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ORIGINAL PAPER

First- and second-order optimality conditions for multiobjective fractional programming

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Abstract We consider nonsmooth multiobjective fractional programming on normed spaces. Using first- and second-order approximations as generalized derivatives, first- and second-order optimality conditions are established. Unlike the existing results, we avoid completely convexity assumptions. Our results can be applied even in infinite-dimensional cases, involving non-Lipschitz maps.

Keywords Multiobjective fractional programming · First and second-order approximations · Weak solutions · Firm solutions · Optimality conditions · Asymptotical pointwise compactness

Mathematics Subject Classfication 90C32 · 90C29 · 49K99

1 Introduction

Fractional programming has been an intensively developed topic in optimization, see, e.g., research papers (Borwein 1976; Schaible 1982; Singh 1981, 1986), a basic presentation in a handbook (Schaible 1995), and bibliographies (Schaible 1982; Stancu-Minasian 2006). Along with numerous contributions to multiobjective optimization, a very important area with significant practical applications in science, economics and engineering, multiobjective problems of fractional programming has also become

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attractive to many researchers, see, e.g., (Bector et al. 1993; Kim et al. 2005; Kuk et al. 2001; Liang et al. 2001; Lyall et al. 1997; Nobakhtian 2008; Reedy and Mukherjee 2001; Soleimani-Damaneh 2008; Zalmai 2006). In these papers, increasing efforts of dealing with nonsmooth problems, relying on various generalized derivatives, can be recognized. Severe convexity requirements, especially in sufficient optimality conditions, have been gradually reduced, using relaxed convexity notions. However, we observe that almost no contributions to problems in infinite-dimensional spaces and that convexity assumptions have not been completely removed so far.

Inspired by these observations, we consider in this paper a nonsmooth multiobjective fractional programming problem in normed spaces. To avoid completely convexity restrictions, we employ first- and second-order approximations as generalized derivatives. This kind of derivatives proved to be effective in problems with a high level of nonsmoothness and without convexity of the data, see Khanh and Tuan (2006, 2008, 2009, 2011, 2014).

The organization of this paper is as follows. In Sect. 2, we state our fractional problem and recall notions needed in the sequel. Section 3 is devoted to properties and calculus rules for first- and second-order approximations for later use. First-order optimality conditions are discussed in Sect. 4. The last Sect. 5 deals with second-order optimality conditions in both first-order differentiable cases and completely nonsmooth cases.

2 Preliminaries

Throughout the paper, if not otherwise specified, let spaces under consideration like X, Y, and Y_i (for i in a given index set) be normed spaces, $K \subseteq Y$ and $C \subseteq \mathbb{R}^m$ be proper closed convex cones with nonempty interior, C being pointed. For $A \subseteq X$, $\mathrm{int}A$, $\mathrm{cl}A$, $\mathrm{bd}A$, A_∞ and $\mathrm{cone}A$ denote its interior, closure, boundary, recession cone (i.e., the cone $\{\lim t_n a_n \mid a_n \in A, t_n \downarrow 0\}$) and the cone generated by A (i.e., $\{tx| \ x \in A, \ t \geq 0\}$), respectively (shortly, respectively). X^* is the dual space of X, B_X stands for the closed unit ball in X, and $B(x_0, \epsilon)$ is the open ball of center x_0 and radius ϵ. We consider the following multiobjective fractional programming problem

$$\min \varphi(x) := \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_m(x)}{g_m(x)}\right) \text{ s.t. } h(x) \in -K, \tag{P}$$

where $f_i, g_i : X \to \mathbb{R}$, $h : X \to Y$ with g_i being continuous and nonzero-valued for $i = 1, \ldots, m$.

Set $f(x) := (f_1(x), \ldots, f_m(x))$, $g(x) := (g_1(x), \ldots, g_m(x))$ and $S := \{x \in X \mid h(x) \in -K\}$ (the feasible set).

Definition 2.1 (e.g., Khanh and Tuan 2008)

(i) A point $x_0 \in S$ is called a local weak solution (local Pareto solution) of (P) if there exists a neighborhood U of x_0 such that, for every $x \in U \cap S$,

$$\varphi(x) - \varphi(x_0) \notin -\mathrm{int}C \ (\varphi(x) - \varphi(x_0) \notin -C\backslash\{0\}, \text{respectively}).$$

The set of all local weak (local Pareto, respectively) solutions of (P) is denoted by $\mathrm{LWE}(\varphi, S)$ ($\mathrm{LE}(\varphi, S)$, respectively).

(ii) For $k \in \mathbb{N}$, $x_0 \in S$ is called a local firm solution of order k of (P), denoted by $x_0 \in \mathrm{LFE}(k, \varphi, S)$, if there are $\gamma > 0$ and neighborhood U of x_0 such that, for all $x \in U \cap S \backslash \{x_0\}$,

$$(\varphi(x) + C) \cap B_{\mathbb{R}^m}(\varphi(x_0), \gamma \|x - x_0\|^k) = \emptyset.$$

Note that a firm solution is known in the literature also as an isolated solution or strict solution. Observe that, for $p \geq m$,

$$\mathrm{LFE}(m, \varphi, S) \subseteq \mathrm{LFE}(p, \varphi, S) \subseteq \mathrm{LE}(\varphi, S) \subseteq \mathrm{LWE}(\varphi, S).$$

So, necessary conditions for the right-most term hold true also for the others and sufficient conditions for the left-most term are valid for the others as well.

Let $L(X, Y)$ be the space of the continuous linear mappings from X into Y and $B(X, X, Y)$ that of the continuous bilinear mappings from $X \times X$ into Y. For $A \subseteq L(X, Y)$ and $x \in X$ ($B \subseteq L(X, X, Y)$ and $x, z \in X \times X$), denote $A(x) := \{M(x) \mid M \in A\}$ $(B(x, z) := \{N(x, z) \mid N \in B\})$. $o(t^k)$, for $t > 0$ and $k \in \mathbb{N}$, stands for a moving point such that $o(t^k)/t^k \to 0$ as $t \downarrow 0$. For a cone $K \subseteq Y$, the positive polar cone of K is

$$K^* := \{y^* \in Y^* \mid \langle y^*, y \rangle \geq 0, \ \forall y \in K\}.$$

Definition 2.2 (*Classic*) Let $x_0, v \in X$ and $S \subseteq X$.

(i) The contingent (or Bouligand) cone of S at x_0 is

$$T(S, x_0) := \{v \in X \mid \exists t_n \downarrow 0, \ \exists v_n \to v, \forall n \in \mathbb{N}, x_0 + t_n v_n \in S\}.$$

(ii) The second-order contingent set of S at (x_0, v) is

$$T^2(S, x_0, v) := \left\{ w \in X \mid \exists t_n \downarrow 0, \ \exists w_n \to w, \forall n \in \mathbb{N}, x_0 + t_n v + \frac{1}{2} t_n^2 w_n \in S \right\}.$$

(iii) The asymptotic second-order tangent cone of S at (x_0, v), see Penot (2000), is

$$T''(S, x_0, v) := \left\{ w \in X \mid \exists (t_n, r_n) \downarrow (0, 0) : \frac{t_n}{r_n} \to 0, \ \exists w_n \to w, \right.$$
$$\left. \forall n \in \mathbb{N}, x_0 + t_n v + \frac{1}{2} t_n r_n w_n \in S \right\}.$$

A subset $S \subseteq X$ is said to be polyhedral if it is the intersection of a finite number of closed half-spaces.

Definition 2.3 (Jourani and Thibault 1993) Let $f : X \to Y$.

(i) A set $A_f(x_0) \subseteq L(X, Y)$ is said to be a first-order approximation of f at $x_0 \in X$ if there exists a neighborhood U of x_0 such that, for all $x \in U$,

$$f(x) - f(x_0) \in A_f(x_0)(x - x_0) + o(\|x - x_0\|);$$

(ii) A set $(A_f(x_0), B_f(x_0)) \subseteq L(X, Y) \times L(X, X, Y)$ is called a second-order approximation of f at x_0 if
  (a) $A_f(x_0)$ is a first-order approximation of f at x_0;
  (b) $f(x) - f(x_0) \in A_f(x_0)(x - x_0) + B_f(x_0)(x - x_0, x - x_0) + o(\|x - x_0\|^2)$.

This kind of generalized derivatives contains a major part of known notions of derivatives as special cases (see Khanh and Tuan 2006, 2008). Furthermore, it is advantageous that even an infinitely discontinuous map may have approximations as is shown by the following example.

Example 2.1 Let $X = Y = \mathbb{R}$, $x_0 = 0$, and

$$f(x) = \begin{cases} -\frac{1}{x}, & \text{if } x > 0, \\ -x, & \text{if } x \le 0. \end{cases}$$

Then, f is infinitely discontinuous at x_0, but it admits $A_f(x_0) =]-\infty, \alpha[$ with $-1 < \alpha < 0$ as an approximation. Indeed, consider x close to x_0. If $x \le 0$, then $f(x) = -x = (-1)(x - 0) + o(|x|)$ with $o(|x|) = 0$ and $-1 \in A_f(x_0) = (-\infty, \alpha)$. Now consider $x > 0$. We need to show that $f(x) = -1/x \in A_f(x_0)(x - 0) + o(|x|)$, i.e., there exists $\beta_x \in A_f(x_0)$ such that $o(|x|)/|x| = -1/x^2 - \beta_x$. If $0 < x < \delta$, then $-\infty < -1/x^2 < -1/\delta^2$, and hence for $\delta > 0$ satisfying $-1/\delta^2 < \alpha$, one has $-1/x^2 \in (-\infty, \alpha)$. Hence, for any $\epsilon > 0$, there exists $\delta > 0$ satisfying $-1/\delta^2 < \alpha$ and $\beta_x \in A_f(x_0) = (-\infty, \alpha)$ such that $-1/x^2 - \beta_x < \epsilon$ for all $x \in (0, \delta)$, and we are done.

