \equiv (\theta_i)_{i \in N}$ is the vector of (unit) waiting costs. For a sequencing problem, each agent is characterized by the processing time and the waiting cost. A *queueing problem* is obtained by assuming that agents need the same amount of processing time, that is, for each $i \in N$, $r_i = 1$. On the other hand, a *scheduling problem* is obtained by assuming that agents have the same waiting cost, that is, for each $i \in N$, $\theta_i = 1$. The sequencing problems[4] has been studied by Suijs (1996), Mitra (2002), Hain and Mitra (2004), van den Brink and Chun (2012), Parikshit and Mitra (2017), and others, and the scheduling problem by Cres and Moulin (2001), Moulin (2007), and others.

2.2 Basic axioms

Now we introduce basic axioms which we will impose on rules. *Queue-efficiency* requires that the rule should choose a queue which minimizes the aggregate waiting costs. *Budget-balance* requires that the sum of all transfers should be equal to zero. It is a strengthening of *no budget deficit,* or *no-deficit,* which requires that the sum of all transfers should not be positive.

Queue-efficiency For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, $\sigma \in E(\theta)$.

Budget-balance For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, $\sum_{i \in N} t_i = 0$.

No-deficit For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, $\sum_{i \in N} t_i \leq 0$.

[4] Also, see Curiel et al. (1989) for a sequencing problem with an initial queue and Chun (2011) for a sequencing problem with bilateral transfers.

Equal treatment of equals requires that two agents with the same waiting cost should end up with the same utilities. The *identical preferences lower bound* (Moulin 1990, 1991) requires that each agent should be at least as well off as she would be, under *queue-efficiency*, *budget-balance,* and *equal treatment of equals,* if all other agents had the same preferences as her. Note that if a rule satisfies *queue-efficiency, budget-balance,* and *equal treatment of equals* and all agents have the same waiting costs as agent i, then all agents end up with the same utilities of $-\frac{|N|-1}{2}\theta_i$. Therefore, the *identical preferences lower bound* requires that all agents should be better off by not having the same waiting cost.

Equal treatment of equals For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta)$, and all i, $j \in N$, if $\theta_i = \theta_j$, then $u(\sigma_i, t_i; \theta_i) = u(\sigma_j, t_j; \theta_j)$.

Identical preferences lower bound For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta)$, and all $i \in N$, $u(\sigma_i, t_i; \theta_i) \geq -\frac{|N|-1}{2}\theta_i$.

2.3 Rules

We first consider a two-agent queueing problem. Suppose that there are two agents denoted by agents 1 and 2 such that $\sigma_1 < \sigma_2$. If agent 2 moves up, then her utility gains are θ_2. She enjoys the same utility whether she receives $\frac{\theta_2}{2}$ at σ_2 or pays the same amount at σ_1. On the other hand, if agent 1 is served later, then her utility losses are θ_1. She enjoys the same utility whether she pays $\frac{\theta_1}{2}$ at σ_1 or receives the same amount at σ_2. Therefore, it is natural to expect the actual transfer will be determined by these two bounds. The following two rules select an efficient queue and transfer either the minimum or the maximum of these two bounds for two-agent problems.

The minimal transfer rule (Maniquet 2003), which chooses the minimum of the two bounds for two-agent problems, selects an efficient queue and transfers to each agent a half of her unit waiting cost multiplied by the number of her predecessors minus a half of the sum of the unit waiting costs of her followers:

Minimal transfer rule, φ^M: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$,

$$\varphi^M(\theta) = \left\{ (\sigma^M, t^M) \in Z(\theta) |\ \sigma^M \in E(\theta) \text{ and } \forall i \in N, \right.$$
$$\left. t_i^M = (\sigma_i^M - 1)\frac{\theta_i}{2} - \sum_{j \in F_i(\sigma^M)} \frac{\theta_j}{2} \right\}.$$

On the other hand, the maximal transfer rule (Chun 2006a), which chooses the maximum of the two bounds for two-agent problems, selects an efficient queue and transfers to each agent a half of the sum of the unit waiting costs of her predecessors minus a half of her unit waiting cost multiplied by the number of her followers:

Maximal transfer rule, φ^C: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$,

$$\varphi^C(\theta) = \left\{ (\sigma^C, t^C) \in Z(\theta) | \ \sigma^C \in E(\theta) \ \text{ and } \ \forall i \in N, \right.$$
$$\left. t_i^C = \sum_{j \in P_i(\sigma^C)} \frac{\theta_j}{2} - (|N| - \sigma_i^C) \frac{\theta_i}{2} \right\}.$$

Both the minimum transfer and the maximum transfer rules satisfy *queue-efficiency, budget-balance, equal treatment of equals,* and the *identical preferences lower bound*. Moreover, these two rules are obtained by applying the Shapley value (Shapley 1953), one of the most widely discussed solution concept in cooperative games, to appropriately defined queueing games (see Sect. 3 for details).

The next three rules satisfy *strategyproofness,* which requires truth telling to be a dominant strategy for each agent and for each (announced) state. The first rule, the symmetrically balanced VCG rule,[5] was first introduced by Suijs (1996) and Mitra (2001) and later characterized by Kayi and Ramaekers (2015), Chun, Mitra, and Mutuswami (2014a, in press), and Hashimoto and Saitoh (2012).

Symmetrically balanced VCG rule, φ^B: For all $N \in \mathcal{N}$ with $|N| \geq 3$ and all $\theta \in \mathcal{Q}^N$,

$$\varphi^B(\theta) = \left\{ (\sigma^B, t^B) \in Z(\theta) | \ \sigma^B \in E(\theta) \text{ and } \forall i \in N, \right.$$
$$\left. t_i^B = \sum_{j \in P_i(\sigma^B)} \frac{\sigma_j^B - 1}{|N| - 2} \theta_j - \sum_{k \in F_i(\sigma^B)} \frac{|N| - \sigma_k^B}{|N| - 2} \theta_k \right\}.$$

The symmetrically balanced VCG rule satisfies *queue-efficiency, budget-balance, equal treatment of equals,* and the *identical preferences lower bound*.

Mitra and Mutuswami (2011) introduce and characterize the family of pivotal rules on the basis of *pairwise strategyproofness,* which requires that any pair of agents can not benefit by deviating from truth-telling. The pivotal and the reward based pivotal rules belong to this family of pivotal rules (see Subsect. 5.3 for details):

Pivotal rule, φ^P: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$,

$$\varphi^P(\theta) = \left\{ (\sigma^P, t^P) \in Z(\theta) | \ \sigma^P \in E(\theta) \ \text{ and } \ \forall i \in N, \ \ t_i^P = - \sum_{j \in F_i(\sigma^P)} \theta_j \right\}.$$

[5] The family of VCG rules is due to Vickrey (1961), Clarke (1971), and Groves (1973).

