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ORIGINAL PAPER

New characterizations of the Owen and Banzhaf–Owen values using the intracoalitional balanced contributions property

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Abstract In this paper, several characterizations of the Owen and the Banzhaf–Owen values are provided. All the characterizations make use of a property based on the principle of balanced contributions. This property is called the intracoalitional balanced contributions property and was defined by Calvo et al. (Math Soc Sci 31:171–182, 1996).

Keywords Cooperative game · Shapley value · Banzhaf value · Coalition structure · Balanced contributions

Mathematics Subject Classification 91A12

1 Introduction

One of the main objectives in the Cooperative Game Theory field is the study of solutions (or values) for cooperative games with transferable utility (TU-games). The Shapley value (Shapley 1953) and the Banzhaf value (Banzhaf 1965) are two of the best known concepts in this context.

A coalition structure is a partition of the set of players. There are many situations where the coalition structures make sense. It could be the case, for instance, of parliamentary coalitions formed by different political parties, alliances among countries in a negotiation process or unions of workers trying to improve a collective bargaining agreement.

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Cooperative games with coalition structure (TU-games with coalition structure) were introduced by Aumann and Drèze (1974). They incorporate the given coalition structure to the classical notion of TU-game. As in the class of TU-games, it is interesting to find solutions (coalitional values) to obtain a suitable assignment for each player.

The coalition structure can be interpreted by the coalitional values in different ways. The coalitional value defined in Aumann and Drèze (1974) assigns to each player the Shapley value in the subgame played by the union he/she belongs to. A different approach is taken into account by Owen (1977) to define the so-called Owen value. In this case, the unions play a TU-game among themselves, called the quotient game, and after that the players in each union play an internal game. In the Owen value, the payoffs for the unions in the quotient game and the payoffs for the players inside the union are given by the Shapley value.

The Banzhaf–Owen value (also known as the modified Banzhaf value) was defined by Owen (1982) following a similar procedure. This coalitional value only differs from the Owen value in the fact that the payoffs for the unions in the quotient game and the payoffs for the players within each union are computed by means of the Banzhaf value.

This framework is focused on the study of the Owen and the Banzhaf–Owen values. The first characterization of the Owen value can be found in Owen (1977), where it is also defined. However, there are many other characterizations of this value in the literature (see, for example, the papers by Hart and Kurz 1983; Calvo et al. 1996; Vázquez-Brage et al. 1997; Hamiache 1999, 2001; Albizuri 2008; Khmelnitskaya and Yanovskaya 2007; Casajus 2010; Gómez-Rúa and Vidal-Puga 2010; Alonso-Meijide et al. 2016, or Lorenzo-Freire 2016). As far as the Banzhaf–Owen value is concerned, although the first characterization of this value was given in Albizuri (2001) for the class of simple games, this coalitional value was first characterized in the whole class of cooperative games with coalition structure in the paper by Amer et al. (1995). Later on, Alonso-Meijide et al. (2007) and Alonso-Meijide et al. (2016) proposed alternative characterizations of this coalitional value.

In this paper, both coalitional values are compared by means of appealing properties. Moreover, new characterizations of these two values are provided. All the characterizations make use of an interesting property, called intracoalitional balanced contributions and introduced in Calvo et al. (1996). It is based on the principle of balanced contributions, introduced by Myerson (1980) to characterize the Shapley value, which can be useful in different contexts. According to the intracoalitional balanced contributions property, if we consider two players in the same coalition, the losses or gains for both agents when the other leaves the game are equal.

The paper is organized as follows. In Sect. 2 we introduce notation and previous definitions required along the paper. The following two sections are devoted to study the relation between the property of intracoalitional balanced contributions and the Owen and Banzhaf–Owen values. So, in Sect. 3 we investigate the characteristics of the coalitional values satisfying intracoalitional balanced contributions, obtaining an expression in terms of the Shapley value for all these coalitional values. Finally, in Sect. 4 we provide the characterizations of the paper, where both values are characterized, trying to identify the similarities and differences of both coalitional values.

2 Notation and definitions

2.1 TU-games

A *cooperative game with transferable utility* (or *TU-game*) is a pair (N, v) defined by a finite set of players $N \subset \mathbb{N}$ (usually, $N = \{1, 2, \ldots, n\}$) and a function $v : 2^N \to \mathbb{R}$, that assigns to each coalition $S \subseteq N$ a real number $v(S)$, called the worth of S, and such that $v(\emptyset) = 0$. For any coalition $S \subseteq N$, we assume that $s = |S|$. In the sequel, $\mathscr{G}_N$ will denote the family of all TU-games on a given N and $\mathscr{G}$ will denote the family of all TU-games. Given $S \subseteq N$, we denote the restriction of a TU-game $(N, v) \in \mathscr{G}_N$ to S as (S, v).

Let us fix a TU-game (N, v) and two players $i, j \in N$ with $i \neq j$. Let the set $\{i, j\}$ be considered as a new player $i^* \notin N$ (it means that the players i and j are amalgamated into one player i^*) and let us denote $N^{\{i,j\}} = (N \backslash \{i, j\}) \cup \{i^*\}$. The $\{i, j\}$*-amalgamated game* $(N^{\{i,j\}}, v^{\{i,j\}}) \in \mathscr{G}$ is defined for all $S \subseteq N^{\{i,j\}}$ by

$$v^{\{i,j\}}(S) = \begin{cases} v((S \backslash \{i^*\}) \cup \{i, j\}) & \text{if } i^* \in S \\ v(S) & \text{otherwise.} \end{cases}$$

A *value* is a map f that assigns to every TU-game $(N, v) \in \mathscr{G}$ a vector $f(N, v) = (f_i(N, v))_{i \in N}$, where each component $f_i(N, v)$ represents the payoff of i when he/she participates in the game.

The Shapley and the Banzhaf values are two widely-known tools in this context. In both cases, the payoff for each player can be computed as the weighted mean of the marginal contributions of the player. But, whereas in the Shapley value the weights are obtained assuming that all the orders of the players are equally likely, in the case of the Banzhaf value the weights are calculated by taking into account that the player is equally likely to join any coalition.

Definition 1 (Shapley 1953) The *Shapley value* is the value defined for all $(N, v) \in \mathscr{G}$ and all $i \in N$ by

$$Sh_i(N, v) = \sum_{S \subseteq N \backslash \{i\}} \frac{s!(n-s-1)!}{n!} [v(S \cup \{i\}) - v(S)].$$

Definition 2 (Banzhaf 1965) The *Banzhaf value* is the value defined for all $(N, v) \in \mathscr{G}$ and all $i \in N$ by

$$Bz_i(N, v) = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{2^{n-1}} [v(S \cup \{i\}) - v(S)].$$

2.2 TU-games with coalition structure

Let us consider a finite set of players N. A *coalition structure* over N is a partition of N, i.e., $C = \{C_1, \ldots, C_m\}$ is a coalition structure over N if it satisfies that $\bigcup_{h \in M} C_h = N$, where $M = \{1, \ldots, m\}$, and $C_h \cap C_r = \emptyset$ when $h \neq r$. There are two *trivial coalition structures*: $C^n = \{\{i\} : i \in N\}$, where each union is a singleton, and $C^N = \{N\}$, where the grand coalition forms.

Given $i \in N$, $\mathcal{C}(i)$ denotes the family of coalition structures over N where $\{i\}$ is a singleton union, that is, $C \in \mathcal{C}(i)$ if and only if $\{i\} \in C$.

Given $T \subseteq C_h$, $C_{|T}$ is the coalition structure where the union C_h is replaced by the subset T, i.e., $C_{|T} = (C \backslash \{C_h\}) \cup \{T\}$.