For $m \in \mathbb{N}$, $f : X \to Y$ is said to be m-calm at x_0 if there exists $L > 0$ and neighborhood U of x_0 such that, for all $x \in U$,

$$\|f(x) - f(x_0)\| \le L\|x - x_0\|^m.$$

In this case, L is called the coefficient of calmness of f. (1-calmness is called simply calmness). Of course, if f is m-calm at x_0, then f is continuous at x_0, for any $m \in \mathbb{N}$.

Let M_α and M be in $L(X, Y)$. The net $\{M_\alpha\}$ is said to pointwise converge to M, and written as $M_\alpha \xrightarrow{p} M$ or $M = \text{p-lim} M_\alpha$, if $\lim M_\alpha(x) = M(x)$ for all $x \in X$. A similar definition is adopted for $N_\alpha, N \in L(X, X, Y)$. Note that the pointwise convergence topology is not metrizable. If $Y = \mathbb{R}$, this topology collapses to the star-weak topology. A subset $A \subseteq L(X, Y)$ ($B \subseteq L(X, X, Y)$) is called asymptotically pointwise compact (shortly asymptotically p-compact) if (see Bector et al. 1993; Kim et al. 2005)

(a) each bounded net $\{M_\alpha\} \subseteq A$ ($\subseteq B$, respectively) has a subnet $\{M_\beta\}$ and $M \in L(X, Y)$ ($M \in L(X, X, Y)$) such that $M = \text{p-lim} M_\beta$;
(b) for each net $\{M_\alpha\} \subseteq A$ ($\subseteq B$, respectively) with $\lim \|M_\alpha\| = \infty$, the net $\{M_\alpha/\|M_\alpha\|\}$ has a subnet converging pointwise to some $M \in L(X, Y)\backslash\{0\}$ ($M \in L(X, X, Y)\backslash\{0\}$).

If pointwise convergence is replaced by convergence (in the norm topology), the term "asymptotic compactness" is used. If X and Y are finite dimensional, every subset is asymptotically p-compact and asymptotically compact. But, in infinite dimensions, the asymptotical p-compactness is weaker than asymptotical compactness.

For $A \subseteq L(X, Y)$ and $B \subseteq L(X, X, Y)$, we adopt the notations:

$$
\begin{aligned}
&p - clA := \{M \in L(X, Y) \mid \exists\{M_\alpha\} \subseteq A,\ M = p - \lim M_\alpha\},\\
&p - clB := \{N \in L(X, X, Y) \mid \exists\{N_\alpha\} \subseteq B,\ N = p - \lim N_\alpha\},\\
&A_\infty := \{M \in L(X, Y) \mid \exists\{M_\alpha\} \subseteq A,\ \exists t_\alpha \downarrow 0, M = \lim t_\alpha M_\alpha\},\\
&p - A_\infty := \{M \in L(X, Y) \mid \exists\{M_\alpha\} \subseteq A,\ \exists t_\alpha \downarrow 0, M = p - \lim t_\alpha M_\alpha\},\\
&p - B_\infty := \{N \in L(X, X, Y) \mid \exists\{N_\alpha\} \subseteq B,\ \exists t_\alpha \downarrow 0, N = p - \lim t_\alpha N_\alpha\}.
\end{aligned}
$$

Observe that, p-cl$A$, p-cl$B$ are the pointwise closures, A_∞ is the recession cone and p-A_∞, p-B_∞ are the pointwise recession cones of the given sets.

3 Properties and calculus rules of approximations

First, some properties of approximations of maps with regular characters are collected in the following proposition.

Proposition 3.1 *Let $f : X \to Y$.*

(i) *Suppose $(\{0\}, B_f(x_0))$ is a second-order approximation of f at x_0 and $B_f(x_0)$ is bounded. Then, f is 2-calm at x_0.*

(ii) *Let $Y = \mathbb{R}$. If the Fréchet derivative f' exists in a convex neighborhood U of x_0 is calm at x_0 with coefficient L, and $f'(x_0) = 0$, then f is 2-calm at x_0 with the same coefficient L.*

(iii) *If f is 2-calm at x_0, then $f'(x_0) = 0$.*

Proof (i) By the assumption, there exists $L > 0$ such that $\|M\| \le L$ for all $M \in B_f(x_0)$. Furthermore, there is a neighborhood U of x_0 such that, for all $x \in U$, there exists
$M_x \in B_f(x_0)$ with

$$
\begin{aligned}
\|f(x) - f(x_0)\| &= \|M_x(x - x_0, x - x_0) + o(\|x - x_0\|^2)\|\\
&\le L.\|x - x_0\|.\|x - x_0\| + \|o(\|x - x_0\|^2)\| \le (L + \epsilon)\|x - x_0\|^2,
\end{aligned}
$$

for some $\epsilon > 0$. Hence, f is 2-calm at x_0.

(ii) From the mean value theorem, for $x \in U$, there exists $c := \alpha x_0 + (1 - \alpha)x$ with $\alpha \in [0, 1]$ such that $f(x) - f(x_0) = f'(c)(x - x_0)$. Hence,

$$
\begin{aligned}
|f(x) - f(x_0)| &= \|f'(c)\|.\|x - x_0\| = \|f'(c) - f'(x_0)\|.\|x - x_0\|\\
&\le L\|c - x_0\|.\|x - x_0\|\\
&= L\|\alpha x_0 + (1 - \alpha)x - x_0\|.\|x - x_0\| = L(1 - \alpha)\|x - x_0\|^2\\
&\le L\|x - x_0\|^2.
\end{aligned}
$$

(iii) We have

$$\begin{aligned}\lim_{x\to x_0}\frac{\|f(x)-f(x_0)-0.(x-x_0)\|}{\|x-x_0\|}&=\lim_{x\to x_0}\frac{\|f(x)-f(x_0)\|}{\|x-x_0\|}\\&=\lim_{x\to x_0}\frac{\|f(x)-f(x_0)\|}{\|x-x_0\|^2}.\|x-x_0\|\\&\le\lim_{x\to x_0}L\|x-x_0\|=0.\end{aligned}$$

Therefore, f is Fréchet differentiable at x_0 and $f'(x_0)=0$. □

In the following two propositions, some simple calculus rules, needed in establishing optimality conditions for problem (P), are developed.

Proposition 3.2 *Let* $f_i : X \to Y_i$, $f := (f_1, \dots, f_k) : X \to Y_1 \times \cdots \times Y_k$ *and* $\lambda_i \in \mathbb{R}$ *for* $i = 1, \dots, k$. *Let* $A_{f_i}(x_0)$ *be a first-order approximation of* f_i *at* x_0. *Then, the following assertions hold.*

(i) $\sum_{i=1}^k \lambda_i A_{f_i}(x_0)$ *is a first-order approximation of* $\sum_{i=1}^k \lambda_i f_i$ *at* x_0.
(ii) *Let* $f = (f_1, f_2, \dots, f_k)$ *and* $A_{f_1}(x_0), \dots, A_{f_k}(x_0)$ *be first-order approximations of* $f_1, \dots, f_k$, *respectively, at* x_0. *Then,* $A_{f_1}(x_0) \times \ldots \times A_{f_k}(x_0)$ *is a first-order approximation of* f *at that point.*
(iii) *Let* Y *be a Hilbert space,* $f, g : X \to Y$ *and* $\langle f, g\rangle(x) := \langle f(x), g(x)\rangle$. *If* $A_f(x_0)$, $A_g(x_0)$ *are first-order bounded approximations of* f *and* g *at* x_0, *then* $\langle g(x_0), A_f(x_0)\rangle + \langle f(x_0), A_g(x_0)\rangle$ *is a first-order approximation of* $\langle f, g\rangle$ *at* x_0.
(iv) *Let* $f : X \to Y$ *and* $g : Y \to Z$. *If* $A_f(x_0)$, $A_g(f(x_0))$ *are bounded approximations, then* $A_g(f(x_0)) \circ A_f(x_0)$ *is a first-order approximation of* $g \circ f$ *at* x_0.