Reward-based pivotal rule, φ^R: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$,

$$\varphi^R(\theta) = \left\{ (\sigma^R, t^R) \in Z(\theta) | \, \sigma^R \in E(\theta) \text{ and } \forall i \in N, \; t_i^R = \sum_{j \in P_i(\sigma^R)} \theta_j \right\}.$$

Both the pivotal and the reward-based pivotal rules satisfy *queue-efficiency* and *equal treatment of equals,* but fail to satisfy *budget-balance*. The pivotal rule does not satisfy the *identical preferences lower bound,* but the reward-based pivotal rule does.

We note that all five rules assign a unique allocation if and only if all agents have different weighting costs. If two agents have the same waiting cost, then the efficient queue is not unique, and consequently the allocations chosen by the rule are not unique either. However, agents' utilities do not depend on the choice of efficient queues if the transfer is determined according to the five rules. Thus, all rules are *essentially single-valued* in the sense that for a given problem, each agent's utility is the same at all allocations that each rule chooses. As a consequence, any efficient queue can be chosen to calculate the utilities assigned by the rules.

3 Cooperative game theoretic approach

We discuss how the queueing problem can be solved by applying solutions developed in cooperative game theory:

3.1 Cooperative games

Let $N = \{1, \ldots, n\}$ be the set of *players.* A nonempty subset $S \subseteq N$ is a *coalition*. A cooperative game with transferable utility, or a *game,* is a real-valued function v defined on all coalitions $S \subseteq N$ satisfying $v(\emptyset) = 0$. The number $v(S)$ is the *worth* of S. Let Γ^N be the class of games with player set N. A *solution* is a function $\phi \colon \Gamma^N \to \mathbb{R}^N$, which associates with every game $v \in \Gamma^N$ a vector $\phi(v) = (\phi_i(v))_{i \in N} \in \mathbb{R}^N$. The number $\phi_i(v)$ represents the payoff to player i in game v.

For all $v \in \Gamma^N$, let $X(v) = \{x \in \mathbf{R}^N | \sum_{i \in N} x_i = v(N)\}$ be the set of *efficient allocations* and $I(v) = \{x \in X(v) | \, x_i \geq v(\{i\})$ for all $i \in N\}$ be the set of *imputations*. For all $x \in I(v)$, its *excess vector* $e(v, x) \in \mathbf{R}^{2^N}$ is defined by setting for all $S \subseteq N$, $e_S(v, x) = v(S) - \sum_{i \in S} x_i$. Each coordinate of the excess vector measures the amount by which the worth of the coalition exceeds its total payoff at x. The *core* is the set of imputations at which no excess is greater than zero: $Core(v) = \{x \in I(v) |$ for all $S \subset N$, $\sum_{i \in S} x_i \geq v(S)\}$.

For all $y \in \mathbf{R}^{2^N}$, let $\widetilde{y} \in \mathbf{R}^{2^{|N|}}$ be obtained by rearranging the coordinates of y in non-increasing order. For all $y, z \in \mathbf{R}^{2^N}$, *$y$ is lexicographically smaller than z* if either (i) $\widetilde{y}_1 < \widetilde{z}_1$ or (ii) there exists $\ell > 1$ such that $\widetilde{y}_\ell < \widetilde{z}_\ell$ and for all $k < \ell$, $\widetilde{y}_k = \widetilde{z}_k$.

The most well-known solutions for games are the Shapley value (Shapley 1953) and the nucleolus (Schmeidler 1969). The Shapley value assigns to each player a payoff equal to a weighted average of her marginal contributions to all possible coalitions, with weights being determined by the sizes of coalitions. The nucleolus chooses the unique allocation from the set of imputations which minimizes the excess of coalitions in the lexicographic way.

Shapley value, ***Sh***: For all $v \in \Gamma^N$ and all $i \in N$,

$$Sh_i(v) = \sum_{S \subseteq N, S \ni i} \frac{(|S|-1)!|N \backslash S|!}{|N|!} \big[v(S) - v(S \backslash \{i\}) \big].$$

Nucleolus, ***Nu***: For all $v \in \Gamma^N$ such that $I(v) \neq \emptyset$,

$$Nu(v) = \left\{ x \in I(v) \,\middle|\, \begin{array}{l} \text{for all } x' \in I(v)\backslash\{x\}, e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,x') \end{array} \right\}$$

The *prenucleolus* chooses the unique allocation from the set of efficient allocations which minimizes the excess of the coalitions in the lexicographic way. For some cooperative games, the set of imputations can be empty and this is the reason we focus on the prenucleolus.

Prenucleolus, ***PN***: For all $v \in \Gamma^N$,

$$PN(v) = \left\{ x \in X(v) \,\middle|\, \begin{array}{l} \text{for all } x' \in X(v)\backslash\{x\}, e(v,x) \text{ is} \\ \text{lexicographically smaller than } e(v,x') \end{array} \right\}$$

A game is *convex* if for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. It is well-known that a convex game has a non-empty core. Moreover, the Shapley value and the nucleolus select allocations in the core.

3.2 The shapley value in queueing games

To apply the solutions of games to queueing problems, we need to define a worth of each coalition. First, it can be defined as the minimum waiting cost incurred by its members under the optimistic assumption that they are served before the non-coalitional members. That is, for all $S \subseteq N$, its worth $v_O(S)$ of the optimistic queueing game is defined by setting:

$$v_O(S) = -\sum_{i \in S} (\sigma_i^* - 1)\theta_i,$$

where $\theta_S = (\theta_i)_{i \in S}$ and $\sigma^* \in E(\theta_S)$. By applying the Shapley value to the optimistic queueing game $v_O = (v_O(S))_{S \subseteq N}$, we can show that the resulting payoff to each player is equal to the utility assigned by the minimal transfer rule.

Theorem 3.1 (Maniquet 2003) *Let $\theta \in \mathcal{Q}^N$. Let $z = (\sigma, t) \in Z(\theta)$ be such that agents' utilities at z are equal to the payoff vector obtained by applying the Shapley value to v_O. Then, $\sigma \in E(\theta)$ and for all $i \in N$, $t_i = (\sigma_i - 1)\frac{\theta_i}{2} - \sum_{j \in F_i(\sigma)} \frac{\theta_j}{2}$.*

Alternatively, the worth of each coalition can be defined as the minimum waiting cost incurred by its members under the pessimistic assumption that they are served after the non-coalitional members. That is, for all $S \subseteq N$, its worth $v_P(S)$ is defined by setting:

$$v_P(S) = -\sum_{i \in S}(|N| - |S| + \sigma_i^* - 1)\theta_i,$$

where $\theta_S = (\theta_i)_{i \in S}$ and $\sigma^* \in E(\theta_S)$. Now we apply the Shapley value to pessimistic queueing game $v_P = (v_P(S))_{S \subseteq N}$ and obtain the maximal transfer rule.