A *cooperative game with coalition structure* (or *TU-game with coalition structure*) is a triple (N, v, C) where (N, v) is a TU-game and C is a coalition structure over N. The set of all TU-games with coalition structure will be denoted by $\mathscr{CG}$, and by $\mathscr{CG}_N$ the subset where N is the player set.

Given $S \subseteq N$, such that $S = \bigcup_{h \in M} S_h$, with $\emptyset \neq S_h \subseteq C_h$ for all $h \in M$, we will denote the restriction of $(N, v, C) \in \mathscr{CG}_N$ to S as the TU-game with coalition structure (S, v, C_S), where $C_S = \{C_1, \ldots, C_m\}$.

If $(N, v, C) \in \mathscr{CG}$ and $C = \{C_1, \ldots, C_m\}$, the *quotient game* (M, v^C) is the TU-game defined as $v^C(R) = v(\bigcup_{r \in R} C_r)$ for all $R \subseteq M$. It means that the quotient game is the game played by the unions, i.e., the TU-game induced by (N, v, C) by considering the elements of C as players.

A *coalitional value* is a map g that assigns to every TU-game with coalition structure (N, v, C) a vector $g(N, v, C) = (g_i(N, v, C))_{i \in N} \in \mathbb{R}^N$, where $g_i(N, v, C)$ is the payoff for each player $i \in N$.

Given a value f on $\mathscr{G}$, a coalitional value g on $\mathscr{CG}$ is a *coalitional f-value for singletons* when $g(N, v, C^n) = f(N, v)$. Thus, a coalitional value g is a *coalitional Shapley value for singletons* when $g(N, v, C^n) = Sh(N, v)$ and g is a *coalitional Banzhaf value for singletons* when $g(N, v, C^n) = Bz(N, v)$.

The Owen and Banzhaf–Owen values are two coalitional values which extend the Shapley and Banzhaf values to the context of cooperative games with coalition structure. Both coalitional values take into account that the players in the same union act together and, in this way, only contributions of each player to coalitions formed by full unions and agents in the union of the player are considered.

Definition 3 (Owen 1977) The *Owen value* is defined for all $(N, v, C) \in \mathscr{CG}$ and all $i \in N$ by

$$Ow_i(N, v, C) = \sum_{R \subseteq M \backslash \{h\}} \sum_{T \subseteq C_h \backslash \{i\}} \frac{r!(m-r-1)!}{m!} \frac{t!(c_h - t - 1)!}{c_h!} \times [v(Q \cup T \cup \{i\}) - v(Q \cup T)],$$

where $C_h \in C$ is the union such that $i \in C_h$, $m = |M|$, $c_h = |C_h|$, $t = |T|$, $r = |R|$ and $Q = \bigcup_{r \in R} C_r$.

Another way to obtain the Owen value consists of computing the Shapley value twice, first applied to a quotient game and then applied to a TU-game inside the unions. According to this procedure, for all $(N, v, C) \in \mathscr{CG}$, all $C_h \in C$ and all $i \in C_h$,

$$Ow_i(N, v, C) = Sh_i(C_h, \hat{v}^{Sh,C}) \qquad (1)$$

and $(C_h, \hat{v}^{Sh,C})$ is the TU-game such that $\hat{v}^{Sh,C}(T) = Sh_h(M, v^{C_{|T}})$ for all $T \subseteq C_h$, $T \neq \emptyset$.

Since the Owen value satisfies that $Ow(N, v, C^n) = Sh(N, v)$, the Owen value is a *coalitional Shapley value for singletons*.

Definition 4 (Owen 1982) The *Banzhaf–Owen value* is defined for all $(N, v, C) \in \mathscr{CG}$ and all $i \in N$ by

$$BzOw_i(N, v, C) = \sum_{R \subseteq M\setminus\{h\}} \sum_{T \subseteq C_h\setminus\{i\}} \frac{1}{2^{m-1}} \frac{1}{2^{c_h-1}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)],$$

where $C_h \in C$ is the union such that $i \in C_h$, $m = |M|$, $c_h = |C_h|$ and $Q = \bigcup_{r \in R} C_r$.

The Banzhaf–Owen value can be obtained according to a procedure in two stages. It is similar to the one used to compute the Owen value, just replacing the Shapley value with the Banzhaf value. Then, for all $(N, v, C) \in \mathscr{CG}$, all $C_h \in C$ and all $i \in C_h$,

$$BzOw_i(N, v, C) = Bz_i(C_h, \hat{v}^{Bz,C}) \tag{2}$$

and $(C_h, \hat{v}^{Bz,C})$ is the TU-game such that $\hat{v}^{Bz,C}(T) = Bz_h(M, v^{C_{|T}})$ for all $T \subseteq C_h$, $T \neq \emptyset$.

Given that $BzOw(N, v, C^n) = Bz(N, v)$, the Banzhaf–Owen value is a *coalitional Banzhaf value for singletons*.

3 The intracoalitional balanced contributions property

The property of intracoalitional balanced contributions was introduced by Calvo et al. (1996) and states that, given two players in the same union, the amounts that both players gain or lose when the other leaves the coalitional game should be equal. This property is satisfied by the Owen and Banzhaf–Owen values, but also by many other coalitional values.

Intracoalitional balanced contributions (IBC). For all $(N, v, C) \in \mathscr{CG}$ and all $i, j \in C_h \in C$, $i \neq j$,

$$g_i(N, v, C) - g_i(N\setminus\{j\}, v, C_{N\setminus\{j\}}) = g_j(N, v, C) - g_j(N\setminus\{i\}, v, C_{N\setminus\{i\}}).$$

In Calvo et al. (1996), a characterization of the Owen value is provided by means of efficiency, coalitional balanced contributions and intracoalitional balanced contributions. The last two properties are based on the property of balanced contributions for TU-games. A value f satisfies this property if, for all $(N, v) \in \mathscr{G}$ and all $i, j \in N$, $f_i(N, v) - f_i(N\setminus\{j\}, v) = f_j(N, v) - f_j(N\setminus\{i\}, v)$. This property says that, given two players, the gains or losses obtained by both players when the other leaves the game coincide. Myerson (1980) used it, together with efficiency, to characterize the Shapley value.

In Sánchez (1997) and Calvo and Santos (1997) it is proved that if a value on TU-games satisfies the balanced contributions property then this value can be expressed as the Shapley value of a particular TU-game, that is, if f satisfies the balanced contributions property then $f(N, v) = Sh(N, v^f)$, where $v^f(S) = \sum_{i \in S} f_i(S, v)$ for all $S \subseteq N$. Following a similar reasoning, we can prove that the payoff of each player, according to any coalitional value satisfying the intracoalitional balanced contributions property, can be computed by means of the Shapley value applied to a TU-game restricted to the union the player belongs to.

Definition 5 For all $(N, v, C) \in \mathscr{CG}$ and all $C_h \in C$, given a coalitional value g, we define the TU-game $(C_h, v^{g,C})$ as

$$v^{g,C}(T) = \sum_{i \in T} g_i((N \backslash C_h) \cup T, v, C_{|T}) \quad \text{for all} \quad T \subseteq C_h, \quad T \neq \emptyset.$$

According to this TU-game, the worth of each subset of players in the union is given by the sum of their values in the coalitional game where the union is replaced with the subset.

Proposition 1 (a) *A coalitional value g satisfies IBC if and only if, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,*

$$g_i(N, v, C) = \frac{v^{g,C}(C_h) - v^{g,C}(C_h \backslash \{i\}) + \sum_{j \in C_h \backslash \{i\}} g_i(N \backslash \{j\}, v, C_{N \backslash \{j\}})}{|C_h|} \tag{3}$$

(b) *A coalitional value g satisfies IBC if and only if, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$, $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$.*

In the first part of the proposition, we propose a necessary and sufficient condition to be fulfilled by any coalitional value satisfying the property of IBC. This condition provides an interpretation of the coalitional value g for a player $i \in C_h$ as the average of $|C_h|$ quantities: the amounts player i gets if another player in the union leaves the game and the marginal contribution of i to his/her union in the TU-game $v^{g,C}$.

