TOP (2014) 22:1103–1122
DOI 10.1007/s11750-014-0314-3

ORIGINAL PAPER

Multi-criteria decision-making method based on normal intuitionistic fuzzy-induced generalized aggregation operator

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Received: 22 June 2013 / Accepted: 7 January 2014 / Published online: 31 January 2014

Abstract Normal intuitionistic fuzzy numbers (NIFNs), which use normal fuzzy numbers to express their membership and non-membership functions, can reflect the evaluation information exactly in different dimensions. In this paper, we are committed to apply NIFNs to multi-criteria decision-making (MCDM) problems, and meanwhile some new aggregation operators are proposed, including normal intuitionistic fuzzy weighted arithmetic averaging operator, normal intuitionistic fuzzy weighted geometric averaging operator, normal intuitionistic fuzzy-induced ordered weighted averaging operator, normal intuitionistic fuzzy-induced ordered weighted geometric averaging operator and normal intuitionistic fuzzy-induced generalized ordered weighted averaging operator (NIFIGOWA). Based on the NIFIGOWA operator, an approach is introduced to solve MCDM problems where the criteria values are NIFNs and the criteria weight information is fixed. Finally, the proposed method is compared to the existing methods by virtue of a numerical example to verify its feasibility and rationality.

Keywords Multi-criteria decision-making · Normal intuitionistic fuzzy number · Normal intuitionistic fuzzy aggregation operator · Normal intuitionistic fuzzy-induced generalized aggregation operator

Mathematics Subject Classification 90B50

1 Introduction

Ever since the fuzzy set theory was first proposed by Zadeh (1965), fuzzy sets (FSs), especially triangular fuzzy numbers (TFN) and trapezoidal fuzzy numbers (TrFN) have

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been widely studied, developed, and applied to various fields, such as multi-criteria decision-making (MCDM), logic programming and pattern recognition. With regard to an arbitrary fuzzy set A, the degree of membership of the element x in a universe X is a single value, which mixes the evidence for $x \in X$ with that against $x \in X$. Therefore, the FS theory cannot tackle the cases of incomplete information referring to a fuzzy concept, i.e., the sum of the degrees of membership and non-membership is less than one. Atanassov' intuitionistic fuzzy sets (AIFSs), which were introduced by Atanassov (1986) and widely studied by many researchers (Atanassov 2000, 2007; Burillo et al. 1994), use two characteristic functions to express the degree of membership and non-membership respectively. With the supplement of non-membership function, AIFSs can deal with the presence of vagueness and hesitancy originating from imprecise knowledge or information.

The talent of AIFSs in describing fuzziness makes it popular with researchers quickly, and quite a lot of work has already been done in MCDM problems (Kavita 2011) and optimization problem (Angelov 1997). Atanassov and Gargov (1989) introduced the concept of interval-valued Atanassov' intuitionistic fuzzy sets (IVAIFSs), in which the membership and non-membership both took the form of interval numbers. Liu and Yuan (2007) brought the definition of fuzzy number AIFSs (FNAIFSs), and introduced the corresponding operations. Angelov (1995) introduced some definitions of crispification, which was an analog to the basic operations of AIFSs—defuzzification, and many scholars proposed a lot of relative decision-making methods based on these concepts. Liu and Li (2009) defined some fuzzy number amelioration operators to optimize Atanassov' intuitionistic fuzzy numbers. Boran et al. (2009) proposed a TOPSIS method combined with AIFSs to select an appropriate supplier in group decision-making environment. Li (2010a) developed a method based on the extended generalized ordered weighted averaging (GOWA) operators to solve MCDM problems with AIFSs. Zhao et al. (2010) introduced some generalized Atanassov' intuitionistic fuzzy aggregation operators to deal with Atanassov' intuitionistic fuzzy and interval-valued Atanassov' intuitionistic fuzzy information. Xu (2010) used the Choquet integral to propose some Atanassov' intuitionistic fuzzy aggregation operators that could reflect the correlations among the elements. Wei (2010) introduced some induced geometric aggregation operators with Atanassov' intuitionistic fuzzy information, and then applied them to group decision-making problems. Park et al. (2010) extended the VIKOR method for dynamic intuitionistic fuzzy MCDM problems. Feng and Qian (2010) presented a method of grey-related analysis for handling MCDM problems based on AIFSs. Chen et al. (2011) proposed an approach to deal with the multi-criteria group decision-making (MCGDM) problem based on the interval-valued Atanassov' intuitionistic fuzzy preference relation. Yadav and Kumar (2009) extended the TOPSIS method to solve supplier selection problems with IVAIFSs. Tan (2011) investigated an MCDM technique with IVAIFSs based on an extension of the TOPSIS method in group decision-making environment, where the inter-dependent or interactive characteristics were taken into account. Ye (2011) studied the fuzzy cross-entropy of IVAIFSs to deal with MCDM problems. Xu and Xia (2011) applied the Choquet integral and Dempster–Shafer evidence theory to aggregate inuitionistic fuzzy information. Li (2011) developed a methodology for solving MCDM problems with AIFSs.