Theorem 3.2 (Chun 2006a) *Let $\theta \in \mathcal{Q}^N$. Let $z = (\sigma, t) \in Z(\theta)$ be such that agents' utilities at z are equal to the payoff vector obtained by applying the Shapley value to v_P. Then, $\sigma \in E(\theta)$ and for all $i \in N$, $t_i = \sum_{j \in P_i(\sigma)} \frac{\theta_j}{2} - (|N| - \sigma_i)\frac{\theta_i}{2}$.*

These results show the importance of the definition of the worth of a coalition in queueing problems. For some classes of problems,[6] it makes no difference whether the coalitional members have priority over the non-coalitional members, or the non-coalitional members have priority over the coalitional members. If the Shapley value is applied, we obtain the same recommendation. However, for queueing problems, this is not the case: depending upon who has priority, the resulting rule has very different properties.

3.3 Coincidence of solutions in queueing games

We apply the nucleolus to the pessimistic queueing game and identify the resulting rule. Interestingly, we end up with the same rule: the Shapley value and the nucleolus coincide for the pessimistic queueing game (Chun and Hokari 2007). To show this, we introduce an *auxiliary pessimistic queueing game* $\tilde{v}_P$, in which the worth of coalition S is obtained by adding $\sum_{i \in S}(n-1)\theta_i$ to $v_P(S)$, that is, for all $S \subseteq N$, $\tilde{v}_P(S) = v_P(S) + \sum_{i \in S}(n-1)\theta_i$. Note that $\tilde{v}_P$ satisfies following conditions:

(i) for each $i \in N$, $\tilde{v}_P(\{i\}) = 0$,
(ii) for each $S \subseteq N$ such that $|S| \geq 2$, $\tilde{v}_P(S) = \sum_{T \subseteq S, |T|=2} \tilde{v}_P(T)$ and $\tilde{v}_P(S) \geq 0$.

As shown in Deng and Papadimitriou (1994) and van den Nouweland et al. (1996), these two conditions are sufficient to guarantee the coincidence of the Shapley value and the nucleolus. Finally, the coincidence for the pessimistic queueing game follows from the fact that both the Shapley value and the nucleolus satisfy *zero-independence,* which requires that adding a constant to the worth of coalitions containing player i should affect her payoff by the constant.

Remark 3.3 For all $v \in \Gamma^N$ and all $i \in N$, let $M_i(v) \equiv v(N) - v(N \backslash \{i\})$ and $m_i(v) \equiv v(\{i\})$. Then, the τ-value (Tijs 1987) selects the maximal feasible allocation on the line connecting $M(v) \equiv (M_i(v))_{i \in N}$ and $m(v) \equiv (m_i(v))_{i \in N}$. The auxiliary pessimistic queueing game $\tilde{v}_P$ is convex and $m(\tilde{v}_P) = 0$. Moreover, it is easy to see

[6] For example, the bankruptcy problem discussed in Thomson (2003).

that for all $j \in N$, $\tilde{v}_P(N) - \tilde{v}_P(N\backslash\{j\}) = \sum_{S \ni j, |S|=2} \tilde{v}_P(S)$ and the τ-value chooses the middle point on the line connecting $M(\tilde{v})$ and $m(\tilde{v})$. By using the fact that the τ-value satisfies *zero-independence,* we can show that the τ-value coincides with the Shapley value and the nucleolus for pessimistic queueing games.

For the optimistic queueing game, the set of imputations is empty. However, the optimistic queueing game satisfies the following two conditions:

(i) for each $i \in N$, $v_O(\{i\}) = 0$,
(ii) for each $S \subseteq N$ such that $|S| \geq 2$, $v_O(S) = \sum_{T \subseteq S, |T|=2} v_O(T)$.

As shown in Kar et al. (2009), these two conditions are sufficient to guarantee the coincidence of the Shapley value and the prenucleolus. Therefore, these two solutions make the same recommendation in the optimistic queueing game.[7]

Remark 3.4 As shown in Chun and Hokari (2007), the minimal transfer rule coincides with the serial cost sharing rule[8] (Moulin and Shenker 1992) and the maximal transfer rule coincides with the decreasing serial cost sharing rule (de Frutos 1998).

4 Normative approach

We present characterizations of the minimal and the maximal transfer rules by imposing various axioms specifying how a rule should respond to changes in the waiting cost or population. Also, we explore the implications of *no-envy* (Foley 1967) and discuss whether three fairness requirements, *no-envy,* the *identical preferences lower bound,* and *egalitarian equivalence,* are compatible in the queueing problem.

4.1 Independence

Suppose that an agent's waiting cost changes. One could take two different perspectives with regards to how the allocation should be affected by this change: (i) an increase in an agent's waiting cost would affect her predecessors, but not her followers, or (ii) a decrease in an agent's waiting cost would affect her followers, but not her predecessors. *Independence of preceding costs* reflects the first perspective whereas *independence of following costs* reflects the second perspective.

Independence of preceding costs: For all $N \in \mathcal{N}$, all $\theta, \theta' \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta)$, all $(\sigma', t') \in \varphi(\theta')$, and all $k \in N$, if for all $i \in N\backslash\{k\}$, $\theta_i = \theta'_i$ and $\theta_k < \theta'_k$, then for all $j \in N$ such that $\sigma_j > \sigma_k$, $u_j(\sigma_j, t_j; \theta_j) = u_j(\sigma'_j, t'_j; \theta'_j)$.

Independence of following costs: For all $N \in \mathcal{N}$, all $\theta, \theta' \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta)$, all $(\sigma', t') \in \varphi(\theta')$, and all $k \in N$, if for all $i \in N\backslash\{k\}$, $\theta_i = \theta'_i$ and $\theta_k > \theta'_k$, then for all $j \in N$ such that $\sigma_j < \sigma_k$, $u_j(\sigma_j, t_j; \theta_j) = u_j(\sigma'_j, t'_j; \theta'_j)$.

7 Kar et al. (2009), by taking a general set of queueing games that includes all convex combinations of the optimistic queueing game and the pessimistic queueing game, obtained the coincidence between the Shapley value and the prenucleolus.