In the second part, we express any coalitional value satisfying IBC as the Shapley value of a particular game. This condition is related to the first part of the proposition and it will we used in the characterizations of the next section.

Proof (a) First of all, it will be proved that if a coalitional value g satisfies IBC then (3) is true. Given $i \in C_h$, by IBC we know that for all $j \in C_h \backslash \{i\}$,

$$g_i(N, v, C) - g_i(N \backslash \{j\}, v, C_{N \backslash \{j\}}) = g_j(N, v, C) - g_j(N \backslash \{i\}, v, C_{N \backslash \{i\}}).$$

If we add these equations over $j \in C_h \backslash \{i\}$, we obtain that

$$
\begin{aligned}
&(|C_h|-1)g_i(N,v,C) - \sum_{j\in C_h\backslash\{i\}} g_i(N\backslash\{j\},v,C_{N\backslash\{j\}})\\
&\quad = \sum_{j\in C_h\backslash\{i\}} g_j(N,v,C) - \sum_{j\in C_h\backslash\{i\}} g_j(N\backslash\{i\},v,C_{N\backslash\{i\}}).
\end{aligned}
$$

Taking into account the definition of the game $(C_h, v^{g,C})$, the previous equation can be replaced by

$$
\begin{aligned}
&(|C_h|-1)g_i(N,v,C) - \sum_{j\in C_h\backslash\{i\}} g_i(N\backslash\{j\},v,C_{N\backslash\{j\}})\\
&\quad = v^{g,C}(C_h) - g_i(N,v,C) - v^{g,C}(C_h\backslash\{i\}),
\end{aligned}
$$

which lead us to deduce (3).

Suppose now that g is a coalitional value that satisfies (3). We will prove that g satisfies IBC. The proof will be done by induction on $|C_h|$.

Suppose that $C_h=\{i,j\}$. Equation (3) can be written as

$$
\begin{aligned}
g_i(N,v,C) &= \frac{v^{g,C}(C_h) - v^{g,C}(C_h\backslash\{i\}) + g_i(N\backslash\{j\},v,C_{N\backslash\{j\}})}{2}\\
&= \frac{g_i(N,v,C) + g_j(N,v,C) - g_j(N\backslash\{i\},v,C_{N\backslash\{i\}}) + g_i(N\backslash\{j\},v,C_{N\backslash\{j\}})}{2},
\end{aligned}
$$

what implies that IBC is satisfied.

Suppose now that $|C_h|>2$. Let us choose $i, j \in C_h$. Then

$$
\begin{aligned}
&|C_h|[g_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) - g_j(N\backslash\{i\},v,C_{N\backslash\{i\}})]\\
&\quad = (|C_h|-1)[g_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) - g_j(N\backslash\{i\},v,C_{N\backslash\{i\}})]\\
&\qquad + g_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) - g_j(N\backslash\{i\},v,C_{N\backslash\{i\}})\\
&\quad = v^{g,C_{N\backslash\{j\}}}(C_h\backslash\{j\}) - v^{g,C_{N\backslash\{j\}}}(C_h\backslash\{i,j\})\\
&\qquad + \sum_{k\in C_h\backslash\{i,j\}} g_i(N\backslash\{j,k\},v,C_{N\backslash\{j,k\}}) - v^{g,C_{N\backslash\{i\}}}(C_h\backslash\{i\})\\
&\qquad + v^{g,C_{N\backslash\{i\}}}(C_h\backslash\{i,j\}) - \sum_{k\in C_h\backslash\{i,j\}} g_j(N\backslash\{i,k\},v,C_{N\backslash\{i,k\}})\\
&\qquad + g_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) - g_j(N\backslash\{i\},v,C_{N\backslash\{i\}})\\
&\quad = v^{g,C_{N\backslash\{j\}}}(C_h\backslash\{j\}) + \sum_{k\in C_h\backslash\{i,j\}} g_i(N\backslash\{j,k\},v,C_{N\backslash\{j,k\}})\\
&\qquad + g_i(N\backslash\{j\},v,C_{N\backslash\{j\}}) - v^{g,C_{N\backslash\{i\}}}(C_h\backslash\{i\})\\
&\qquad - \sum_{k\in C_h\backslash\{i,j\}} g_j(N\backslash\{i,k\},v,C_{N\backslash\{i,k\}}) - g_j(N\backslash\{i\},v,C_{N\backslash\{i\}}).
\end{aligned}
$$

On the other hand, by the induction hypothesis we know that for all $k \in C_h \backslash \{i, j\}$,

$$g_i(N\backslash\{j,k\}, v, C_{N\backslash\{j,k\}}) - g_j(N\backslash\{i,k\}, v, C_{N\backslash\{i,k\}})$$
$$= g_i(N\backslash\{k\}, v, C_{N\backslash\{k\}}) - g_j(N\backslash\{k\}, v, C_{N\backslash\{k\}}).$$

Thus, by induction hypothesis and (3),

$$\begin{aligned}
&|C_h|[g_i(N\backslash\{j\}, v, C_{N\backslash\{j\}}) - g_j(N\backslash\{i\}, v, C_{N\backslash\{i\}})] \\
&= \sum_{k\in C_h\backslash\{i\}} g_i(N\backslash\{k\}, v, C_{N\backslash\{k\}}) - v^{g,C_{N\backslash\{i\}}}(C_h\backslash\{i\}) \\
&\quad - \sum_{k\in C_h\backslash\{j\}} g_j(N\backslash\{k\}, v, C_{N\backslash\{k\}}) + v^{g,C_{N\backslash\{j\}}}(C_h\backslash\{j\}) \\
&= \sum_{k\in C_h\backslash\{i\}} g_i(N\backslash\{k\}, v, C_{N\backslash\{k\}}) + v^{g,C}(C_h) - v^{g,C_{N\backslash\{i\}}}(C_h\backslash\{i\}) \\
&\quad - \sum_{k\in C_h\backslash\{j\}} g_j(N\backslash\{k\}, v, C_{N\backslash\{k\}}) - v^{g,C}(C_h) + v^{g,C_{N\backslash\{j\}}}(C_h\backslash\{j\}) \\
&= |C_h|[g_i(N, v, C) - g_j(N, v, C)].
\end{aligned}$$

□

(b) Taking into account (a), the result will be proved if we prove that, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$, a coalitional value g satisfies (3) if and only if $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$.

In Hart and Mas-Collel (1996), a recursive formula to compute the Shapley value is provided. According to this formula, we know that for all $(N, v) \in \mathscr{G}$ and all $i \in N$,

$$Sh_i(N, v) = \frac{v(N) - v(N\backslash\{i\}) + \sum_{j\in N\backslash\{i\}} Sh_i(N\backslash\{j\}, v)}{n}.$$

Suppose that for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$, with $C_h \in C$, we know that $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$. By the recursive formula we have that

$$\begin{aligned}
g_i(N, v, C) &= \frac{v^{g,C}(C_h) - v^{g,C}(C_h\backslash\{i\}) + \sum_{j\in C_h\backslash\{i\}} Sh_i(C_h\backslash\{j\}, v^{g,C_{N\backslash\{j\}}})}{|C_h|} \\
&= \frac{v^{g,C}(C_h) - v^{g,C}(C_h\backslash\{i\}) + \sum_{j\in C_h\backslash\{i\}} g_i(N\backslash\{j\}, v, C_{N\backslash\{j\}})}{|C_h|}.
\end{aligned}$$

Next, we will prove that, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$, a coalitional value g satisfying (3) can be expressed as $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$. The proof will be done by induction on $|C_h|$.