Proof (i) For each $i = 1, \dots, k$, there exists a neighborhood U_i of x_0 such that, for all $x \in U_i$,

$$f_i(x) - f(x_0) \in A_{f_i}(x_0)(x - x_0) + o_i(\|x - x_0\|).$$

Hence, for all $x \in U := \bigcap_{i=1}^k U_i$,

$$\sum_{i=1}^k f_i(x) - \sum_{i=1}^k f_i(x_0) \in \sum_{i=1}^k A_{f_i}(x_0)(x - x_0) + o(\|x - x_0\|),$$

where $o(\|x - x_0\|) = \sum_{i=1}^k o_i(\|x - x_0\|)$.
(ii) This is immediate.
(iii) There exists a neighborhood U of x_0 such that, for all $x \in U$,

$$\begin{aligned}&\langle f, g\rangle(x) - \langle f, g\rangle(x_0)\\&\quad= (\langle f(x), g(x)\rangle - \langle f(x), g(x_0)\rangle) + (\langle f(x), g(x_0)\rangle - \langle f(x_0), g(x_0)\rangle)\end{aligned}$$

$$\begin{aligned}
&= \langle f(x_0), g(x) - g(x_0)\rangle + \langle f(x) - f(x_0), g(x) - g(x_0)\rangle \\
&\quad + \langle g(x_0), f(x) - f(x_0)\rangle \in \langle f(x_0), A_g(x_0)(x - x_0) \\
&\quad + o_1(\|x - x_0\|)\rangle + \langle g(x_0), A_f(x_0)(x - x_0) \\
&\quad + o_2(\|x - x_0\|)\rangle + \langle f(x) - f(x_0), g(x) - g(x_0)\rangle \\
&= (\langle f(x_0), A_g(x_0)\rangle + \langle g(x_0), A_f(x_0)\rangle)(x - x_0) + \langle f(x_0), o_1(\|x - x_0\|)\rangle \\
&\quad + \langle g(x_0), o_2(\|x - x_0\|)\rangle + \langle f(x) - f(x_0), g(x) - g(x_0)\rangle.
\end{aligned}$$

By the boundedness of $A_f(x_0)$, $A_g(x_0)$, we have L_1 and L_2 such that, for x close to x_0, $\|f(x) - f(x_0)\| \le L_1\|x - x_0\|$ and $\|g(x) - g(x_0)\| \le L_2\|x - x_0\|$. Hence,

$$\begin{aligned}
\|\langle f(x) - f(x_0), g(x) - g(x_0)\rangle\| &\le \|f(x) - f(x_0)\|.\|g(x) - g(x_0)\| \\
&\le L_1 L_2 \|x - x_0\|^2.
\end{aligned}$$

Summarizing the above estimates we get, for some $o(\|x - x_0\|)$,

$$\begin{aligned}
\langle f, g\rangle(x) - \langle f, g\rangle(x_0) = (\langle f(x_0), A_g(x_0)\rangle + \langle g(x_0), A_f(x_0)\rangle)(x - x_0) \\
+ o(\|x - x_0\|).
\end{aligned}$$

(iv) There exists a neighborhood U of x_0 and V of $f(x_0)$ such that, for all $x \in U \cap f^{-1}(V)$,

$$\begin{aligned}
&(g \circ f)(x) - (g \circ f)(x_0) \in A_g(f(x_0))(f(x) - f(x_0)) + o_2(\|f(x) - f(x_0\|) \\
&\subseteq A_g(f(x_0))[A_f(x_0)(x - x_0) + o_1(\|x - x_0\|)] + o_2(\|f(x) - f(x_0\|) \\
&= [(A_g(f(x_0)) \circ A_f(x_0)](x - x_0) + A_g(f(x_0))o_1(\|x - x_0\|) \\
&\quad + o_2(\|f(x) - f(x_0\|).
\end{aligned}$$

Let $u \in A_g(f(x_0))o_1(\|x - x_0\|) + o_2(\|f(x) - f(x_0\|)$. We need to prove that $u\|x - x_0\|^{-1} \to 0$. Indeed, by the boundedness of $A_f(x_0)$, $A_g(x_0)$, there exist L_1 and L_2 such that $\|M\| \le L_1$ for all $M \in A_g(f(x_0))$ and, for x close to x_0, $\|f(x) - f(x_0)\| \le L_2\|x - x_0\|$. Hence, with $M_u \in A_g(f(x_0))$,

$$\begin{aligned}
\frac{\|u\|}{\|x - x_0\|} &= \left\| M_u \frac{o_1(\|x - x_0\|)}{\|x - x_0\|} + \frac{o_2(\|f(x) - f(x_0)\|)}{\|f(x) - f(x_0)\|}.\frac{\|f(x) - f(x_0)\|}{\|x - x_0\|} \right\| \\
&\le L_1 \frac{\|o_1(\|x - x_0\|)\|}{\|x - x_0\|} + L_2 \frac{\|o_2(\|f(x) - f(x_0)\|)\|}{\|f(x) - f(x_0)\|} \to 0.
\end{aligned}$$

□

For $f, g : X \to \mathbb{R}$, we define $f.g$ and f/g as usual: $(f.g)(x) := f(x).g(x)$ and $(f/g)(x) := f(x).(g(x))^{-1}$ for $x \in X$.

Proposition 3.3 *Let $f, g : X \to \mathbb{R}$ and $A_f(x_0)$, $A_g(x_0)$ be first-order approximations of f and g, respectively, at x_0. Then, the following assertions hold.*

(i) *If g is continuous at x_0 and $A_f(x_0)$ is bounded, then $g(x_0)A_f(x_0)+f(x_0)A_g(x_0)$ is a first-order approximation of $f.g$ at x_0.*
(ii) *If $A_f(x_0)$, $A_g(x_0)$ are bounded and $g(x_0) \neq 0$, then $[g(x_0)A_f(x_0) - f(x_0) A_g(x_0)]/g^2(x_0)$ is a first-order approximation of f/g at x_0.*

Proof (i) From the assumptions, there exists a neighborhood U of x_0 such that, for all $x \in U$,

$$\begin{aligned}(f.g)(x) - (f.g)(x_0) &= g(x)[f(x) - f(x_0)] + f(x_0)[g(x) - g(x_0)]\\ &\in g(x)[A_f(x_0)(x - x_0) + o_1(\|x - x_0\|)] + f(x_0)[A_g(x_0)(x - x_0)\\ &\quad +o_2(\|x - x_0\|)]\\ &= [g(x_0)A_f(x_0) + f(x_0)A_g(x_0)](x - x_0) + (g(x) - g(x_0))A_f(x_0)(x - x_0)\\ &\quad +g(x)o_1(\|x - x_0\|) + f(x_0)o_2(\|x - x_0\|)].\end{aligned}$$

We have to show that, for any u in the set being the last line above, $u\|x-x_0\|^{-1} \to 0$ when $x \to x_0$. Indeed, there is $x^* \in A_f(x_0)$ (depending on$x$) such that

$$\begin{aligned}u\|x - x_0\|^{-1} = [(g(x) - g(x_0))\langle x^*, x - x_0\rangle + g(x)o_1(\|x - x_0\|)\\ +f(x_0)o_2(\|x - x_0\|].\|x - x_0\|^{-1}.\end{aligned}$$

Because of the boundedness of $A_f(x_0)$, there exists $L > 0$ such that $\|x^*\| \leq L, \forall x^* \in A_f(x_0)$. Passing to limit, one has $u\|x - x_0\|^{-1} \to 0$, since

$$\begin{aligned}\lim_{x\to x_0} |g(x) - g(x_0)|.\frac{|\langle x^*, x - x_0\rangle|}{\|x - x_0\|} &\leq \lim_{x\to x_0} |g(x) - g(x_0)|\frac{\|x^*\|.\|x - x\|}{\|x - x_0\|}\\ &\leq \lim_{x\to x_0} L.|g(x) - g(x_0)| = 0.\end{aligned}$$

(ii) We have

$$\begin{aligned}\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0) &= \frac{g(x_0)(f(x) - f(x_0)) - f(x_0)(g(x) - g(x_0))}{g(x)g(x_0)}\\ &\in \frac{1}{g(x)g(x_0)}[g(x_0)(A_f(x_0)(x-x_0) + o_1(\|x-x_0\|))\\ &\quad - f(x_0)(A_g(x_0)(x-x_0) + o_2(\|x - x_0\|))]\\ &= \frac{g(x_0)A_f(x_0) - f(x_0)A_g(x_0)}{g(x)g(x_0)}(x - x_0)\\ &\quad + \frac{g(x_0)o_1(\|x - x_0\|) - f(x_0)o_2(\|x - x_0\|)}{g(x)g(x_0)}\\ &= \frac{g(x_0)A_f(x_0)- f(x_0)A_g(x_0)}{g^2(x_0)}(x - x_0)-[(g(x_0)A_f(x_0)\\ &\quad - f(x_0)A_g(x_0))(\frac{g(x) - g(x_0)}{g(x)g^2(x_0)})](x - x_0)\end{aligned}$$

$$+\frac{g(x_0)o_1(\|x-x_0\|)-f(x_0)o_2(\|x-x_0\|)}{g(x)g(x_0)}$$
$$\in \frac{g(x_0)A_f(x_0)-f(x_0)A_g(x_0)}{g^2(x_0)}(x-x_0)-(g(x_0)A_f(x_0)$$
$$-f(x_0)A_g(x_0))(x-x_0)\frac{g(x)-g(x_0)}{g(x)g^2(x_0)}$$
$$+\frac{g(x_0)o_1(\|x-x_0\|)-f(x_0)o_2(\|x-x_0\|)}{g(x)g(x_0)}.$$

Similarly as in (i), one gets the required conclusion, by the boundedness of $A_f(x_0), A_g(x_0)$. □

Proposition 3.4 (i) *Let $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1 \neq 0$. If $A_1, A_2 \subseteq L(X, Y)$ are asymptotically p-compact sets with A_2 being bounded, then $\lambda_1 A_1 + \lambda_2 A_2$ is an asymptotically p-compact set.*

(ii) *For asymptotically p-compact sets $A_i \subseteq L(X, Y_i), i = 1, \ldots, k$, $\prod_{i=1}^k A_i \subseteq L(X, \prod_{i=1}^k Y_i)$ is also asymptotically p-compact.*

Proof (i) Let $\{\lambda_1 M_\alpha + \lambda_2 N_\alpha\}$ be a net in $\lambda_1 A_1 + \lambda_2 A_2$. Since $\{N_\alpha\}$ is bounded, we assume that $N_\alpha \xrightarrow{p} N$. If $\{M_\alpha\}$ is bounded, $\{\lambda_1 M_\alpha + \lambda_2 N_\alpha\}$ admits also a pointwise convergent subnet. If $\{M_\alpha\}$ is unbounded, we may assume that $\|M_\alpha\| \to \infty$ and $M_\alpha \|M_\alpha\|^{-1} \xrightarrow{p} M$ with $M \in L(X, Y)\backslash\{0\}$. Since $N_\alpha/\|M_\alpha\| \xrightarrow{p} 0$, one has

$$\frac{\lambda_1 M_\alpha + \lambda_2 N_\alpha}{\|\lambda_1 M_\alpha + \lambda_1 N_\alpha\|} \xrightarrow{p} \frac{\lambda_1 M}{\|\lambda_1 M\|} \in L(X, Y)\backslash\{0\}.$$

Consequently, $\lambda_1 A + \lambda_2 B$ is an asymptotically p-compact set.