8 Moulin (2007) makes the same observation for the scheduling problem.

We are ready to state our characterization results on the basis of independence requirements.

Theorem 4.1 *(1) (Maniquet 2003) The minimal transfer rule is the only rule satisfying queue-efficiency, budget-balance, equal treatment of equals, and independence of preceding costs.*[9]
(2) (Chun 2006a) The maximal transfer rule is the only rule satisfying queue-efficiency, budget-balance, equal treatment of equals, and independence of following costs.

4.2 Monotonicity and equal responsibility

Once again, suppose that the waiting cost of one agent increases. One could take two different perspectives with regards to how the allocation should be affected by this change: (i) one may feel that she deserves greater compensation for her waiting, which will affect other agents in a negative direction or (ii) one may feel that she should be required to pay more for having the service, which will affect other agents in a positive direction. *Negative cost monotonicity* requires that an increase in an agent's waiting cost should cause all other agents to weakly lose. On the other hand, *positive cost monotonicity* requires that an increase in an agent's waiting cost should cause all other agents to weakly gain.

Negative cost monotonicity: For all $N \in \mathcal{N}$, all $\theta, \theta' \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta)$, all $(\sigma', t') \in \varphi(\theta')$, and all $k \in N$, if for all $i \in N\backslash\{k\}$, $\theta_i = \theta'_i$ and $\theta_k < \theta'_k$, then for all $i \in N\backslash\{k\}$, $u_i(\sigma_i, t_i; \theta_i) \geq u_i(\sigma'_i, t'_i; \theta'_i)$.

Positive cost monotonicity: For all $N \in \mathcal{N}$, all $\theta, \theta' \in \mathcal{Q}^N$, all $(\sigma, t) \in \varphi(\theta)$, all $(\sigma', t') \in \varphi(\theta')$, and all $k \in N$, if for all $i \in N\backslash\{k\}$, $\theta_i = \theta'_i$ and $\theta_k < \theta'_k$, then for all $i \in N\backslash\{k\}$, $u_i(\sigma_i, t_i; \theta_i) \leq u_i(\sigma'_i, t'_i; \theta'_i)$.

The next two axioms are concerned with changes in the population. If some agent in the queue leaves, then under *queue-efficiency,* the queue can be assumed to be affected minimally, that is, her precedents remain at the same position, but her followers move forward by one position. However, the monetary compensations may need to be adjusted. *Last-agent equal responsibility* requires that upon the departure of the agent served last; all other agents should remain at the same position and their transfers should be affected by the same amount. On the other hand, *first-agent equal responsibility* requires that upon the departure of the agent served first, all other agents should move forward by one position and their transfers should be affected by the same amount. For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $k \in N$, let $\theta_{N\backslash\{k\}} = (\theta_i)_{i \in N\backslash\{k\}}$. Note that $\theta_{N\backslash\{k\}} \in \mathcal{Q}^{N\backslash\{k\}}$.

Last-agent equal responsibility: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, if agent $k \in N$ is such that $\sigma_k = |N|$, then there exists $(\sigma', t') \in \varphi(\theta_{N\backslash\{k\}})$ such that for all $i \in N\backslash\{k\}$, $\sigma'_i = \sigma_i$ and $t'_i = t_i + \frac{t_k}{|N|-1}$.

[9] Since we assume an existence of a tie-breaking rule, we always have a unique queue satisfying *queue-efficiency*. As a consequence, we do not need either *Pareto-indifference* or *anonymity* in the statement of our theorems.

First-agent equal responsibility: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, if agent $k \in N$ is such that $\sigma_k = 1$, then there exists $(\sigma', t') \in \varphi(\theta_{N \setminus \{k\}})$ such that for all $i \in N \setminus \{k\}$, $\sigma'_i = \sigma_i - 1$ and $t'_i = t_i + \frac{t_k}{|N|-1}$.

Our second characterizations of the two rules are based on *cost monotonicity* and *equal responsibility* axioms.

Theorem 4.2 (1) (Maniquet 2003) *The minimal transfer rule is the only rule satisfying no-deficit, the identical preferences lower bound, negative cost monotonicity, and last-agent equal responsibility.*
(2) (Chun 2006a) *The maximal transfer rule is the only rule satisfying no-deficit, the identical preferences lower bound, positive cost monotonicity, and first-agent equal responsibility.*

4.3 No-envy

No-envy, introduced by Foley (1967), requires that no agent should end up with a higher utility by consuming what any other agent consumes. It is a standard fairness requirement studied in a wide class of problems (Thomson and Varian 1985; Thomson 2013). Given $N \in \mathcal{N}$ and $\theta \in \mathcal{Q}^N$, an allocation $(\sigma, t) \in Z(\theta)$ satisfies *no-envy* if for all $i, j \in N$, $u_i(\sigma_i, t_i; \theta_i) \geq u_i(\sigma_j, t_j; \theta_i)$. Let $F(\theta)$ be the set of all no-envy allocations for $\theta \in \mathcal{Q}^N$.

No-envy: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, $(\sigma, t) \in F(\theta)$.

Remark 4.3 As shown in Svensson (1983) in economies with indivisible goods and Chun et al. (2014b) for queueing problems,[10] *no-envy* implies *queue-efficiency.* Also, *no-envy* is equivalent to *group no-envy,*[11] and the set of *envy-free* allocations coincides with the set of *equal income Walrasian allocations.*[12]

Now we present a simple way of checking whether a rule satisfies *no-envy.*:

Theorem 4.4 (Chun 2006b) *A rule φ satisfies no-envy if and only if for all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, $\sigma \in E(\theta)$ and for all $\sigma_i = 1, \ldots, |N| - 1$, $\theta_{\sigma_i} \geq t_{\sigma_i+1} - t_{\sigma_i} \geq \theta_{\sigma_i+1}$.*

Example 4.5 The minimal and the maximal transfer rules do not satisfy *no-envy.* On the other hand, the symmetrically balanced VCG rule, the pivotal rule, and the reward-based pivotal rule satisfy *no-envy*.

We investigate whether there is a rule satisfying *budget-balance* and *no-envy* together with either one of two *cost monotonicity* axioms (Subsect. 4.2) or either one of two *independence* axioms (Subsect. 4.1). The answer is no.

Theorem 4.6 (Chun 2006b) *Let $|N| \geq 3$. Then, there is no rule satisfying budget-balance, no-envy, and either negative or positive cost monotonicity together. Also, there is no rule satisfying budget-balance, no-envy, and either independence of preceding costs or independence of following costs together.*

10 For this, a position in a queue is considered as an indivisible good.

11 *Group no-envy* extends the notion of *no-envy* to groups. See Svensson (1983) for details.