Suppose that $C_h = \{i\}$. Then, in this case, $Sh_i(C_h, v^{g,C}) = Sh_i(\{i\}, v^{g,C}) = v^{g,C}(\{i\}) = g_i(N, v, C)$.

Suppose now that $|C_h| > 1$. By Eq. (3) and the induction hypothesis, we deduce that

$$\begin{aligned}
&|C_h| g_i(N, v, C) \\
&= v^{g,C}(C_h) - v^{g,C}(C_h \backslash \{i\}) + \sum_{j \in C_h \backslash \{i\}} g_i(N \backslash \{j\}, v, C_{N \backslash \{j\}}) \\
&= v^{g,C}(C_h) - v^{g,C}(C_h \backslash \{i\}) + \sum_{j \in C_h \backslash \{i\}} Sh_i(C_h \backslash \{j\}, v^{g,C_{N \backslash \{j\}}}) \\
&= v^{g,C}(C_h) - v^{g,C}(C_h \backslash \{i\}) \\
&\quad + \sum_{j \in C_h \backslash \{i\}} \sum_{T \subseteq C_h \backslash \{i,j\}} \frac{t!(c_h - t - 2)!}{(c_h - 1)!} [v^{g,C_{N \backslash \{j\}}}(T \cup \{i\}) - v^{g,C_{N \backslash \{j\}}}(T)] \\
&= v^{g,C}(C_h) - v^{g,C}(C_h \backslash \{i\}) \\
&\quad + \sum_{j \in C_h \backslash \{i\}} \sum_{T \subseteq C_h \backslash \{i,j\}} \frac{t!(c_h - t - 2)!}{(c_h - 1)!} [v^{g,C}(T \cup \{i\}) - v^{g,C}(T)] \\
&= v^{g,C}(C_h) - v^{g,C}(C_h \backslash \{i\}) \\
&\quad + \sum_{T \subsetneq C_h \backslash \{i\}} \sum_{j \in C_h \backslash (T \cup \{i\})} \frac{t!(c_h - t - 2)!}{(c_h - 1)!} [v^{g,C}(T \cup \{i\}) - v^{g,C}(T)] \\
&= v^{g,C}(C_h) - v^{g,C}(C_h \backslash \{i\}) \\
&\quad + \sum_{T \subsetneq C_h \backslash \{i\}} \frac{t!(c_h - t - 1)!}{(c_h - 1)!} [v^{g,C}(T \cup \{i\}) - v^{g,C}(T)].
\end{aligned}$$

Therefore

$$g_i(N, v, C) = \sum_{T \subseteq C_h \backslash \{i\}} \frac{t!(c_h - t - 1)!}{c_h!} [v^{g,C}(T \cup \{i\}) - v^{g,C}(T)] = Sh_i(C_h, v^{g,C}).$$

□

Remark 1
- Since the Owen value satisfies IBC, by Proposition 1 we obtain an alternative expression for this coalitional value. Then, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,

  $$Ow_i(N, v, C) = Sh_i(C_h, v^{Ow,C}),$$

  where $v^{Ow,C}(T) = \sum_{i \in T} Ow_i((N \backslash C_h) \cup T, v, C_{|T}) = Sh_h(M, v^{C_{|T}}) = \hat{v}^{Sh,C}(T)$ for all $T \subseteq C_h$.
  It implies that $Ow_i(N, v, C) = Sh_i(C_h, \hat{v}^{Sh,C})$, which coincides with the expression already given in (1).

- It is straightforward to prove that the Banzhaf–Owen value satisfies IBC. Then, by Proposition 1, we can also obtain an alternative expression for the Banzhaf–Owen value. Thus, for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$ with $C_h \in C$,

$$BzOw_i(N, v, C) = Sh_i(C_h, v^{BzOw,C}), \tag{4}$$

where $v^{BzOw,C}(T) = \sum_{i \in T} BzOw_i((N \backslash C_h) \cup T, v, C_{|T})$ for all $T \subseteq C_h$.
Note that this expression is different from (2), although both of them lead to the Banzhaf–Owen value.

The notion of potential function was first introduced by Hart and Mas-Collel (1989) in the case of cooperative games, characterizing the Shapley value in terms of a potential function. Later on, Winter (1992) characterized the Owen and the Aumann and Drèze coalitional values by means of a potential function.

Calvo and Santos (1997) establish the conditions that a value f must satisfy to admit what they define as a potential function. Following a similar reasoning, we will consider that a coalitional value g admits a potential when there exists a function P_g that assigns to each $(N, v, C) \in \mathcal{CG}$ a vector $(P_g^h(N, v, C))_{h \in M} \in \mathbb{R}^m$ and satisfies that $P_g^h(N, v, C) - P_g^h(N \backslash \{i\}, v, C_{N \backslash \{i\}}) = g_i(N, v, C)$ for all $i \in C_h$, with $C_h \in C$.

We will also consider a normalization condition that says that, for each $h \in M$, $P_g^h(S, v, C_S) = 0$ for all $S \subseteq N$ such that $S \cap C_h = \emptyset$.

It is straightforward to prove that any coalitional value admits at most one potential.

Calvo and Santos (1997) also show that a rule satisfies the property of balanced contributions if and only if it admits a potential function. In the next theorem, we show a similar result for the case of the coalitional values.

Theorem 1 *Let g be a coalitional value. The following statements are equivalent*:

(a) *g satisfies IBC.*
(b) *For all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$ with $C_h \in C$, $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$.*
(c) *g admits an associated vector potential function $(P_g^h(N, v, C))_{h \in M}$.*

Proof In Proposition 1 part (b) it is shown that (a) $\Longleftrightarrow$ (b). Thus, the proof will be finished if we prove that (b) $\Longleftrightarrow$ (c). This proof follows the same spirit than the proof of Theorem A in Calvo and Santos (1997).

Let us suppose that, for all $i \in C_h$ with $C_h \in C$, $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$. Then, we will consider for all $h \in M$, $P_g^h(N, v, C) = P_{Sh}(C_h, v^{g,C})$, where P_{Sh} denotes the potential function of the Shapley value.[1] Hence,

$$\begin{aligned} P_g^h(N, v, C) - P_g^h(N \backslash \{i\}, v, C_{N \backslash \{i\}}) &= P_{Sh}(C_h, v^{g,C}) - P_{Sh}(C_h \backslash \{i\}, v^{g,C}) \\ &= Sh_i(C_h, v^{g,C}) \\ &= g_i(N, v, C). \end{aligned}$$

Assume now that g admits the vector potential function $(P_g^h(N, v, C))_{h \in M}$. We will prove that $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$ for all $i \in C_h$, with $C_h \in C$, by induction on $|C_h|$.

[1] Given a TU-game (N, v), $P_{Sh}(N, v) = \sum_{S \subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S)$.