(ii) Let $\{(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)\}$ be a net in $A_1 \times A_2 \times \cdots \times A_k$. If $\{M_\alpha^i\}, i = 1, \ldots, k$, are bounded, then clearly $\{(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)\}$ has a pointwise convergent subnet. If at least one of $\{M_\alpha^i\}, i = 1, \ldots, k$, is unbounded, say all are unbounded, we may assume that $\|M_\alpha^i\| \to \infty, i = 1, \ldots, k$, and $M_\alpha^i/\|M_\alpha^i\| \xrightarrow{p} M^i, i = 1, \ldots, k$, with $M^i \in L(X, Y_i)\backslash\{0\}$. Since $\{\|M_\alpha^i\|\}, i = 1, \ldots, k$ are nonnegative sequences in $\mathbb{R}$, there exist only three cases (using subsequences).

Case 1. There exists $\|M_\alpha^{i_0}\|$ such that $\|M_\alpha^i\|/\|M_\alpha^{i_0}\| \to 0, \forall i \in \{1, \ldots, k\}\backslash\{i_0\}$. One has

$$\frac{(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)}{\|(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)\|} = \frac{(M_\alpha^1/\|M_\alpha^{i_0}\|, \ldots, M_\alpha^k)/\|M_\alpha^{i_0}\|)}{\|(M_\alpha^1/\|M_\alpha^{i_0}\|, \ldots, M_\alpha^k/\|M_\alpha^{i_0}\|)\|}$$
$$\xrightarrow{p} \frac{(0, \ldots, M^{i_0}, \ldots, 0)}{\|(0, \ldots, M^{i_0}, \ldots, 0)\|} \in L(X, \prod_{i=1}^k Y_i)\backslash\{0\}.$$

Case 2. There exists $\|M_\alpha^{i_0}\|$ such that $\|M_\alpha^i\|/\|M_\alpha^{i_0}\| \to a_i > 0$ for all $i \in \{1, \ldots, k\}\backslash\{i_0\}$. Since $M_\alpha^i/\|M_\alpha^{i_0}\| \xrightarrow{p} a_i M^i$, one gets

$$\frac{(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)}{\|(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)\|} = \frac{(M_\alpha^1/\|M_\alpha^{i_0}\|, \ldots, M_\alpha^k/\|M_\alpha^{i_0}\|)}{\|(M_\alpha^1/\|M_\alpha^{i_0}\|, \ldots, M_\alpha^k/\|M_\alpha^{i_0}\|)\|}$$
$$\xrightarrow{p} \frac{(a_1 M^1, \ldots, M^{i_0}, \ldots, a_k M^k)}{\|(a_1 M^1, \ldots, M^{i_0}, \ldots, a_k M^k)\|} \in L(X, \prod_{i=1}^k Y_i)\backslash\{0\}.$$

Case 3. There exists $\|M_\alpha^{i_0}\|$ such that $\|M_\alpha^i\|/\|M_\alpha^{i_0}\| \to 0$ for all $i \in I_1$, and $\|M_\alpha^i\|/\|M_\alpha^{i_0}\| \to a_i > 0$ for all $i \in I_2$, with $I_1 \cup I_2 = \{1, \ldots, k\}\backslash\{i_0\}$, $I_1 \cap I_2 = \emptyset$. Then, there exists $N \in L(X, \prod_{i=1}^k Y_i)\backslash\{0\}$ such that

$$\frac{(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)}{\|(M_\alpha^1, M_\alpha^2, \ldots, M_\alpha^k)\|} = \frac{(M_\alpha^1/\|M_\alpha^{i_0}\|, \ldots, M_\alpha^k/\|M_\alpha^{i_0}\|)}{\|(M_\alpha^1/\|M_\alpha^{i_0}\|, \ldots, M_\alpha^k/\|M_\alpha^{i_0}\|)\|} \xrightarrow{p} N.$$

Therefore, $\prod_{i=1}^k A_i$ is asymptotically p-compact. □

Now, we pass to calculus rules for second-order approximations. In the following proposition, when Y is a Hilbert space, $y \in Y$, $A_1, A_2 \subseteq L(X, Y)$, and $B \subseteq L(X, X, Y)$, we denote $\langle y, A_1\rangle(.) := \langle y, A_1(.)\rangle$, $\langle y, B\rangle(., .) := \langle y, B(., .)\rangle$ and $\langle A_1, A_2\rangle(., .) := \langle A_1(.), A_2(.)\rangle$.

Proposition 3.5 (i) *Let $f_i : X \to Y$, $\lambda_i \in \mathbb{R}$, and $(A_{f_i}(x_0), B_{f_i}(x_0))$ be a second-order approximation of f_i at x_0 for $i = 1, \ldots, k$. Then, $\left(\sum_{i=1}^k \lambda_i A_{f_i}(x_0), \sum_{i=1}^k \lambda_i B_{f_i}(x_0)\right)$ is a second-order approximation of $\sum_{i=1}^k \lambda_i f_i$ at x_0.*

(ii) *Let $f_i : X \to Y_i$, $i = 1, \ldots, k$, $f := (f_1, f_2, \ldots, f_k)$, and $(A_{f_1}(x_0), B_{f_1}(x_0))$, $\ldots$, $(A_{f_k}(x_0), B_{f_k}(x_0))$ be second-order approximations of $f_1, \ldots, f_k$, respectively, at x_0. Then, $(A_{f_1}(x_0) \times \cdots \times A_{f_k}(x_0), B_{f_1}(x_0) \times \cdots \times B_{f_k}(x_0))$ is a second-order approximation of f at that point.*

(iii) *Let Y be a Hilbert space and $f, g : X \to Y$. If $(A_f(x_0), B_f(x_0))$ and $(A_g(x_0), B_g(x_0))$ are second-order approximations of f and g, respectively, at x_0 and $A_f(x_0), A_g(x_0)$ are bounded at x_0, then $(\langle g(x_0), A_f(x_0)\rangle + \langle f(x_0), A_g(x_0)\rangle, \langle g(x_0), B_f(x_0)\rangle + \langle f(x_0), B_g(x_0)\rangle + \langle A_f(x_0), A_g(x_0)\rangle)$ is a second-order approximation of $\langle f, g\rangle$ at x_0.*

Proof (i) and (ii) are easy consequences of Proposition 3.2.

(iii) Also by Proposition 3.2, $\langle g(x_0), A_f(x_0)\rangle + \langle f(x_0), A_g(x_0)\rangle$ is a first-order approximation of $\langle f, g\rangle$ at x_0. Furthermore, from the boundedness of $A_f(x_0), A_g(x_0)$, one gets

$$\begin{aligned}
&\langle f, g\rangle(x) - \langle f, g\rangle(x_0)\\
&\quad = \langle f(x_0), g(x) - g(x_0)\rangle + \langle g(x_0), f(x) - f(x_0)\rangle + \langle f(x) - f(x_0), g(x) - g(x_0)\rangle\\
&\quad \in \langle f(x_0), A_g(x_0)(x - x_0) + B_g(x_0)(x - x_0, x - x_0) + o_1(\|x - x_0\|^2)\rangle\\
&\qquad + \langle g(x_0), A_f(x_0)(x - x_0) + B_f(x_0)(x - x_0, x - x_0) + o_2(\|x - x_0\|^2\rangle\\
&\qquad + \langle A_f(x_0)(x - x_0), A_g(x_0)(x - x_0)\rangle + \langle A_f(x_0)(x - x_0), o_2(\|x - x_0\|)\rangle\\
&\qquad + \langle o_1(\|x - x_0\|), A_g(x_0)(x - x_0)\rangle + o_3(\|x - x_0\|^2)
\end{aligned}$$

$$= (\langle g(x_0), A_f(x_0)\rangle + \langle f(x_0), A_g(x_0)\rangle)(x - x_0) + (\langle g(x_0), B_f(x_0)\rangle$$
$$+\langle f(x_0), B_g(x_0)\rangle + \langle A_f(x_0), A_g(x_0)\rangle))(x - x_0, x - x_0) + o(\|x - x_0\|^2).$$

By the definition of a second-order approximation, the proof is complete. □

Proposition 3.6 *Let $f, g : X \to \mathbb{R}$, g be 2-calm at x_0, and $(A_f(x_0), B_f(x_0))$, $(0, B_g(x_0))$ be second-order approximations of f and g, respectively, at x_0. Then,*

- (i) *if $A_f(x_0)$, $B_f(x_0)$ are bounded, then $(f(x_0)A_g(x_0) + g(x_0)A_f(x_0), g(x_0)B_f(x_0) + f(x_0)B_g(x_0))$ is a second-order approximation of $f.g$ at x_0;*
- (ii) *if $A_f(x_0)$, $B_f(x_0)$, $B_g(x_0)$ are bounded and $g(x_0) \neq 0$, then*

$$\left(\frac{A_f(x_0)}{g(x_0)}, \frac{g(x_0)B_f(x_0) - f(x_0)B_g(x_0)}{g^2(x_0)}\right)$$

is a second-order approximation of f/g at x_0.