12 These are allocations that can be supported as Walrasian equilibrium with an equal implicit income.

4.4 No-envy, the identical preferences lower bound, and egalitarian equivalence

Although *no-envy* plays an important role in the literature on fairness, there are other interesting concepts. The following are the main ones: the *identical preferences lower bound* and *egalitarian equivalence* (Pazner and Schmeidler 1978). *Egalitarian equivalence* requires that there should be a reference bundle such that each agent enjoys the same utility between her bundle and that reference bundle. Formally, given $N \in \mathcal{N}$ and $\theta \in \mathcal{Q}^N$, an allocation $(\sigma, t) \in Z(\theta)$ is *egalitarian equivalent* if there is a reference bundle (σ_0, t_0) such that for all $i \in N$, $u_i(\sigma_i, t_i; \theta_i) = u_i(\sigma_0, t_0; \theta_i)$. Let $EE(\theta)$ be the set of all egalitarian equivalent allocations for $\theta \in \mathcal{Q}^N$.

Egalitarian equivalence: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, and all $(\sigma, t) \in \varphi(\theta)$, $(\sigma, t) \in EE(\theta)$.

Remark 4.7 In economies with indivisible goods, when there are as many objects as agents, *budget-balance* and *no-envy* together imply the *identical preferences lower bound* (Bevia 1996). Moreover, if there are only two agents, then *budget-balance* and the *identical preferences lower bound* together imply *no-envy.* A similar observation can be made for queueing problems. However, without *budget-balance,* no logical relation exists between *no-envy* and the *identical preferences lower bound.*

Now we investigate whether a rule can satisfy *no-envy* and *egalitarian equivalence* together. If there are only two agents, then any rule satisfying *queue-efficiency, budget-balance,* and *egalitarian equivalence* satisfies *no-envy*. Moreover, if there are only three agents, then by choosing the middle position as a part of the reference bundle, we can establish the existence of a rule satisfying *budget-balance, no-envy,* and *egalitarian equivalence*. However, the positive result does not generalize to problems with more than three agents.

Proposition 4.8 (Chun et al. 2014b) *Let* $|N| \geq 4$. *Then, there is no rule satisfying no-envy and egalitarian equivalence together.*

In economies with indivisible goods, there is no rule satisfying *object-efficiency,*[13] *egalitarian equivalence,* and the *identical preferences lower bound* (Thomson 1990). However, in queueing problems, we can construct a rule satisfying *queue-efficiency, budget-balance, egalitarian equivalence,* and the *identical preferences lower bound* (Chun 2006b).

5 Strategic approach

Strategy-proofness requires that an agent should not have an incentive to misrepresent her waiting cost no matter what she believes other agents to be doing. We investigate its implications in the context of queueing problems.[14]

[13] *Object-efficiency* requires that there is no feasible allocation which makes every agent better off and at least one agent strictly better off.

[14] The literature on *strategy-proofness* is too large to give a comprehensive list of references. A recent review of this literature, along with a list of references, can be found in Barberà (2011) and Thomson (2013).

5.1 Strategy-proofness and the VCG rules

Here we fix the set of agents and change the profile of waiting costs. To indicate the dependence on the profile θ, we denote the allocation as $\mu(\theta) = (\sigma(\theta), t(\theta))$.[15] For each agent $i \in N$, let $\mu_i(\theta) = (\sigma_i(\theta), t_i(\theta))$ be agent i's assignment for the problem θ and $u_i(\mu_i(\theta); \theta_i') = -(\sigma_i(\theta) - 1)\theta_i' + t_i(\theta)$ be agent i's utility when the profile of announced waiting costs is θ and her true waiting cost is θ_i'.

Strategy-proofness: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, all $i \in N$, and all $\theta_i' \in \mathbf{R}_+$, $u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_i(\theta_i', \theta_{N\setminus\{i\}}); \theta_i)$.

Remark 5.1 Holmström (1979) shows that when preferences are quasi-linear and the domain of types is convex, the VCG rules are the only ones satisfying *queue-efficiency* and *strategy-proofness*. For queueing problems, the preferences are completely specified by the profile of waiting costs, which is $\mathbb{R}^n_+$. Therefore, it follows that a rule satisfies *queue-efficiency* and *strategy-proofness* if and only if it is a VCG rule.

We use the following notation: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$, suppose there is an initial queue $\sigma(\theta)$ and agent $i \in N$ leaves the queue. We define the "induced" queue $\sigma(\theta_{N\setminus\{i\}})$ (of length $n-1$) for the agents in $N\setminus\{i\}$ as follows:

$$\sigma_j(\theta_{N\setminus\{i\}}) = \begin{cases} \sigma_j(\theta) & \text{if } j \in P_i(\theta), \\ \sigma_j(\theta) - 1 & \text{if } j \in F_i(\theta). \end{cases}$$

In words, $\sigma(\theta_{N\setminus\{i\}})$ is the queue formed by removing agent i and moving all agents behind her up by one position.

We now formally define the VCG rules.

VCG rule associated with g_i, μ^{g_i}: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$, $\mu^{g_i}(\theta) = (\sigma(\theta), t(\theta))$ is defined as: $\sigma(\theta) \in E(\theta)$ and for all $i \in N$,

$$t_i(\theta) = -\sum_{j \in F_i(\sigma)} \theta_j + g_i(\theta_{N\setminus\{i\}}). \tag{5.1}$$

Remark 5.2 The standard way of specifying the VCG transfers is as follows:

$$t_i(\theta) = -\sum_{j \neq i} (\sigma_j(\theta) - 1)\theta_j + h_i(\theta_{N\setminus\{i\}}). \tag{5.2}$$

This is equivalent to (5.1) since we can write without loss of generality

$$h_i(\theta_{N\setminus\{i\}}) = \sum_{j \neq i} (\sigma_j(\theta_{N\setminus\{i\}}) - 1)\theta_j + g_i(\theta_{N\setminus\{i\}}). \tag{5.3}$$

Substituting (5.3) in (5.2) and simplifying gives us (5.1).

[15] Since t depends on the choice of the queue, we should denote the transfers by $t(\sigma(\theta))$ instead of $t(\theta)$. Note that our (single-valued) rule chooses a unique queue which in turn determines the unique transfers. Therefore, we abuse the notation and write $t(\theta)$.