If $C_h = \{i\}$, then $Sh_i(C_h, v^{g,C}) = v^{g,C}(\{i\}) = g_i(N, v, C)$. Otherwise, let us suppose that $|C_h| > 1$. We know that for all $i \in C_h$

$$g_i(N, v, C) - Sh_i(C_h, v^{g,C}) = P_g^h(N, v, C) - P_g^h(N\backslash\{i\}, v, C_{N\backslash\{i\}}) \\ - P_{Sh}(C_h, v^{g,C}) + P_{Sh}(C_h\backslash\{i\}, v^{g,C}).$$

By the induction hypothesis, $P_g^h(N\backslash\{i\}, v, C_{N\backslash\{i\}}) = P_{Sh}(C_h\backslash\{i\}, v^{g,C})$ for all $i \in C_h$. It implies that

$$|C_h|[P_g^h(N, v, C) - P_{Sh}(C_h, v^{g,C})] = \sum_{i\in C_h} g_i(N, v, C) - \sum_{i\in C_h} Sh_i(C_h, v^{g,C}) \\ = v^{g,C}(C_h) - v^{g,C}(C_h) = 0.$$

Then, for all $i \in C_h$

$$\begin{aligned} g_i(N, v, C) &= P_g^h(N, v, C) - P_g^h(N\backslash\{i\}, v, C_{N\backslash\{i\}}) \\ &= P_{Sh}(C_h, v^{g,C}) - P_{Sh}(C_h\backslash\{i\}, v^{g,C}) \\ &= Sh_i(C_h, v^{g,C}). \end{aligned}$$

□

Remark 2 At the view of the results, it is easy to obtain that the potential for the Owen value is given by $(P_{Ow}^h(N, v, C))_{h\in M}$, where for each $h \in M$,

$$P_{Ow}^h(N, v, C) = \sum_{R\subseteq M\backslash\{h\}} \sum_{T\subseteq C_h} \frac{r!(m-r-1)!}{m!} \frac{(t-1)!(c_h-t)!}{c_h!} [v(Q\cup T) - v(Q)],$$

with $Q = \bigcup_{r\in R} C_r$.

It is straightforward to prove that this formula coincides with the expression for the potential function of the Owen value given in Winter (1992).

Now, let us define a coalitional value g as follows. Given $(N, v, C) \in \mathcal{CG}$, there exist two values f^1 and f^2 such that

$$g_i(N, v, C) = f_i^1(C_h, \hat{v}^{f^2,C}), \quad \text{for all} \quad i \in C_h \in C, \tag{5}$$

where $\hat{v}^{f^2,C}(T) = f_h^2(M, v^{C_{|T}})$ for all $T \subseteq C_h$, $T \neq \emptyset$.

To define this coalitional value we consider that the unions play a TU-game among themselves, the so-called quotient game, and after that the players in each union play an internal game. To assign the payoffs for the unions in the quotient game and the payoffs for the players inside the union, two values are used. Both values could coincide or not. Note that, according to expressions (1) and (2), the Owen and the Banzhaf–Owen values can be defined in this way.

The following corollary can be directly obtained by applying Theorem 1 to the coalitional values expressed according to (5).

Corollary 1 *Let us consider a coalitional value g defined according to the procedure described by formula* (5). *If the value f^1 satisfies the property of balanced contributions then*:

(a) *For all* $(N, v, C) \in \mathcal{CG}$ *and all* $i \in C_h$ *with* $C_h \in C$, $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$.
(b) *g satisfies IBC.*
(c) *g admits an associated vector potential function* $(P_g^h(N, v, C))_{h\in M}$.

Proof By Theorem 1, it is sufficient to prove that all the coalitional values satisfying (5) can be also computed with the formula of part (a).

Since f^1 satisfies balanced contributions, applying the formula obtained in Sánchez (1997) and Calvo and Santos (1997), we know that

$$g_i(N, v, C) = f_i^1(C_h, \hat{v}^{f^2,C}) = Sh_i(C_h, (\hat{v}^{f^2,C})^{f^1}).$$

As $(\hat{v}^{f^2,C})^{f^1}(T) = \sum_{i\in T} f_i^1(T, \hat{v}^{f^2,C}) = \sum_{i\in T} g_i((N\backslash C_h) \cup T, v, C_{|T}) = v^{g,C}(T)$ for all $T \subseteq C_h$, we obtain that $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$ for all $i \in C_h$ with $C_h \in C$. □

4 The characterizations

Below, we introduce the properties that, together with the property of intracoalitional balanced contributions, will be used to study the behavior of the Owen and Banzhaf–Owen values.

Efficiency (E). For all $(N, v, C) \in \mathcal{CG}$, $\sum_{i\in N} g_i(N, v, C) = v(N)$.
Efficiency says that the worth of the grand coalition should be distributed among the players.
1-Player efficiency (1-E). If $N = \{i\}$, then $g_i(N, v, C) = v(N)$.
1-Player efficiency is just the property of Efficiency applied to 1-player sets.
Neutrality for amalgamated players (NAP). For all $(N, v, C) \in \mathcal{CG}$, any $C_h \in C$, and $i, j \in C_h$, $i \neq j$,

$$g_i(N, v, C) + g_j(N, v, C) = g_{i^*}(N^{\{i,j\}}, v^{\{i,j\}}, C^{\{i,j\}}),$$

where $(N^{\{i,j\}}, v^{\{i,j\}})$ is the $\{i, j\}$-amalgamated game of (N, v) and $C^{\{i,j\}} = \{C_1^{\{i,j\}}, \ldots, C_m^{\{i,j\}}\}$ is the coalition structure over $N^{\{i,j\}}$ such that $C_h^{\{i,j\}} = (C_h\backslash\{\{i, j\}\}) \cup \{i^*\}$ and $C_r^{\{i,j\}} = C_r$ for $r \neq h$.
According to this property, the sum of the payoffs of two players in the same union coincides with the payoff of their representative i^* in the amalgamated game.
Coherence (C). For all $(N, v) \in \mathcal{G}$, $g(N, v, C^N) = g(N, v, C^n)$.
Coherence means that the situations where all the players belong to the same union and when all of them act as singletons are indistinguishable.
Independence of amalgamation in other unions (IA). If $(N, v, C) \in \mathcal{CG}$ and $i, j \in C_h \in C$, with $i \neq j$, then $g_k(N, v, C) = g_k(N^{\{i,j\}}, v^{\{i,j\}}, C^{\{i,j\}})$ for all $k \in$

$N \backslash C_h$.
Independence of amalgamation in other unions says that if two players in the same union join their forces then the players outside the union are not affected.
1-Quotient game (1-QG). If $(N, v, C) \in \mathscr{CG}$ and $C \in \mathcal{C}(i)$ for some $i \in N$, then $g_i(N, v, C) = g_h(M, v^C, C^m)$, where $C_h = \{i\}$.
This property says that the amount an isolated player gets in the original game with coalition structure coincides with the payoff of the union he forms in the quotient game.
Quotient game (QG). For all $(N, v, C) \in \mathscr{CG}$ and all $C_h \in C$, $\sum_{i \in C_h} g_i(N, v, C) = g_h(M, v^C, C^m)$.
In the case of the quotient game property, the sum of the payoffs of the players in a union coincides with the payoff of this union in the quotient game.

Both NAP and IA are referred to the amalgamated game. However, whereas NAP deals with the players in the same union who merge, IA is focused on the study of the payoffs of the players outside the union. NAP was proposed in Alonso-Meijide et al. (2007), which is called 2-Efficiency within unions, and extends the property of 2-Efficiency, introduced by Lehrer (1988) and used by Nowak (1997) to characterize the Banzhaf value. IA was introduced in Alonso-Meijide et al. (2014) with the name of Neutrality for the reduced game and was used to give a new characterization of the Banzhaf–Owen coalitional value.

The property of coherence was considered in Calvo and Gutiérrez (2010) to obtain a characterization of the coalitional value known as the two-step Shapley value.

The property 1-QG was first introduced in Alonso-Meijide et al. (2007) and is weaker than properties IA and QG. In fact, 1-QG is a particular case of property QG for the coalitions formed by one player. An interesting comparison of 1-QG and QG can be also found in Alonso-Meijide et al. (2007).

4.1 Characterizations of the Owen value

In the next proposition, different relations between the properties and the corresponding payoff for each coalition are studied. These results, together with Proposition 1, will be used to give characterizations for the Owen value.