Proof (i) Proposition 3.3 implies that $g(x_0)A_f(x_0)$ is a first-order approximation of $f.g$ at x_0. On the other hand,

$$f(x)g(x) - f(x_0)g(x_0) = g(x)[f(x) - f(x_0)] + f(x_0)[g(x) - g(x_0)]$$
$$\in g(x)[A_f(x_0)(x - x_0) + B_f(x_0)(x - x_0, x - x_0) + o_1(\|x - x_0\|^2)]$$
$$+f(x_0)[B_g(x_0)(x - x_0, x - x_0) + o_2(\|x - x_0\|^2)]$$
$$= g(x_0)A_f(x_0)(x - x_0) + [g(x_0)B_f(x_0) + f(x_0)B_g(x_0)](x - x_0, x - x_0)$$
$$+[(g(x) - g(x_0))(A_f(x_0)(x - x_0) + B_f(x_0)(x - x_0, x - x_0))$$
$$+g(x)o_1(\|x - x_0\|^2) + f(x_0)o_2(\|x - x_0\|^2)].$$

We have to show that $u\|x - x_0\|^{-2} \to 0$, for all u in the set being the last term of the last side above, when $x \to x_0$. For such a u, there are $M_u \in A_f(x_0)$ and $N_u \in B_f(x_0)$ such that

$$u\|x - x_0\|^{-2} = [(g(x) - g(x_0))(\langle M_u, x - x_0\rangle + N_u(x - x_0, x - x_0))$$
$$+ g(x)o_1(\|x - x_0\|^2) + f(x_0)o_2(\|x - x_0\|^2)].\|x - x_0\|^{-2}.$$

Clearly, this element tends to 0 as $x \to x_0$, since g is 2-calm at x_0 and $A_f(x_0)$, $B_f(x_0)$ are bounded.

(ii) We have

$$\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(x_0) = \frac{g(x_0)(f(x) - f(x_0)) - f(x_0)(g(x) - g(x_0))}{g(x)g(x_0)}$$
$$\in \frac{1}{g(x)g(x_0)}[g(x_0)(A_f(x_0)(x - x_0) + B_f(x_0)(x - x_0, x - x_0)$$
$$+o_1(\|x - x_0\|^2)) - f(x_0)(B_g(x_0)(x - x_0, x - x_0) + o_2(\|x - x_0\|^2))]$$

$$
\begin{aligned}
&= \frac{A_f(x_0)}{g(x)}(x-x_0) + \frac{g(x_0)B_f(x_0) - f(x_0)B_g(x_0)}{g(x)g(x_0)}(x-x_0, x-x_0) \\
&\quad + \frac{g(x_0)o_1(\|x-x_0\|^2) - f(x_0)o_2(\|x-x_0\|^2)}{g(x)g(x_0)} \\
&= \frac{A_f(x_0)}{g(x_0)}(x-x_0) + g(x_0)A_f(x_0)(x-x_0)\left[\frac{1}{g(x)g(x_0)} - \frac{1}{g^2(x_0)}\right] \\
&\quad + \frac{g(x_0)B_f(x_0) - f(x_0)B_g(x_0)}{g^2(x_0)}(x-x_0, x-x_0) \\
&\quad + (g(x_0)B_f(x_0) - f(x_0)B_g(x_0))(x-x_0, x-x_0)\left[\frac{1}{g(x)g(x_0)} - \frac{1}{g^2(x_0)}\right] \\
&\quad + \frac{g(x_0)o_1(\|x-x_0\|^2) - f(x_0)o_2(\|x-x_0\|^2)}{g(x)g(x_0)} \\
&= \frac{A_f(x_0)}{g(x_0)}(x-x_0) + \frac{g(x_0)B_f(x_0) - f(x_0)B_g(x_0)}{g^2(x_0)}(x-x_0, x-x_0) \\
&\quad + [g(x_0)A_f(x_0)(x-x_0) + (g(x_0)B_f(x_0) \\
&\quad - f(x_0)B_g(x_0))(x-x_0, x-x_0)]\left[\frac{g(x_0)-g(x)}{g(x)g^2(x_0)}\right] \\
&\quad + \frac{g(x_0)o_1(\|x-x_0\|^2) - f(x_0)o_2(\|x-x_0\|^2)}{g(x)g(x_0)}.
\end{aligned}
$$

It remains to prove that the last two terms of the last side above are of the form $o(\|x-x_0\|^2)$. This proof is similar to the corresponding one in (i). □

The assumption that g is 2-calm at x_0 in Proposition 3.6 (ii) cannot be dispensed as shown by the following example.

Example 3.1 Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + 1$, $g(x) = x + 1$, and $x_0 = 0$. Then, $\varphi(x) := (\frac{f}{g})(x) = x^2 - x + 1,\ x \neq -1$. We can check that g is calm at x_0, but not 2-calm at this point. By direct calculations, we have $f(x_0) = 1$, $g(x_0) = 1$, and

$$
\begin{aligned}
&A_f(x) = \{0\},\ \ B_f(x_0) = \{0\}, \\
&A_g(x_0) = \{1\},\ \ B_g(x_0) = \{0\}, \\
&A_\varphi(x_0) = \{-1\}, B_\varphi(x_0) = \{1\}.
\end{aligned}
$$

So, $\{(-1, 1)\}$ is a second-order approximation of $\varphi(x) = x^2 - x + 1$ at x_0. Hence,

$$
\left(\frac{g(x_0)A_f(x_0) - f(x_0)A_g(x_0)}{g^2(x_0)}, \frac{g(x_0)B_f(x_0) - f(x_0)B_g(x_0)}{g^2(x_0)}\right) = \{(-1, 0)\}
$$

is not a second-order approximation of $\varphi(x) = x^2 - x + 1$ at x_0.

4 First-order optimality conditions

To establish necessary optimality conditions for our fractional problem, we need Lemma 4.1 below on such conditions for local weak solutions to the constrained vector minimization problem (P_1) below. Let X, Y and Z be normed spaces, C and K be proper closed convex cones with nonempty interior in Z and Y, respectively (note that, for (P_1), C does not need to be pointed). Let $F : X \to Z$ and $h : X \to Y$. Consider the vector minimization problem

$$\min F(x)\text{s.t.}h(x) \in -K. \tag{P_1}$$

Denote $\Omega := \{x \in X |\ h(x) \in -K\}$ (the feasible set). We recall that the cone of weak feasible directions to $S \subseteq X$ at $x_0 \in S$ is

$$W_f(S, x_0) := \{u \in X \mid \exists t_n \downarrow 0,\ \forall n,\ x_0 + t_n u \in S\}.$$

Lemma 4.1 *Assume that $A_F(x_0)$ and $A_h(x_0)$ are asymptotically p-compact first-order approximations of F and h, respectively, at x_0.*

If x_0 is a local weak solution of (P_1)*, then, $\forall u \in X$, $\exists P \in$ p-$A_F(x_0)$, $\exists Q \in A_h(x_0)$, $\exists (c^*, d^*) \in C^* \times K^* \backslash \{(0, 0)\}$,*

$$\langle c^*, P(u)\rangle + \langle d^*, Q(u)\rangle \geq 0,\ \ \langle d^*, h(x_0)\rangle = 0.$$

Furthermore, for u satisfying $0 \in \mathrm{int}(Q(u) + h(x_0) + K)$ for all $Q \in A_k(x_0)$, we have $c^* \neq 0$.

Proof Let x_0 be a local weak solution of (P_1). For $u \in X$, there are two cases.

Case 1. $u \in W_f(\Omega, x_0)$. Then, there exists $P \in$ p-$A_F(x_0)$ such that $P(u) \notin -\mathrm{int}C$. Indeed, for the sequence $t_n \downarrow 0$ associated with u (in the definition of $W_f(\Omega, x_0)$) and n large enough, we have $F(x_0 + t_n u) - F(x_0) \notin -\mathrm{int}C$. Therefore, there is $P_n \in A_F(x_0)$ such that

$$P_n u + \frac{o(t_n)}{t_n} \notin -\mathrm{int}C. \tag{1}$$

We have two possibilities. If $\{P_n\}$ is bounded, we can assume that $P_n \xrightarrow{p} P \in$ p-cl$A_F(x_0)$. Passing (1) to limit, we obtain $Pu \notin -\mathrm{int}C$ as required. If $\{P_n\}$ is unbounded, we can assume that $P_n/\|P_n\| \xrightarrow{p} P \in$ p-$A_F(x_0)_\infty \backslash \{0\}$. Dividing (1) by $\|P_n\|$ and letting $n \to \infty$, we also obtain $P(u) \notin -\mathrm{int}C$.

Case 2. $u \notin W_f(\Omega, x_0)$. Then, $\forall t_n \downarrow 0$, $\exists n$, $h(x_0 + t_n u) \notin -K$. We claim the existence of $Q \in A_h(x_0)$ such that $Q(u) \notin -\mathrm{int}K - h(x_0)$. Indeed, suppose to the contrary that $A_h(x_0)(u) \subseteq -\mathrm{int}K - h(x_0)$. Then, for n large enough,

$$h\left(x_0 + \frac{1}{n}u\right) - h(x_0) \in \frac{1}{n}A_h(x_0)(u) + o\left(\frac{1}{n}\|u\|\right) \in \frac{1}{n}A_h(x_0)(u) + r_n B_Z,$$

where $r_n n \to 0$, and B_Z is the closed unit ball of Z. Hence,

$$h\left(x_0 + \frac{1}{n}u\right) \in \left(1 - \frac{1}{n}\right) h(x_0) + \frac{1}{n}(h(x_0) + A_k(x_0)(u) + r_n n B_Z).$$

By the contradiction assumption that $A_h(x_0)(u) \subseteq -\mathrm{int}K - h(x_0)$, one gets $h(x_0) + A_h(x_0)(u) + r_n n B_Z \subseteq -K$, for n large enough. Hence, $h(x_0 + \frac{1}{n}u) \in -K$. This leads to a contradiction with the assumption that $u \notin W_f(\Omega, x_0)$.