Remark 5.3 For a VCG rule, an agent's utility is independent of the tie-breaking rule. By *queue-efficiency,* all agents whose waiting cost is θ_i occupy the same set of consecutive queue positions in all efficient queues. By (5.1), the utility of agent i in two different efficient queues can differ only through differences in $g_i(\theta_{N\setminus\{i\}})$ across tie-breaking rules but since g_i is independent of θ_i, it cannot depend on the tie-breaking rule either.

We investigate the implications of imposing *equal treatment of equals* together with *queue-efficiency* and *strategyproofness* and characterize the family of anonymous VCG rules.

Anonymous VCG rule associated with g, μ^g: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$,

(1) μ^g is a VCG rule.
(2) For all $i \in N$, g_i is symmetric, i.e., $g_i(x) = g_i(y)$ whenever x and y are permutations of one another.
(3) For all $i, j \in N$ such that $\theta_i = \theta_j$, $g_i(\theta_{N\setminus\{i\}}) = g_j(\theta_{N\setminus\{j\}})$.

Remark 5.4 Given (2) and (3), we can write $g_i = g$ for all $i \in N$.

Our characterization result follows.

Proposition 5.5 (Chun et al. 2014a) *A rule satisfies queue-efficiency, equal treatment of equals, and strategy-proofness if and only if it is an anonymous VCG rule.*

Remark 5.6 *Anonymity in welfare* requires that a permutation of waiting costs implies a permutation of welfare also. Hashimoto and Saitoh (2012) show that *anonymity in welfare* and *strategy-proofness* together imply *queue-efficiency*. It is natural to ask whether *anonymity in welfare* can be weakened to *equal treatment of equals*. However, *equal treatment of equals* and *strategy-proofness* together do not imply *queue-efficiency* (Chun et al. 2014a).

5.2 Further characterizations of VCG rules

We discuss how to single out some interesting rules from the class of anonymous VCG rules characterized in Proposition 5.5. We begin with the symmetrically balanced VCG rule which satisfies many nice properties. It is *queue-efficient, budget-balanced,* and *strategy-proof,* and hence "first-best" implementable (Suijs 1996; Mitra 2001). It also satisfies *no-envy* (Chun 2006b). Recently, Kayi and Ramaekers (2015) characterize the symmetrically balanced VCG rule and Chun, Mitra, and Mutuswami (in press) provide an alternative simple proof for the characterization.

We note that all these characterizations, Kayi and Ramaekers (2015) and Chun, Mitra, and Mutuswami (in press), assume that a rule is *multi-valued* and impose *Pareto-indifference* in their characterizations. *Pareto indifference* requires that if an allocation is chosen by a rule, then all other *queue-efficient* and *budget-balanced* allocations which assign the same utilities to each agent should be chosen by the rule. On the other hand, since we assume that a rule is *single-valued,* we can characterize the symmetrically balanced VCG rule without imposing *Pareto indifference,*

Theorem 5.7 (Chun et al. 2014a) *Let $n \geq 3$. A rule satisfies queue-efficiency, budget-balance, equal treatment of equals, and strategy-proofness if and only if it is the symmetrically balanced VCG rule.*

To characterize the pivotal and the reward-based pivotal mechanisms, we use two independence axioms introduced in Subsect. 4.1 together with a mild regularity condition on the transfers saying that if all agents have zero waiting costs, then their transfers must add up to zero.

Weak budget-balance: For all $N \in \mathcal{N}$ and all $\theta \in \mathcal{Q}^N$, if $\theta = (0, \ldots, 0)$, then $\sum_{i \in N} t_i(\theta) = 0$.

A characterization theorem follows:

Theorem 5.8 (Chun et al. 2014a) (1) *The pivotal rule is the only rule satisfying queue-efficiency, equal treatment of equals, strategy-proofness, independence of preceding costs, and weak budget-balance.*
(2) *The reward-based pivotal rule is the only rule satisfying queue-efficiency, equal treatment of equals, strategy-proofness, independence of following costs, and weak budget-balance.*

5.3 Group strategy-proofness and the *k*-pivotal rules

There are many ways of strengthening *strategy-proofness* to coalitional deviations. Our first strengthening requires that there does not exist a deviation which makes all deviating agents weakly better-off and at least one agent strictly better-off.

Strong group strategy-proofness: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, all $S \subseteq N$, and all $\theta'_S \in \mathbb{R}_+^{|S|}$, $u_i(\mu_i(\theta); \theta_i) \leq u_i(\mu_i(\theta'_S, \theta_{N\setminus S}), \theta_i)$ for all $i \in S$ implies $u_i(\mu_i(\theta); \theta_i) = u_i(\mu_i(\theta'_S, \theta_{N\setminus S}), \theta_i)$ for all $i \in S$.

With *strong group strategy-proofness,* we have a negative result.

Theorem 5.9 (Mitra and Mutuswami 2011) *There is no rule satisfying queue-efficiency and strong group strategy-proofness together.*

We thus look at a weaker notion of *group strategy-proofness,* which requires that there does not exist a deviation which makes all deviating agents strictly better-off.

Weak group strategy-proofness: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, all $S \subseteq N$, and all $\theta'_S \in \mathbb{R}_+^{|S|}$, $u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_i(\theta'_S, \theta_{N\setminus S}), \theta_i)$ for some $i \in S$.

For any positive integer k and any $N \in \mathcal{N}$ such that $|N| > k$, we define the transfer function as follows: For all $\theta \in \mathcal{Q}^N$,

$$\bar{t}_i^k(\theta) = \begin{cases} -\sum_{j:\sigma_i(\theta)<\sigma_j(\theta)\leq k} \theta_j & \text{if } \sigma_i(\theta) < k, \\ 0 & \text{if } \sigma_i(\theta) = k, \\ \sum_{j:k\leq\sigma_j(\theta)<\sigma_i(\theta)} \theta_j & \text{if } \sigma_i(\theta) > k. \end{cases}$$

k-pivotal rule, φ^k: Given any positive integer k, for all $N \in \mathcal{N}$ such that $|N| \geq k$ and all $\theta \in \mathcal{Q}^N$, $\varphi^k(\theta) = \left\{(\sigma^k, t^k) \in Z(\theta) | \, \sigma^k \in E(\theta) \text{ and } \forall i \in N, t_i^k(\theta) = \bar{t}_i^k(\theta)\right\}$.

It is not difficult to show that all k-pivotal rules satisfy *weak group strategy-proofness*. Also, the pivotal rule is obtained by setting $k = n$ and the reward-based pivotal rule by setting $k = 1$.

To characterize the k-pivotal rules, we use a weaker notion of *weak group strategy-proofness, pairwise strategy-proofness,* and a technical property called *weak linearity* together with standard requirements of *queue-efficiency* and *equal treatment of equals*. *Pairwise strategy-proofness* requires that there does not exist a deviation making all deviating agents strictly better-off for a coalition of size at most two.