Proposition 2 (a) *If g is a coalitional value that satisfies E and IA then g satisfies QG.*

(b) *A coalitional value g is a coalitional Shapley value for singletons and satisfies QG if and only if $\sum_{i \in C_h} g_i(N, v, C) = Sh_h(M, v^C)$ for all $C_h \in C$.*

Proof
(a) Let us consider a coalitional value g that satisfies the properties E and IA. We will prove that g satisfies QG.

Then, let us fix a TU-game with coalition structure $(N, v, C) \in \mathscr{CG}$ and $C_h \in C$. We distinguish several cases:

– *First case* $C = C^n$. In this particular case, $C_h = \{i\}$ and $g_i(N, v, C^n) = g_h(M, v^C, C^m)$.

– *Second case* $|C_r| = 1$ for all $r \neq h$ and $|C_h| > 1$. In this case we apply a recursive procedure, which consists of amalgamating two players in the union C_h and then a third player in the same union is amalgamated to the new one, and so on. The procedure is applied $c_h - 1$ times to build the TU-game with coalition structure $(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)})$, where $|C_h^{(c_h-1)}| = 1$, $C_r^{(c_h-1)} = C_r$ for all $r \neq h$ and $v^{(c_h-1)}(S) = \begin{cases} v((S \backslash C_h^{(c_h-1)}) \cup C_h) & \text{if } C_h^{(c_h-1)} \subseteq S \\ v(S) & \text{otherwise.} \end{cases}$
Moreover, if we apply IA at each step, at the end of the procedure we obtain that for all $i \in N \backslash C_h$,

$$g_i(N, v, C) = g_i(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}).$$

Note that the game $(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)})$ is now in the conditions of the first case. Then, for all $C_r \in C$, with $r \neq h$, and $i \in C_r$,

$$g_i(N, v, C) = g_i(N^{(c_h-1)}, v^{(c_h-1)}, C^{(c_h-1)}) = g_r(M, v^C, C^m).$$

By E we know that $\sum_{i \in C_h} g_i(N, v, C) = v(N) - \sum_{i \in N \backslash C_h} g_i(N, v, C)$.
So, it is easy to deduce that

$$\begin{aligned} \sum_{i \in C_h} g_i(N, v, C) &= v(N) - \sum_{i \in N \backslash C_h} g_i(N, v, C) \\ &= v^C(M) - \sum_{r \in M \backslash \{h\}} g_r(M, v^C, C^m) \\ &= g_h(M, v^C, C^m). \end{aligned}$$

– *Third case* There exists at least one union $C_r \in C$, $r \neq h$, such that $|C_r| > 1$.
Let us consider the set $M_h = \{r \in M \backslash \{h\} : |C_r| > 1\}$, with $m_h = |M_h|$. In this case we apply a procedure, which consists of m_h stages. The procedure at each stage is similar to the recursive procedure used in the second case until we get, after $\sum_{r \in M_h} c_r - m_h$ steps, the TU-game with coalition structure denoted by $(N^{(\sum_{r \in M_h} c_r - m_h)}, v^{(\sum_{r \in M_h} c_r - m_h)}, C^{(\sum_{r \in M_h} c_r - m_h)})$.
This TU-game with coalition structure satisfies that $|C_r^{(\sum_{r \in M_h} c_r - m_h)}| = 1$ for all $r \in M_h$ and $C_r^{(\sum_{r \in M_h} c_r - m_h)} = C_r$ for all $r \in M \backslash M_h$. Moreover, if we apply IA at each step, we finally have that, for all $i \in C_h$,

$$g_i\left(N^{\left(\sum_{r \in M_h} c_r - m_h\right)}, v^{\left(\sum_{r \in M_h} c_r - m_h\right)}, C^{\left(\sum_{r \in M_h} c_r - m_h\right)}\right) = g_i(N, v, C).$$

Since the game $(N^{(\sum_{r\in M_h} c_r - m_h)}, v^{(\sum_{r\in M_h} c_r - m_h)}, C^{(\sum_{r\in M_h} c_r - m_h)})$ is in the conditions of the first or second case, it is easy to prove that

$$\begin{aligned}\sum_{i\in C_h} g_i(N, v, C) &= \sum_{i\in C_h} g_i\left(N^{\left(\sum_{r\in M_h} c_r - m_h\right)}, v^{\left(\sum_{r\in M_h} c_r - m_h\right)}, C^{\left(\sum_{r\in M_h} c_r - m_h\right)}\right)\\ &= v^C(M) - \sum_{r\in M\backslash\{h\}} g_r(M, v^C, C^m)\\ &= g_h(M, v^C, C^m).\end{aligned}$$

□

(b) Let us consider the TU-game with coalition structure $(N, v, C) \in \mathscr{CG}$ and $C_h \in C$. It is straightforward to prove that any coalitional value g satisfying $\sum_{i\in C_h} g_i(N, v, C) = Sh_h(M, v^C)$ is a coalitional value for singletons and satisfies QG.

On the other hand, if g is a coalitional value for singletons and satisfies QG, then

$$\sum_{i\in C_h} g_i(N, v, C) = g_h(M, v^C, C^m) = Sh_h(M, v^C).$$

□

Theorem 2 (a) *The Owen value is the only coalitional Shapley value for singletons that satisfies QG and IBC.*

(b) *The Owen value is the only coalitional Shapley value for singletons that satisfies E, IA and IBC.*

(c) *The Owen value is the only coalitional value that satisfies E, C, IA and IBC.*

(d) *The Owen value is the only coalitional value that satisfies 1-E, C, QG and IBC.*

Proof As it is asserted in Alonso-Meijide et al. (2007), the Owen value satisfies 1-E, E, C, IA and QG. Moreover, according to Calvo et al. (1996) it also satisfies IBC. Then, it only remains to prove the uniqueness.

(a) Since g is in the conditions of Proposition 1 we can assume that, for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$, $g_i(N, v, C) = Sh_i(C_h, v^{g,C})$.

On the other hand, since g is in the conditions of part (b) from Proposition 2, we know that for all $T \subseteq C_h$,

$$v^{g,C}(T) = \sum_{i\in T} g_i((N\backslash C_h) \cup T, v, C_{|T}) = Sh_h(M, v^{C_{|T}}) = \hat{v}^{Sh,C}(T).$$

Then, we know that for all $(N, v, C) \in \mathscr{CG}$ and all $i \in C_h$ with $C_h \in C$,

$$g_i(N, v, C) = Sh_i(C_h, \hat{v}^{Sh,C}) = Ow_i(N, v, C).$$

□

(b) By Proposition 2 part (a), this result can be seen as a consequence of part (a) of this theorem. □

(c) First of all, we will prove that any coalitional value satisfying E, C and IBC is a coalitional Shapley value for singletons. To get it, we consider two coalitional values g^1 and g^2 satisfying these properties and we will show that $g^1(N, v, C^n) = g^2(N, v, C^n)$ for all $(N, v) \in \mathcal{G}$. The proof will be done by induction on $|N|$.