Now, in both cases, one has some $P \in$ p-cl$A_F(x_0)$ and $Q \in A_h(x_0)$ with $(P(u), Q(u)) \notin -\mathrm{int}[C \times (K + h(x_0))]$. According to a classic separation theorem from convex analysis, we have $(c^*, d^*) \in C^* \times K^* \backslash \{(0, 0)\}$ such that, for all $u \in X$,

$$\langle c^*, P(u)\rangle + \langle d^*, Q(u)\rangle \geq 0, \ \langle d^*, h(x_0)\rangle = 0.$$

Now let u satisfy $0 \in \mathrm{int}(Q(u) + h(x_0) + K)$ for all $Q \in A_h(x_0)$, and suppose to the contrary that $c^* = 0$. Then, the separation result collapses to

$$\langle d^*, Q(u)\rangle \geq 0, \quad \langle d^*, h(x_0)\rangle = 0.$$

This implies that $d^*(Q(u) + h(x_0) + d) \geq 0$ for all $d \in K$. So, $0 \notin \mathrm{int}(Q(u) + h(x_0) + K)$, contradicting the assumption. □

Note that, Lemma 4.1 sharpens Theorem 3.1 of Khanh and Tuan (2009), by removing the assumed boundedness of $A_h(x_0)$ and adding the case $c^* \neq 0$. This removal is important, since a map with a bounded first-order approximation at a point must be continuous (even calm) at this point. Now, we pass to our fractional programming. Denote

$$A_\varphi(x_0) := \prod_{i=1}^{m} \frac{g_i(x_0)A_{f_i}(x_0) - f_i(x_0)A_{g_i}(x_0)}{g_i^2(x_0)}.$$

Theorem 4.1 (Necessary condition) *For problem* (P)*, let $A_{f_i}(x_0)$, $A_{g_i}(x_0)$, $A_h(x_0)$ be asymptotically p-compact first-order approximations of f_i, g_i and h, respectively, at x_0, with $A_{f_i}(x_0)$, $A_{g_i}(x_0)$ being bounded, for $i = 1, \ldots, m$. If x_0 is a local weak solution of* (P)*, then, $\forall u \in X$, $\exists P \in$* p-cl$A_\varphi(x_0) \cup ($p-$A_\varphi(x_0)_\infty \backslash \{0\})$*, $\exists Q \in$* cl$A_h(x_0)$*, $\exists (c^*, d^*) \in C^* \times K^* \backslash \{(0, 0)\}$,*

$$\langle c^*, Pu\rangle + \langle d^*, Qu\rangle \geq 0, \ \langle d^*, h(x_0)\rangle = 0.$$

Furthermore, for u satisfying $0 \in \mathrm{int}(Q(u) + h(x_0) + K)$ *for all $Q \in A_h(x_0)$, we have $c^* \neq 0$.*

Proof Since $A_{g_i}(x_0)$, $i = 1, \ldots, m$, are bounded, all g_i are calm at x_0. By Propositions 3.2 and 3.3, $A_\varphi(x_0)$ is a first-order approximation of φ at x_0. As $A_{f_i}(x_0)$, $A_{g_i}(x_0)$ are bounded and $g_i(x_0) \neq 0$, $i = 1, \ldots, m$, from Proposition 3.4, we see that $A_\varphi(x_0)$ is asymptotically p-compact. To complete the proof, invoke Lemma 4.1 for $F = \varphi$. □

Note that in most of the known optimality conditions for fractional problems, X is assumed to be finite dimensional. Furthermore, when applied to the finite-dimensional case, Theorem 4.1 is also advantageous, since f is not required to be Lipschitz continuous. In the following example, the results for cases with assumed Lipschitz continuity in (Kim et al. 2005; Kuk et al. 2001; Nobakhtian 2008; Reedy and Mukherjee 2001; Soleimani-Damaneh 2008; Bao et al. 2007; Chinchuluun et al. 2007; Liu and Feng 2007) or with continuous differentiability in Singh (1981); Liang et al. (2001); Zalmai (2006); Mishra (1997); Cambini et al. (2005); Husain and Jabeen (2005) are not applicable, while Theorem 4.1 works well.

Example 4.1 Let $X = \mathbb{R}$, $m = 1$, $Y = \mathbb{R}$, $C = K = \mathbb{R}_+$, $x_0 = 0$,

$$f(x) = \begin{cases} -x(|\sin(1/x)| + 1), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

$g(x) = x^2 + 1$, and $h(x) = -\sqrt[3]{x} + x^2$. We can take approximations $A_g(x_0) = \{0\}$ and $A_h(x_0) =]-\infty, \beta[$ with $\beta < 0$ being arbitrary and fixed. Since f is not Lipschitz at $x_0 = 0$, the mentioned known results [e.g., Theorem 2.1 of Kim et al. (2005), Theorem 4.2 of Bao et al. (2007)] are not in use. Since $g(x_0) = 1$, $f(x_0) = 0$ and $A_f(x_0) = [-2, -1]$, one has $A_\varphi(x_0) = A_f(x_0)$, $\mathrm{cl}A_\varphi(x_0) \cup (A_\varphi(x_0)_\infty \backslash \{0\}) = [-2, -1]$. For $u = 1$, we see that, $\forall P \in \mathrm{cl}A_\varphi(x_0) \cup (A_\varphi(x_0)_\infty \backslash \{0\})$, $\forall Q \in \mathrm{cl}A_h(x_0)$, $\forall (c^*, d^*) \in C^* \times K^* \backslash \{(0, 0)\} = \mathbb{R}^2_+ \backslash \{(0, 0)\}$ with $\langle d^*, h(x_0)\rangle = 0$,

$$\langle c^*, Pu\rangle + \langle d^*, Qu\rangle = c^* P + \beta d^* < 0.$$

According to Theorem 4.1, x_0 is not a local weak solution of (P).

Theorem 4.2 (Sufficient condition) *Let $X = \mathbb{R}^n$ and $x_0 \in h^{-1}(-K)$. Assume that, for $i = 1, \ldots, m$, $A_{f_i}(x_0)$, $A_{g_i}(x_0)$, $A_h(x_0)$ are asymptotically p-compact first-order approximations of f_i, g_i and h, respectively, at x_0, with all $A_{f_i}(x_0)$, $A_{g_i}(x_0)$ being bounded. Suppose that, for all $u \in T(h^{-1}(K), x_0)$ with norm one, $P \in \mathrm{cl}A_\varphi(x_0) \cup (A_\varphi(x_0)_\infty \backslash \{0\})$ and $Q \in$ p-cl $A_h(x_0) \cup$ (p- $A_h(x_0)_\infty \backslash \{0\})$, there exists $(y^*, z^*) \in C^* \times K^* \backslash \{(0, 0)\}$ such that*

$$\langle y^*, Pu\rangle + \langle z^*, Qu\rangle > 0, \ \langle z^*, h(x_0)\rangle = 0.$$

Then, x_0 is a local firm solution of order 1 of (P).

Proof Since $A_{g_i}(x_0)$ is bounded, g_i is calm at x_0 for $i = 1, \ldots, m$. By Propositions 3.2 and 3.3, $A_\varphi(x_0)$ is a first-order approximation of φ at x_0. Since $A_\varphi(x_0)$ is finite dimensional, it is asymptotically p-compact. Now, apply Theorem 3.3 of Khanh and Tuan (2009) to complete the proof. □

Example 4.2 Let $n = 1$, $m = 2$, $Y = \mathbb{R}$, $C = \mathbb{R}^2_+$, $K = \mathbb{R}_+$, $x_0 = 0$, $f(x) = (x, f_1(x))$,

$$f_1(x) = \begin{cases} x(|\sin(1/x)| + 1), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

$g(x) = (e^{-x}, 1)$, and $h(x) = x^2 - 2x$. Then, $T(h^{-1}(-K), x_0) = [0, \infty[$, and f, g and h admit first-order approximations $A_f(x_0) = \{(1, \alpha) \in \mathbb{R}^2 | \alpha \in [1, 2]\}$, $A_g(x_0) = \{(-1, 0)\}$, and $A_h(x_0) = \{-2\}$, respectively, for any fixed $\alpha > 0$. Hence,

$$A_\varphi(x_0) = A_f(x_0),\ \mathrm{cl}A_\varphi(x_0) = A_\varphi(x_0)\ ,\ A_\varphi(x_0)_\infty = \{(0, 0)\}.$$

Choosing $(y^*, z^*) = ((0, 1), 0) \in C^* \times K^* \backslash \{(0, 0)\}$, one sees that, for all $u \in T(h^{-1}(K), x_0)$ with norm one, $P \in \mathrm{cl}A_\varphi(x_0) \cup (A_\varphi(x_0)_\infty \backslash \{0\})$, and $Q \in \mathrm{cl}A_h(x_0) \cup (A_h(x_0)_\infty \backslash \{0\})$,

$$\langle y^*, Pu\rangle + \langle z^*, Qu\rangle > 0,\ \langle z^*, h(x_0)\rangle = 0.$$

In view of Theorem 4.2, x_0 is a local firm solution of order 1 of (P).