Pairwise strategy-proofness: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, all $S \subseteq N$ such that $|S| \leq 2$, and all $\theta'_S \in \mathbb{R}_+^{|S|}$, $u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_i(\theta'_S, \theta_{N\setminus S}), \theta_i)$ for some $i \in S$.

Weak linearity: For all $N \in \mathcal{N}$, all $\theta \in \mathcal{Q}^N$, all $j \in N$, and all θ'_j such that $\sigma(\theta) = \sigma(\theta'_j, \theta_{N\setminus\{j\}})$, $t_i(\lambda\theta + (1-\lambda)(\theta'_j, \theta_{N\setminus\{j\}})) = \lambda t_i(\theta) + (1-\lambda)t_i(\theta'_j, \theta_{N\setminus\{j\}})$ for all $\lambda \in [0, 1]$ and all $i \in N$.

Now we present a characterization of the k-pivotal rules.

Theorem 5.10 (Mitra and Mutuswami 2011) *A rule satisfies queue-efficiency, equal treatment of equals, pairwise strategy-proofness, and weak linearity if and only if it is a k-pivotal rule.*

One can relax *weak linearity* to obtain a larger class of *group strategy-proof* rules that includes discontinuous VCG transfers (Mukherjee 2013).

We conclude this subsection by characterizing the subset of k-pivotal rules that satisfy *no-deficit*. Let $\langle x \rangle_+$ denote the smallest integer greater than or equal to x.

Proposition 5.11 (Mitra and Mutuswami 2011) *A k-pivotal rule satisfies no-deficit if and only if* $k \geq \left\langle \frac{n+1}{2} \right\rangle_+$.

6 Combining strategic and normative approaches

We investigate the implications of normative requirements such as *egalitarian equivalence* (Pazner and Schmeidler 1978) and the *identical preferences lower bound* (Moulin 1990, 1991) together with *queue-efficiency* and *strategy-proofness*.

6.1 Strategy-proofness and egalitarian equivalence

Our next theorem characterizes the complete family of rules satisfying *queue-efficiency, strategy-proofness,* and *egalitarian equivalence*. It shows that the queue position in the reference bundle must be the same for all profiles. Furthermore, the choice of the queue position determines the transfers at all profiles up to a type-independent constant.

Theorem 6.1 (Chun et al. 2014b) *A rule $\mu = (\sigma, t)$ satisfies queue-efficiency, strategy-proofness, and egalitarian equivalence if and only if it is a VCG rule and there exists $\sigma_0 \in \{1, \ldots, n\}$ and $c \in \mathbb{R}$ such that for all $\theta \in \mathcal{Q}^N$ and all $i \in N$,*

$$t_i(\theta) = \sum_{j \in N \setminus \{i\}} (\sigma_0 - \sigma_j(\theta))\theta_j + c. \tag{6.1}$$

We now examine whether the rules characterized in Theorem 6.1 satisfy additional desirable properties. One such property is *budget-balance* which requires that there be no net transfer into or out of the problem. As it turns out, none of the rules characterized in Theorem 6.1 satisfies *budget-balance*. While *budget-balanced* rules are not possible, it turns out that there are rules satisfying *feasibility* together with *queue-efficiency, strategy-proofness,* and *egalitarian equivalence*. The following theorem characterizes all such rules:

Theorem 6.2 (Chun et al. 2014b) *A rule μ satisfies queue-efficiency, strategy-proofness, egalitarian equivalence, and feasibility if and only if it is a VCG rule such that the transfers satisfy equation (6.1) with $\sigma_0 = 1$ and $c \leq 0$.*

6.2 Strategy-proofness and the identical preferences lower bound

Mitra (2007) showed the existence of a VCG rule meeting the *identical preferences lower bound* together with *queue-efficiency* and *budget-balance*. Here, we provide the full characterization of the class of VCG rules meeting the *identical preferences lower bound*. For each $x > 0$, let $\langle x \rangle_+$ be the smallest integer greater than or equal to x and $\langle x \rangle_-$ the largest integer smaller than or equal to x.

Proposition 6.3 (Chun and Yengin 2017) *(a) If a rule satisfies queue-efficiency, strategy-proofness, and the identical preferences lower bound, then it is a VCG rule $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ such that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,*

$$g_i(\theta_{-i}) \geq \sum_{\ell = \left\langle \frac{n+1}{2} \right\rangle_+}^{n-1} (\theta_{-i})_{[\ell]}. \tag{6.2}$$

(b) If a VCG rule $\mu^{g,\tau} = (\sigma^\tau, t^{g,\tau})$ is such that for each $\theta \in \mathcal{Q}^N$ and each $i \in N$,

$$g_i(\theta_{-i}) \geq \sum_{\ell = \left\langle \frac{n+1}{2} \right\rangle_-}^{n-1} (\theta_{-i})_{[\ell]}, \tag{6.3}$$

then it meets the identical preferences lower bound.

Since the symmetrically balanced VCG rules satisfy the three axioms of Proposition 6.3(a), they also satisfy (6.2). If n is an odd number, then $\left\langle \frac{n+1}{2} \right\rangle_- = \left\langle \frac{n+1}{2} \right\rangle_+ = \frac{n+1}{2}$.

Hence, for problems with an odd number of agents, a rule minimizes the deficit in each problem among all rules satisfying *queue-efficiency, strategy-proofness,* and the *identical preferences lower bound* if and only if it is a k-pivotal rule with $k = \frac{n+1}{2}$.

Now suppose that the center wants all agents to enjoy the highest possible levels of welfare without violating the upper bound on the deficit. First, the center checks whether it is possible to guarantee each agent her utility at the nth queue position with zero transfer without violating the upper bound on the deficit. The *n-welfare lower bound* can easily be met even when the center wants to generate *no-deficit*. Next, the center checks whether it is possible to guarantee each agent her utility at the $(n-1)$th queue position with zero transfer without violating the upper bound on the deficit. If this lower bound is feasible, then the center raises the bar further and checks progressively for each $k \in \{n-2, \ldots, 1\}$, whether it can guarantee each agent her utility at the kth queue position with zero transfer without violating the upper bound on the deficit.

Formally, for each $k \in \{n, \ldots, 1\}$, the *k-welfare lower bound* requires that each agent should be guaranteed her utility at the kth queue position with zero transfer. As k decreases, the lower bound on the utility increases. Hence, if a rule satisfies the *k-welfare lower bound*, then it satisfies the $(k+1)$-*welfare lower bound*.

k-welfare lower bound: For each $\theta \in \mathcal{Q}^N$ and each $i \in N$, $u_i(\mu_i(\theta); \theta_i) \geq -(k-1)\theta_i$.