If $N = \{i\}$, by E $g^1(N, v, C^n) = v(\{i\}) = g^2(N, v, C^n)$. Let us suppose that $|N| \geq 2$ and let us fix $i \in N$. By IBC we obtain that for all $l \in \{1, 2\}$ and all $j \in N \backslash \{i\}$,

$$g_i^l(N, v, C^N) - g_i^l(N \backslash \{j\}, v, C^{N \backslash \{j\}}) = g_j^l(N, v, C^N) - g_j^l(N \backslash \{i\}, v, C^{N \backslash \{i\}}).$$

By induction, we know that for all $j \in N \backslash \{i\}$,

$$\begin{aligned} &g_i^1(N \backslash \{j\}, v, C^{N \backslash \{j\}}) - g_j^1(N \backslash \{i\}, v, C^{N \backslash \{i\}}) \\ &\quad = g_i^2(N \backslash \{j\}, v, C^{N \backslash \{j\}}) - g_j^2(N \backslash \{i\}, v, C^{N \backslash \{i\}}). \end{aligned}$$

Then

$$\begin{aligned} (n-1)g_i^1(N, v, C^N) - \sum_{j \in N \backslash \{i\}} g_j^1(N, v, C^N) &= (n-1)g_i^2(N, v, C^N) \\ - \sum_{j \in N \backslash \{i\}} g_j^2(N, v, C^N). \end{aligned}$$

In addition, by E $\sum_{j \in N} g_j^1(N, v, C^N) = \sum_{j \in N} g_j^2(N, v, C^N)$.

These last two equations imply that $g_i^1(N, v, C^N) = g_i^2(N, v, C^N)$. By C we deduce that $g_i^1(N, v, C^n) = g_i^2(N, v, C^n) = Ow(N, v, C^n) = Sh(N, v)$.

Then, since a coalitional value satisfying E, C, IA and IBC is in the conditions of the part (b) of this theorem, we can assert that this result is true. □

(d) Let us suppose that g is a coalitional value satisfying the four properties. By QG and 1-E we know that for all $(N, v) \in \mathcal{G}$, $\sum_{i \in N} g_i(N, v, C^N) = v(N)$. Taking into account IBC and C and following a similar procedure than in part c), we obtain that g is a coalitional Shapley value for singletons.

Thus, since g is in the conditions of the part (a) of this theorem, this result is true. □

4.2 Characterizations of the Banzhaf–Owen value

In this part several characterizations of the Banzhaf–Owen value are obtained.

Theorem 3 (a) *The Banzhaf–Owen value is the only coalitional Banzhaf value for singletons that satisfies NAP, 1-QG and IBC.*

(b) *The Banzhaf–Owen value is the only coalitional Banzhaf value for singletons that satisfies NAP, IA and IBC.*

(c) *The Banzhaf–Owen value is the only coalitional value that satisfies 1-E, NAP, C, 1-QG and IBC.*
(d) *The Banzhaf–Owen value is the only coalitional value that satisfies 1-E, NAP, C, IA and IBC.*

Proof In Alonso-Meijide et al. (2007) it is said that the Banzhaf–Owen value satisfies NAP, IA, 1-QG. Moreover, it is straightforward to prove that this coalitional value also satisfies 1-E and IBC. Then, it only remains to prove the uniqueness.
(a) Let us fix $(N, v, C) \in \mathscr{CG}$ and $i \in C_h \in C$. We will prove that $g_i(N, v, C) = BzOw_i(N, v, C)$ by induction on the number of players in the union C_h. We will distinguish several cases:

- *First case* $C = C^n$. Since g is a coalitional Banzhaf value for singletons we have that $g(N, v, C) = Bz(N, v) = BzOw(N, v, C)$.
- *Second case* $|C_h| = 1$ and there exists at least one union $C_r, r \neq h$, such that $|C_r| > 1$. Suppose that $C_h = \{i\}$. By 1-QG we obtain that $g_i(N, v, C) = g_h(M, v^C, C^m)$. According to the first case and taking into account that $BzOw$ also satisfies 1-QG, $g_i(N, v, C) = g_h(M, v^C, C^m) = BzOw_h(M, v^C, C^m) = BzOw_i(N, v, C)$.
- *Third case* $|C_h| = c_h > 1$. Thus, let us fix two players $i, j \in C_h$.
By NAP and the induction hypothesis,

$$\begin{aligned} g_i(N, v, C) + g_j(N, v, C) &= g_{i^*}(N^{\{i,j\}}, v^{\{i,j\}}, C^{\{i,j\}}) \\ &= BzOw_{i^*}(N^{\{i,j\}}, v^{\{i,j\}}, C^{\{i,j\}}) \\ &= BzOw_i(N, v, C) + BzOw_j(N, v, C). \end{aligned}$$

In addition, by IBC and the induction hypothesis,

$$\begin{aligned} g_i(N, v, C) - g_j(N, v, C) &= g_i(N\backslash\{j\}, v, C_{N\backslash\{j\}}) - g_j(N\backslash\{i\}, v, C_{N\backslash\{i\}}) \\ &= BzOw_i(N\backslash\{j\}, v, C_{N\backslash\{j\}}) \\ &\quad - BzOw_j(N\backslash\{i\}, v, C_{N\backslash\{i\}}) \\ &= BzOw_i(N, v, C) - BzOw_j(N, v, C). \end{aligned}$$

It means that $g_i(N, v, C) = BzOw_i(N, v, C)$ and $g_j(N, v, C) = BzOw_j(N, v, C)$ for any pair of players $i, j \in C_h$. □

(b) It is straightforward to prove that IA implies 1-QG. Then, this result is a consequence of part (a). □
(c) First of all, we will prove that any coalitional value satisfying 1-E, NAP, C and IBC is a coalitional Banzhaf value for singletons. To this aim, we consider two coalitional values g^1 and g^2 satisfying these properties and we will show that $g^1(N, v, C^n) = g^2(N, v, C^n)$ for all $(N, v) \in \mathscr{G}$. The proof will be done by induction on $|N|$.

If $N = \{i\}$, by 1-E $g^1(N, v, C^n) = v(\{i\}) = g^2(N, v, C^n)$. Let us suppose that $|N| \geq 2$. If we choose two players $i, j \in N$, by IBC we obtain that for all $l \in \{1, 2\}$,

$$g_i^l(N, v, C^N) - g_i^l(N\backslash\{j\}, v, C^{N\backslash\{j\}}) = g_j^l(N, v, C^N) - g_j^l(N\backslash\{i\}, v, C^{N\backslash\{i\}}).$$

By induction, we know that

$$g_i^1(N\backslash\{j\}, v, C^{N\backslash\{j\}}) - g_j^1(N\backslash\{i\}, v, C^{N\backslash\{i\}})$$
$$= g_i^2(N\backslash\{j\}, v, C^{N\backslash\{j\}}) - g_j^2(N\backslash\{i\}, v, C^{N\backslash\{i\}}).$$

Then

$$g_i^1(N, v, C^N) - g_j^1(N, v, C^N) = g_i^2(N, v, C^N) - g_j^2(N, v, C^N).$$

Moreover, by NAP and induction

$$\begin{aligned} g_i^1(N, v, C^N) + g_j^1(N, v, C^N) &= g_{i^*}^1(N^{\{i,j\}}, v^{N^{\{i,j\}}}, C^{N^{\{i,j\}}}) \\ &= g_{i^*}^2(N^{\{i,j\}}, v^{N^{\{i,j\}}}, C^{N^{\{i,j\}}}) \\ &= g_i^2(N, v, C^N) + g_j^2(N, v, C^N). \end{aligned}$$

So, it is easy to deduce that $g^1(N, v, C^N) = g^2(N, v, C^N)$. By C we obtain that $g^1(N, v, C^n) = g^2(N, v, C^n) = Bz(N, v)$.

Therefore, since the coalitional value is a coalitional Banzhaf value for singletons that satisfies NAP, 1-QG and IBC, this result is obtained as a consequence of part (a). □

(d) Since IA implies 1-QG, this part can be seen as a consequence of (c). □

5 Concluding remarks

In this framework we study the behaviour of two well-known coalitional values: the Owen and Banzhaf–Owen values. To achieve this objective, we make use of several properties and compare both coalitional values, trying to deduce their main differences and similarities. As a consequence of this study, some characterizations of these two values are obtained.