Now, we check directly, with $\gamma = 1$ and an arbitrary neighborhood U of $x_0 = 0$, that, for all $x \in U \cap S \backslash \{x_0\}$,

$$(\varphi(x) + C) \cap B_{\mathbb{R}^2}(\varphi(x_0), |x - x_0|) = \emptyset.$$

Indeed, since the feasible set is $S = [0, 2]$, then $x > 0$ for all $x \in U \cap S \backslash \{x_0\}$, and we have

$$\begin{aligned} \|\varphi(x) - \varphi(x_0)\| &= \|(xe^x, x(|\sin(1/x)| + 1)\| \\ &= \sqrt{x^2(e^{2x} + (|\sin(1/x)| + 1)^2)} > |x - x_0|. \end{aligned}$$

Hence, for all $x \in U \cap S \backslash \{x_0\}$,

$$(\varphi(x) + C) \cap B_{\mathbb{R}^2}(\varphi(x_0), \gamma|x - x_0|) = \emptyset.$$

5 Second-order optimality conditions

By the calculus rules obtained in Sect. 3, we can apply easily second-order optimality conditions in Khanh and Tuan (2009, 2011) for vector optimization to multiobjective fractional programming. Hence, the deriving of the results here is immediate. However, since these optimality conditions have advantages in applications, we present them and illustrate applications for the sake of completeness. For necessary conditions, we admit the following notation for problem (P), with $z^* \in K^*$,

$$H(z^*) := \{x \in X \mid h(x) \in -K, \langle z^*, h(x)\rangle = 0\}.$$

5.1 The case f and h are first-order differentiable

In this subsection, assume that f_i and h are Fréchet differentiable at x_0 for $i = 1, \ldots, m$, and set

$$A_\varphi(x_0) := \prod_{i=1}^{m} \frac{f_i'(x_0)}{g_i(x_0)}, \quad B_\varphi(x_0) := \prod_{i=1}^{m} \frac{g_i(x_0)B_{f_i}(x_0) - f_i(x_0)B_{g_i}(x_0)}{g_i^2(x_0)}.$$

Theorem 5.1 (Necessary condition) *Assume that C is polyhedral, g_i are 2-calm at x_0 for $i = 1, \ldots, m$, and $z^* \in K^*$ with $\langle z^*, h(x_0)\rangle = 0$. Impose further that $(f_i'(x_0), B_{f_i}(x_0))$, $(0, B_{g_i}(x_0))$ and $(h'(x_0), B_h(x_0))$ are bounded asymptotically p-compact second-order approximations of f_i, g_i and h, respectively, at x_0, for $i = 1, \ldots, m$.*

If x_0 is a local weak solution of (P)*, then, for any $v \in T(H(z^*), x_0)$, there exists $y^* \in B$, where B is finite and* $\operatorname{cone}(\operatorname{co}B) = C^*$*, such that $\langle y^*, A_\varphi(x_0)v\rangle + \langle z^*, h'(x_0)v\rangle \geq 0$.*

If, furthermore, $y^* \circ A_\varphi(x_0) + z^* \circ h'(x_0) = 0$, we have either $M \in$ p-cl$B_\varphi(x_0)$ and $N \in$ p-cl$B_h(x_0)$ such that $\langle y^*, M(v, v)\rangle + \langle z^*, N(v, v)\rangle \geq 0$, or $M \in$ p-$B_\varphi(x_0)_\infty \setminus \{0\}$ such that $\langle y^*, M(v, v)\rangle \geq 0$.

Proof By Propositions 3.1, 3.5 and 3.6, $(A_\varphi(x_0), B_\varphi(x_0))$ is a second-order approximation of φ at x_0. Furthermore, since $(f_i'(x_0), B_{f_i}(x_0))$, $(0, B_{g_i}(x_0))$ are asymptotically p-compact, $g_i(x_0) \neq 0$, and $B_{f_i}(x_0)$, $B_{g_i}(x_0)$ are bounded, for $i = 1, \ldots, m$, Proposition 3.4 implies that $(A_\varphi(x_0), B_\varphi(x_0))$ is asymptotically p-compact. Now, applying Theorem 4.1 of Khanh and Tuan (2009, 2011) ends the proof. □

Note that in this statement and Theorem 5.2 below, the assumptions imposed on g_i are restrictive. However, in applications we can always rewrite the fractions involved in the problem, with new f_i and g_i, so that the new g_i satisfy these assumptions. We illustrate Theorem 5.1 by the following example.

Example 5.1 Let $X = l^2, m = 1, Y = \mathbb{R}, C = D = \mathbb{R}_+, B = \{1\}$, and $x_0 = 0$. Let $f(x) = -\|x\|^2 = -\sum_{i=1}^{\infty} x_i^2$, $g(x) = \|x\|^{4/3} + 1 = \left(\sum_{i=1}^{\infty} x_i^2\right)^{2/3} + 1$, $h(x) = x_1^2 - x_1$. Then, $g(x_0) = 1 \neq 0$, $g'(x_0) = 0$, i.e., g is Fréchet differentiable at x_0 but g is not 2-calm at x_0, and $B_g(x_0) = \{N_\lambda \in B(l^2, l^2, \mathbb{R}) \mid \lambda > 1\}$, where $N_\lambda(x, y) = \lambda \sum_{i=1}^{\infty} x_i y_i$ for $x, y \in l_2$. We have $f(x_0) = 0$, $f'(x_0) = 0$, $B_f(x_0) = \{-1\}$, $h'(x_0) = \{(-1, 0, 0, \ldots)\}$, and $B_h(x_0) = \{N \in B(l^2, l^2, \mathbb{R}) \mid N(x, y) = x_1.y_1\}$. Therefore, $A_\varphi(x_0) = A_f(x_0)$ and $B_\varphi(x_0) = B_f(x_0)$, and $(A_\varphi(x_0), B_\varphi(x_0))$ is an asymptotically p-compact second-order approximation of φ at x_0. We have p-cl$B_\varphi(x_0) = \{-1\}$ and p-$B_\varphi(x_0)_\infty = \{0\}$. Choose $z^* = 0 \in K^* = \mathbb{R}_+$ and

$$v = (1, 0, 0, \ldots) \in T(H(z^*), x_0) = \{x = (x_1, x_2, \ldots) \in l^2 \mid x_1 \geq 0\}.$$

Then, for any $y^* \in B$, i.e., $y^* = 1$, we have $y^* \circ A_\varphi(x_0) + z^* \circ h'(x_0) = 0$ and

$$\langle y^*, M(v, v)\rangle + \langle z^*, N(v, v)\rangle = -1 < 0$$

for all $M \in$ p-cl$B_\varphi(x_0)$. Due to Theorem 5.1, x_0 is not a local weak solution of problem (P). But, since $X = l^2$ is infinite dimensional, Theorem 4.1 of Reedy and Mukherjee (2001) cannot be applied.

Theorem 5.2 (Sufficient condition) *Assume that X is finite dimensional, $x_0 \in h^{-1}(-K)$, g_i is 2-calm at x_0 and $(f_i'(x_0), B_{f_i}(x_0))$, $(0, B_{g_i}(x_0))$, and $(h'(x_0), B_h(x_0))$ are bounded asymptotically p-compact second-order approximations of f_i, g_i and h, respectively, at x_0, for $i = 1, \ldots, m$. Set*

$$C_0^* \times K_0^* := \{(y^*, z^*) \in C^* \times K^* \backslash \{(0,0)\} \mid y^* \circ A_1(x_0) + z^* \circ h'(x_0) = 0, \langle z^*, h(x_0) \rangle = 0\}.$$

Impose further the existence of $(y^*, z^*) \in C_0^* \times K_0^*$ such that, for all $v \in T(h^{-1}(-K), x_0)$ with $\|v\| = 1$ and $\langle y^*, A_\varphi(x_0)v \rangle = \langle z^*, h'(x_0) \rangle = 0$, one has

(i) *for each $M \in \mathrm{cl}B_\varphi(x_0)$ and $N \in$ p-cl$B_h(x_0)$, $\langle y^*, M(v,v) \rangle + \langle z^*, N(v,v) \rangle > 0$;*
(ii) *for each $M \in$ p-$B_\varphi(x_0)_\infty \backslash \{0\}$, $\langle y^*, M(v,v) \rangle > 0$.*
Then, x_0 is a local firm solution of order 2.

Proof Propositions 3.1, 3.5 and 3.6 together imply that the finite dimensional set $(A_\varphi(x_0), B_\varphi(x_0))$ is an asymptotically p-compact second-order approximation of φ at x_0. Applying Theorem 4.5 of Khanh and Tuan (2009), the conclusion is obtained. □

Example 5.2 Let $n = 1, m = 2$, $Y = \mathbb{R}^2$, $C = \mathbb{R}_+^2$, $K = \mathbb{R}_+$, $x_0 = 0$, $f(x) = (f_1(x), x^2)$ with

$$f_1(x) = \begin{cases} \int_0^x t^2 \sin(1/t^2), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

$g(x) = (-x^2 + 1, \cos^2 x)$, and $h(x) = -x + x^2$. Then, we have $g(x_0) = (1,1)$, $g'(x_0) = (0,0)$, $B_g(x_0) = \{(-1,-1)\}$ (g is 2-calm at x_0), $f(x_0) = 0$, $f'(x_0) = (0,0)$, $h'(x_0) = -1$,
$B_f(x_0) = \{(0,1)\}$, $B_h(x_0) = \{1\}$, $h^{-1}(-K) = [0,1]$, and $T(h^{-1}(-K), x_0) = [0, \infty[$. Hence, $A_\varphi(x_0) = \{(0,0)\}$, $B_\varphi(x_0) = \mathrm{cl}B_\varphi(x_0) = \{(0,1)\}$, $B_\varphi(x_0)_\infty = \{(0,0)\}$, and $C_0^* \times K_0^* = \{(y^*, 0) \mid y^* \in \mathbb{R}_+^2 \backslash \{0\}\}$.