We are searching for rules that generate the minimal deficit in each problem within the class of VCG rules meeting the *k-welfare lower bound*. As it turns out, for each k, the k-pivotal rules achieve our objective. Hence, our next characterization provides an alternative normative justification for these rules.

Theorem 6.4 (Chun and Yengin 2017) *Let $k \in \{1, 2, ..., n\}$. A rule minimizes the deficit in each problem among all rules satisfying queue-efficiency, strategy-proofness, and the k-welfare lower bound if and only if it is a k-pivotal rule.*

7 Bargaining approach

The queueing problem can be solved by adopting a bargaining approach which builds up a natural and intuitive bargaining protocol such that players can negotiate among themselves to resolve the queueing conflicts.[16] Players are assumed to be risk neutral and expected utility maximizers.

The first game, called the *first-served mechanism,* is described as follows: At stage 1, all players participate in a multi-bidding auction competing for the first position of a queue. In this auction, each player submits an $(n-1)$-tuple of numbers, one number for each player (excluding herself). A positive number means a payment she makes to another player and a negative number means a compensation she asks for from another player. The player whose net bid (the difference between the sum of bids made by the player and the sum of bids the other players made to her) is the highest wins the first position while making the payment or receiving the compensation depending on

[16] For various bargaining protocols implementing the Shapley value, see Gul (1989), Hart and Mas-Colell (1996), Ju (2013), Ju and Wettstein (2009), and Pérez-Castrillo and Wettstein (2001).

her bid. At stage 2, the winner has two options. She can either keep the first position or sell it to other players. If she decides to keep the position, then the rest of the players play the game again from the first stage to bargain over the positions after her. If she decides to sell the position, then this sale cannot be a bilateral one because where to locate the winner after the sale affects other players' positions. Therefore, the winner makes a proposal that consists of a queue assigning positions to all players and a vector of transfers specifying the amount each player is supposed to pay or receive. Stage 3 is to approve or disapprove the proposal. The proposal is accepted if all players agree. In case of acceptance, the proposal is implemented so that the queue is formed with transfers in effect to all players. In case of rejection, the proposer retains the first position. Meanwhile, all players except for the rejected proposer start new round of negotiation. This first-served mechanism has a unique subgame perfect equilibrium (SPE) outcome, which coincides with the payoff vector assigned by the maximal transfer rule.

Theorem 7.1 (Ju et al. 2014) *The first-served mechanism has a SPE outcome, which coincides with the payoff vector assigned by the maximal transfer rule.*

On the other hand, the second game, called the *last-served mechanism*, implements the minimal transfer rule. Differently from the first-served mechanism, players compete for the last position. Alternatively, one can think that players are now demanding compensations for them to be served last, which is in the same light as the ALDB (auctioning the leadership with differentiated bids) mechanism (Moulin 1981). The one with the highest net bid (or lowest net compensation if the bids are negative) is selected as the winner. The winner can decide to keep the last position or sell it to the others. For the latter option, she makes a proposal of a queue and a vector of transfers. If the proposal is rejected, she remains at the last position to be served after all the participating players. The last-served mechanism has a unique SPE outcome, which coincides with the payoff vector assigned by the minimal transfer rule.

Theorem 7.2 (Ju et al. 2014) *The last-served mechanism has a unique SPE outcome, which coincides with the payoff vector assigned by the minimal transfer rule.*

In both mechanisms, the players have the same strategies. However, the two mechanisms assign different positions to a winner who decides to keep the position in stage 2 or whose proposal is rejected in stage 3. In the first-served mechanism this player gets the first position (after the already rejected players), while in the last-served mechanism this player gets the last position (in front of the already rejected players).

As it turns out, in SPE, being the proposer in the first-served mechanism is so attractive that the bids to become the proposer are so high that it eventually leads to a combination of bids and offers such that the SPE outcome yields the utility payoffs assigned by the maximal transfer rule. On the other hand, in SPE, being the proposer in the last-served mechanism is so unattractive that the bids to become the proposer are so low (in fact, the players want to be paid to become the proposer) that it eventually leads to a combination of bids and offers such that the SPE outcome yields the utility payoffs assigned by the minimal transfer rule.

8 Generalizations of the queueing problem

Even though the queueing problem has been studied from many different perspectives, its generalizations have not been studied in depth yet. Here, we discuss its possible generalizations to indicate the directions for future research.

8.1 Queueing problems with an initial queue

In many queueing situations, agents are usually served on the first-come first-served basis. Although it is easy to implement this rule, it may not be *queue-efficient* when waiting in a queue is costly for agents and agents differ in their (unit) waiting costs. Trading queue positions can be allowed to overcome the inefficiency resulted from the initial queue. This queueing problem with an initial queue has been studied in Curiel et al. (1989), Gershkov and Schweinzer (2010), and Chun et al. (2017).

8.2 Queueing problems with multiple facilities

This queueing problem with one facility can be generalized to the queueing problem with multiple facilities which allows to handle as many agents as the number of facilities at one time. Chun and Heo (2008) generalize the minimal and the maximal transfer rules to queueing problem with two facilities, but their generalizations to the queueing problem with more than two facilities remain as open questions. On the other hand, Mitra (2005) and Mukherjee (2013) study *strategy-proof* rule for queueing problems with multiple facilities.

8.3 Slot allocation problems

A group of agents must be assigned to a slot located along a line. Only one agent can be assigned to each slot. Agents differ in their (unit) waiting cost and the most preferred slot position, called the peak. Each agent's utility from her assignment is equal to the amount of monetary transfer minus the (unit) waiting cost multiplied by the distance between the peak and her assigned slot. The slot allocation problem tries to find a way of assigning slots to agents and the monetary transfers they should receive. It generalizes the queueing problem by allowing each agent to have a different peak. Chun and Park (2017) studies a special subclass of the slot allocation problem in which all agents have the identical waiting cost. Also, an ordinal version of this problem has been studied by Hougaard et al. (2014) and a related problem of assigning landing slots to airlines by Schummer and Vohra (2013) and Schummer and Abizada (2017).

8.4 Other generalizations

In the queueing problem, each agent has the same rankings over queue positions, but differs in her waiting cost which is assumed to be constant per unit of time. It would be an interesting extension if the (unit) waiting cost varies over queue positions. Also, it would be worth to study another generalization of the queueing problem in which an agent is assumed to have a different arrival time (Ghosh et al. 2018).

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