With the increasing development of AIFS theory, quite a few extensions of Atanassov' intuitionistic fuzzy numbers (AIFNs) such as triangular Atanassov' intuitionistic fuzzy numbers (TAIFNs) (Shu et al. 2006), trapezoid Atanassov' intuitionistic fuzzy numbers (TrAIFNs) (Wang 2008) and Atanassov' intuitionistic linguistic numbers (AILNs) have been proposed (Wang and Li 2010). Shu et al. (2006) discussed the definition of TAIFNs and their operations, and then applied them to the fault-tree analysis. Wang and Li (2010) gave a definition of AILNs as well as the aggregation operators, and then applied them to MCDM problems with AILNs. Some trapezoid Atanassov' intuitionistic fuzzy aggregation operators were defined to deal with MCDM problems with TrAIFNs (Wang and Zhang 2009a,b; Wan and Dong 2010; Wan 2013a; Wang and Nie 2012). Wang et al. (2013) introduced the operators of TAIFNs and applied them in the system fault analysis. With considerations of inter-dependent or interactive characteristics among criteria, Wang and Nie (2011) proposed an Atanassov' intuitionistic triangle fuzzy MCDM method. Wan also did a lot of work on TAIFNs, such as the properties of TAIFNs (Wan et al. 2013a), the extended VIKOR method for MCGDM with TAIFNs (Wan et al. 2013b), and the MCDM method based on possibility variance coefficient of TAIFNs (Wan 2013b). In addition, Wan (2010) made a comprehensive survey on Atanassov' intuitionistic fuzzy MCDM approach according to the forms of AIFSs, such as AIFSs, IVAIFSs, TAIFNs and TrAIFNs. It is worthwhile to note that the domains of AIFSs and IVAIFSs are discrete sets, they are the extensions of fuzzy sets. TAIFNs and TrAIFNs extend the domain of AIFSs from discrete sets to continuous sets, and they are the extensions of fuzzy numbers. Compared to AIFSs, TAIFNs and TrAIFNs express their membership and non-membership functions using TFNs and TrFNs, and make the membership degree and the non-membership degree no longer relative to a fuzzy concept "Excellent" or "Good", but relative to TFNs or TrFNs. Thus, the information of decision makers can be exactly reflected and expressed in different dimensions (Wan 2013a). Hence, TAIFNs and TrAIFNs can better reflect the information of decision problems than IFSs.

The normal intuitionistic fuzzy numbers (NIFNs) which express the membership and non-membership functions by normal fuzzy numbers (NFNs) have much more realistic sense than TAIFNs or TrAIFNs. The definition of NIFNs was first proposed by Wang and Li (2012, 2013), together with the corresponding operations, the score function and the stability factor. In reality, a large number of natural phenomena and social phenomena belong to the normal distribution, and NFNs defined by Yang and Ko (1996) can well express these phenomena. Li and Liu (2004) pointed out that, compared to other fuzzy numbers such as TFNs and TrFNs, NFNs has several advantages. First, the normal distribution widely exists in natural phenomena, social phenomena and production activities. Second, the higher derivative of the normal membership function is continuous. Third, the fuzzy concepts characterized by the normal membership function are much closer to human mind. Consequently, NIFNs are superior than other Atanassov' intuitionistic fuzzy numbers.