In all the characterizations, the property of intracoalitional balanced contributions (IBC) is a key property. In fact, we have provided an expression for all the coalitional values satisfying this property. However, we have taken into account many other interesting properties in this framework, most of them crucial in the context of coalitional values. All the properties and results are summarized in Table 1.

It should be pointed out that parallel characterizations of both values have been obtained, since the parts (b) of Theorems 2 and 3 share the properties of IA and IBC. In Theorem 2 part (b) the Owen is characterized by adding coalitional Shapley value for singletons and E, whereas in Theorem 3 part (b) these properties are replaced with coalitional Banzhaf value for singletons and NAP.

Similarly, Theorem 2 part (c) and Theorem 3 part (d) share the properties of C, IA and IBC. Both theorems only differ in the property of E, used in the case of the Owen value, and the properties of 1-E and NAP, used in the case of the Banzhaf–Owen value. Note that 1-E is a particular case of E. So, it can be asserted that the two characterizations only differ in just one property.

On the other hand, all the properties in Theorems 2 and 3 are independent.

Table 1 Properties satisfied by the Owen and the Banzhaf–Owen values

Properties	Owen	Banzhaf–Owen
Coalitional Shapley value for singletons	$\checkmark^{(2a,2b)}$	$\times$
Coalitional Banzhaf value for singletons	$\times$	$\checkmark^{(3a,3b)}$
Efficiency (E)	$\checkmark^{(2b,2c)}$	$\times$
1-Efficiency (1-E)	$\checkmark^{(2d)}$	$\checkmark^{(3c,3d)}$
Neutrality for amalgamated players (NAP)	$\times$	$\checkmark^{(3a,3b,3c,3d)}$
Coherence (C)	$\checkmark^{(2c,2d)}$	$\checkmark^{(3c,3d)}$
Independence of amalgamation in other unions (IA)	$\checkmark^{(2b,2c)}$	$\checkmark^{(3b,3d)}$
1-Quotient game (1-QG)	$\checkmark$	$\checkmark^{(3a,3c)}$
Quotient game (QG)	$\checkmark^{(2a,2d)}$	$\times$
Intracoalitional balanced contributions (IBC)	$\checkmark^{(2a,2b,2c,2d)}$	$\checkmark^{(3a,3b,3c,3d)}$

- *Independence of the properties in Theorem 2 part (a)*
  - The coalitional value given for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$, with $C_h \in C$, by $\gamma_i^1(N, v, C) = Sh_i(C_h, v_1)$, where (C_h, v_1) is the TU-game such that $v_1(T) = \dfrac{v(T \cup (N \backslash C_h))}{m}$ for all $T \subseteq C_h$, $T \neq \emptyset$, satisfies E, IA and IBC but it is not a coalitional Shapley value for singletons.
  - The coalitional value given for all $(N, v, C) \in \mathcal{CG}$ by $\gamma^2(N, v, C) = Sh(N, v)$ is a coalitional Shapley value for singletons that satisfies IBC and fails QG.
  - The two-step Shapley value defined by Kamijo (2009) and given for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$, with $C_h \in C$, by the formula

$$Sh_i(C_h, v) + \frac{1}{c_h}[Sh_h(M, v^C) - v(C_h)]$$

    is a coalitional Shapley value for singletons that satisfies QG and fails IBC.
- *Independence of the properties in Theorem 2 part (b)*
  - The coalitional value γ^1 satisfies E, IA and IBC but it is not a coalitional Shapley value for singletons.
  - The coalitional value studied by Alonso-Meijide et al. (2014) and defined for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$, with $C_h \in C$, by

$$\sum_{R \subseteq M \backslash \{h\}} \sum_{T \subseteq C_h \backslash \{i\}} \frac{(r+t)!(m+c_h-r-t-2)!}{(m+c_h-1)!}[v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

    is a coalitional Shapley value for singletons that satisfies both IA and IBC but fails E.
  - The coalitional value γ^2 is a coalitional Shapley value for singletons that satisfies E and IBC but fails IA.

  - The two-step Shapley value is a coalitional Shapley value for singletons that satisfies E and IA but fails IBC.
- *Independence of the properties in Theorem 2 part (c)*
  - The Banzhaf–Owen value satisfies C, IA and IBC but fails E.
  - The coalitional value γ^1 does not satisfy C but satisfies E, IA and IBC.
  - The coalitional value γ^2 satisfies E, C and IBC and fails IA.
  - The two-step Shapley value is a coalitional value that satisfies E, C and IA but fails IBC.
- *Independence of the properties in Theorem 2 part (d)*
  - The coalitional value γ^3 defined for all $(N, v, C) \in \mathcal{CG}$ by $\gamma^3(N, v, C) = \alpha \cdot Ow(N, v, C)$, with $\alpha \in \mathbb{R}\backslash\{1\}$, satisfies C, QG and IBC but fails 1-E.
  - The coalitional value γ^1 does not satisfy C but satisfies 1-E, QG and IBC.
  - The coalitional value γ^2 satisfies 1-E, C and IBC and fails QG.
  - The two-step Shapley value is a coalitional value that satisfies 1-E, C and QG but fails IBC.
- *Independence of the properties in Theorem 3 parts (a) and (b)*
  - The coalitional value defined by Amer et al. (1995) and defined for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$, with $C_h \in C$, by

$$\sum_{R \subseteq M\backslash\{h\}} \sum_{T \subseteq C_h\backslash\{i\}} \frac{r!(m-r-1)!}{m!} \frac{1}{2^{c_h-1}} [v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

    satisfies NAP, IA, 1-QG and IBC but, however, it is not a coalitional Banzhaf value for singletons.
  - The symmetric coalitional Banzhaf value introduced by Alonso-Meijide and Fiestras-Janeiro (2002) and defined for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$, with $C_h \in C$, by

$$\sum_{R \subseteq M\backslash\{h\}} \sum_{T \subseteq C_h\backslash\{i\}} \frac{1}{2^{m-1}} \frac{t!(c_h-t-1)!}{c_h!} [v(Q \cup T \cup \{i\}) - v(Q \cup T)]$$

    is a coalitional Banzhaf value for singletons, satisfies IA, 1-QG and IBC but fails NAP.
  - The coalitional value given for all $(N, v, C) \in \mathcal{CG}$ by $\gamma^4(N, v, C) = Bz(N, v)$ is a coalitional Banzhaf value for singletons that satisfies NAP and IBC but fails 1-QG and IA.
  - The coalitional value given, for all $(N, v, C) \in \mathcal{CG}$ and all $i \in C_h$, with $C_h \in C$, by

$$\gamma_i^5(N, v, C) = Bz_i(C_h, v) + \frac{1}{2^{c_h-1}} [Bz_h(M, v^C) - v(C_h)]$$

    is a coalitional Banzhaf value for singletons that satisfies NAP, IA and 1-QG. However, it does not satisfy property IBC.
- *Independence of the properties in Theorem 3 parts (c) and (d)*

- The coalitional value γ^6 defined for all $(N, v, C) \in \mathscr{CG}$ by $\gamma^6(N, v, C) = \alpha \cdot BzOw(N, v, C)$, with $\alpha \in \mathbb{R}\backslash\{1\}$, satisfies NAP, C, 1-QG, IA and IBC but fails 1-E.
- The Owen value satisfies 1-E, C, IA, 1-QG and IBC but fails NAP.
- The coalitional value defined by Amer et al. (1995) satisfies 1-E, NAP, IA, 1-QG and IBC and fails C.
- The coalitional value γ^4 satisfies 1-E, NAP, C and IBC but fails IA and 1-QG.
- The coalitional value γ^5 satisfies 1-E, NAP, C, IA and 1-QG. However, it does not satisfy property IBC.

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