Choose $(y^*, z^*) = ((1,0), 0) \in C_0^* \times K_0^*$. Then, for all $v \in T(h^{-1}(-K), x_0)$ with norm one, i.e., $v = 1$, we see that, for each $M \in \mathrm{cl}B_\varphi(x_0)$ and $N \in \mathrm{cl}B_h(x_0)$,

$$\langle y^*, M(v,v) \rangle + \langle z^*, N(v,v) \rangle = 1 > 0.$$

According to Theorem 5.2, x_0 is a local firm solution order 2 of (P).

5.2 The case f and h are not differentiable

In this general case, we set

$$A_\varphi(x_0) := \prod_{i=1}^m \frac{g_i(x_0) A_{f_i}(x_0) - f_i(x_0) A_{g_i}(x_0)}{g_i^2(x_0)},$$

$$B_\varphi(x_0) := \prod_{i=1}^{m} \frac{g_i(x_0)B_{f_i}(x_0) - f_i(x_0)B_{g_i}(x_0)}{g_i^2(x_0)},$$

$$P(x_0, y^*, z^*) := \{v \in X | \langle y^*, Pv\rangle + \langle z^*, Qv\rangle = 0, \forall P \in A_1(x_0), \forall Q \in A_h(x_0)\}.$$

Theorem 5.3 (Necessary condition) *Let C be polyhedral, g_i be 2-calm at x_0 for $i = 1, \dots, m$, and $z^* \in K^*$ with $\langle z^*, h(x_0)\rangle = 0$. Suppose $(A_{f_i}(x_0), B_{f_i}(x_0))$, $(A_{g_i}(x_0), B_{g_i}(x_0))$, and $(A_h(x_0), B_h(x_0))$ are bounded asymptotically p-compact second-order approximations of f_i, g_i, $i = 1, \dots, m$, and h, respectively, at x_0.*

If x_0 is a local weak solution of (P)*, then, for any $v \in T(H(z^*), x_0)$,*

(i) *for all $w \in T^2(H(z^*), x_0, v)$, there exist $y^* \in B$ with B being finite and* $\mathrm{cone}(\mathrm{co}B) = C^*$, *$P \in$* p-cl$A_\varphi(x_0)$*, and $Q \in$* p-cl$A_h(x_0)$ *such that $\langle y^*, Pv\rangle + \langle z^*, Qv\rangle \geq 0$.*
If, in addition, $v \in P(x_0, y^*, z^*)$, then either there are $P \in$ p-cl$A_\varphi(x_0)$, *$Q \in$* p-cl$A_h(x_0)$, *$M \in$* p-cl$B_\varphi(x_0)$*, and $N \in$* p-cl$B_h(x_0)$ *such that*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v, v)\rangle + 2\langle N(v, v)\rangle \geq 0,$$

or there exists $M \in$ p-$B_\varphi(x_0)_\infty\backslash\{0\}$ *with $\langle y^*, M(v, v)\rangle \geq 0$;*

(ii) *for all $w \in T''(G(z^*), x_0, v)$, there exist $P \in$* p-cl$A_\varphi(x_0)$ *and $Q \in$* p-cl$A_h(x_0)$ *such that $\langle y^*, Pv\rangle + \langle z^*, Qv\rangle \geq 0$.*
If, in addition, $v \in P(x_0, y^*, z^*)$, then either $P \in$ p-cl$A_\varphi(x_0)$, *$Q \in$* p-cl$A_h(x_0)$, *and $M \in$* p-cl$B_\varphi(x_0)_\infty$ *exist such that*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v, v)\rangle \geq 0,$$

or some $M \in$ p-$B_\varphi(x_0)_\infty\backslash\{0\}$ *exists with $\langle y^*, M(v, v)\rangle \geq 0$.*

Proof The proof is similar to that of Theorem 5.1, but now we apply Theorem 4.7 of Khanh and Tuan (2009, 2011) and we do not need Proposition 3.1. □

Theorem 5.3 rejects a candidate for a weak solution in the following illustrative example.

Example 5.3 Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $m = 2$, $C = \mathbb{R}^2_+$, $B = \{y_1^* = (1, 0), y_2^* = (0, 1)\}$, $K = \{0\}, x_0 = (0, 0), f(x, y) = (-y, x + |y|), g(x, y) = (x^2 + 1, y^2 + 1)$, and $h(x, y) = -x^3 + y^2$. Then, we have $g(x_0) = (1, 1)$, $A_g(x_0) = \{0\}$, and $B_g(x_0) = \left\{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}$. So, g is 2-calm at x_0. We have the following approximations

$$A_f(x_0) = \left\{\begin{pmatrix} 0 & -1 \\ 1 & \pm 1 \end{pmatrix}\right\}, \ B_f(x_0) = \{0\},$$

$$A_h(x_0) = \{0\}, \ B_h(x_0) = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

Therefore, $A_\varphi(x_0) = A_f(x_0)$, $B_\varphi(x_0) = \{0\}$. Let $z^* = 0$. Then, $H(z^*) = \{(x, y) \in \mathbb{R}^2 | -x^3 + y^2 = 0\}$, $T(H(z^*), x_0) = \mathbb{R}_+ \times \{0\}$.

Choosing $v = (1, 0) \in T(H(z^*), x_0)$, we have

$$T^2(H(z^*), x_0, v) = \emptyset, \ T''(H(z^*), x_0, v) = \mathbb{R}^2.$$

Now, let $w = (0, 1) \in T''(H(z^*), x_0, v)$. For $y_1^* = (1, 0) \in B$, $\forall P \in \mathrm{cl}A_1(x_0)$, $\forall Q \in \mathrm{cl}A_h(x_0)$, one gets $\langle y_1^*, Pv\rangle + \langle z^*, Qv\rangle \geq 0$, and $v \in P(x_0, y_1^*, z^*) = \{(v_1, v_2) \in \mathbb{R}^2 | v_2 = 0\}$. Hence, for all $P \in \mathrm{cl}A_\varphi(x_0)$, $Q \in \mathrm{cl}A_h(x_0)$ and $M \in B_\varphi(x_0)_\infty$, one has

$$\langle y_1^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y_1^*, M(v, v)\rangle = -1 < 0.$$

For $y_2^* = (0, 1) \in B$, and for all $P \in \mathrm{cl}A_\varphi(x_0)$, all $Q \in \mathrm{cl}A_h(x_0)$, one obtains $\langle y_1^*, Pv\rangle + \langle z^*, Qv\rangle = 1 > 0$, and $v \notin P(x_0, y_2^*, z^*)$.

Taking into account Theorem 5.3, one sees that x_0 is not a local weak solution of (P).

We pass finally to sufficient conditions.

Theorem 5.4 *(Sufficient condition) Let $X = \mathbb{R}^n$, $x_0 \in h^{-1}(-K)$, g_i be 2-calm at x_0, $(y^*, z^*) \in C^* \times K^*$ with $\langle z^*, g(x_0)\rangle = 0$ and $(A_{f_i}(x_0), B_{f_i}(x_0))$, $(A_{g_i}(x_0), B_{g_i}(x_0))$, $(A_h(x_0), B_h(x_0))$ be bounded asymptotically p-compact second-order approximations of f_i, g_i, and h, respectively, at x_0, for $i = 1, \ldots, m$. Then, x_0 is a local firm solution of order 2 of* (P)*, if the following conditions hold*

(i) *for all $v \in T(h^{-1}(-K), x_0)$, $P \in A_1(x_0)$, and $Q \in A_h(x_0)$, one has $\langle y^*, Pv\rangle + \langle z^*, Qv\rangle = 0$;*

(ii) $\forall w \in T^2(h^{-1}(-K), x_0, v) : \|w\| = 1$, $\exists \overline{P} \in \mathrm{cl}A_\varphi(x_0) : \overline{P}w \in -C$, $\exists \overline{Q} \in \text{p-cl}B_h(x_0) : \overline{Q}w \in -K(h(x_0))$, *and* $\forall M \in B_\varphi(x_0)_\infty \backslash \{0\}$, *one has $\langle y^*, M(v, v)\rangle > 0$;*

(ii$_1$) *for all $w \in T^2(h^{-1}(-K), x_0, v) \cap v^\perp$, $P \in \mathrm{cl}A_\varphi(x_0)$, $Q \in \mathrm{cl}A_h(x_0)$, $M \in \mathrm{cl}B_\varphi(x_0)$, and $N \in \text{p-cl}B_h(x_0)$, one has*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + 2\langle y^*, M(v, v)\rangle + 2\langle N(v, v)\rangle > 0;$$

(ii$_2$) *for all $w \in T''(h^{-1}(-K), x_0, v) \cap v^\perp \backslash \{0\}$, $P \in \mathrm{cl}A_\varphi(x_0)$, $Q \in \text{p-cl}A_h(x_0)$, and $M \in \text{p-cl}B_\varphi(x_0)$, one has*

$$\langle y^*, Pw\rangle + \langle z^*, Qw\rangle + \langle y^*, M(w, w)\rangle > 0.$$

Proof By Propositions 3.5 and 3.6, we can apply Theorem 4.9 of Khanh and Tuan (2009) to complete the proof. □

6 Conclusion

Multiobjective fractional programming on finite dimensional spaces has been intensively investigated recently. In this paper, we consider this problem with nonsmooth data, in infinite-dimensional normed spaces. We develop first and second-order optimality conditions in terms of generalized derivatives called approximations. Then, unlike the existing papers on fractional programming which could only weaken convexity assumptions, we can avoid completely such conditions. Furthermore, the maps in our problem may not be Lipschitz, a condition imposed usually in earlier existing results. Our necessary optimality conditions are established for local weak solutions and sufficient conditions are for local firm solutions. The obtained conditions of orders 1 and 2 are expressed in terms of approximations of orders 1 and 2, respectively. We provide also a number of examples to illustrate in detail the results.

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