Due to the talent of NIFNs in reflecting uncertain information, especially in reflecting the inherent fuzziness of uncertain information, NIFNs have wide applications in MCDM problems just as TIFNs or TrIFNs. As the aggregation operators of NIFNs are important tools of information fusion in MCDM problems with normal intuitionistic fuzzy information, some aggregation operators of NIFNs will be introduced in this paper. The rest of this paper is organized as follows. The definition of NIFNs as well as their operations is reviewed in Sect. 2 to make you have a better understanding of this paper. In Sect. 3, some normal intuitionistic fuzzy aggregation operators and their properties are presented. Then in Sect. 4, a normal intuitionistic fuzzy MCDM approach based on normal intuitionistic fuzzy-induced generalized aggregation operator is proposed. In Sect. 5, an illustrative example and the comparison analysis are given to show the feasibility and validity of the approach. In Sect. 6, some conclusions are drawn.

2 NIFNs and the related concepts

For the purpose of making this paper smoother and easier to understand, some basic concepts are here to review.

Definition 1 (Yang and Ko 1996) Let R be a real number set and $\tilde{A} = (a, \sigma)$ be an NFN if its membership function satisfies:

$$\tilde{A}(x) = e^{-\left(\frac{x-a}{\sigma}\right)^2} (\sigma > 0).$$

Here, the set of NFNs is denoted by $\tilde{N}$.

Definition 2 (Xu and Li 2001) Let $\tilde{A}, \tilde{B} \in \tilde{N}$ and $\tilde{A} = (a, \sigma)$, $\tilde{B} = (b, \tau)$, and then two operations between $\tilde{A}$ and $\tilde{B}$ can be defined as follows:

(1) $t\tilde{A} = t(a, \sigma) = (ta, t\sigma)(t > 0)$;
(2) $\tilde{A} + \tilde{B} = (a, \sigma) + (b, \tau) = (a + b, \sigma + \tau)$.

Definition 3 (Li 2008) Let $\tilde{A}, \tilde{B} \in \tilde{N}$ and $\tilde{A} = (a, \sigma)$, $\tilde{B} = (b, \tau)$, and then the distance between $\tilde{A}$ and $\tilde{B}$ is

$$d^2(\tilde{A}, \tilde{B}) = (a - b)^2 + \frac{1}{2}(\sigma - \tau)^2. \quad (1)$$

Definition 4 (Wang and Li 2012, 2013) Let X be an ordinary finite non-empty set and $(a, \sigma) \in \tilde{N}$, $A = \langle (a, \sigma), \mu_A, v_A \rangle$ is an NIFN if its membership function satisfies:

$$\mu_A(x) = \mu_A e^{-\left(\frac{x-a}{\sigma}\right)^2}, \quad x \in X, \quad (2)$$

and its non-membership function satisfies:

$$v_A(x) = 1 - (1 - v_A)e^{-\left(\frac{x-a}{\sigma}\right)^2}, \quad x \in X. \quad (3)$$

where $0 \leq \mu_A \leq 1, 0 \leq v_A \leq 1, \mu_A + v_A \leq 1$. When $\mu_A = 1$ and $\nu_A = 0$, the NIFN will be an NFN. Compared to NFNs, NIFNs add the non-membership function that expresses the degree of alternatives not belonging to (a, σ). In addition, $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ expresses the degree of hesitance.

The set of NIFNs is denoted by NIFNS.

Definition 5 (Wang and Li 2012, 2013) Let $A = \langle(a, \sigma_A), \mu_A, v_A\rangle$ and $B = \langle(b, \sigma_B), \mu_B, v_B\rangle$ be two NIFNs, and some operations between A and B can be defined as follows.

(1) $A + B = \left\langle(a + b, \sigma_A + \sigma_B), \frac{|a|\mu_A+|b|\mu_B}{|a|+|b|}, \frac{|a|v_A+|b|v_B}{|a|+|b|}\right\rangle$;

(2) $A - B = \left\langle(a - b, \sigma_A + \sigma_B), \frac{|a|\mu_A+|b|\mu_B}{|a|+|b|}, \frac{|a|v_A+|b|v_B}{|a|+|b|}\right\rangle$.
When $|a| = 0$ and $|b| = 0$, $\mu_{A+B} = \mu_{A-B} = \frac{\mu_A+\mu_B}{2}$, and $v_{A+B} = v_{A-B} = \frac{v_A+v_B}{2}$.

(3) $\lambda A = \langle(\lambda a, \lambda\sigma_A), \mu_A, v_A\rangle$.

(4) $\frac{1}{A} = \left\langle\left(\frac{1}{a}, \frac{\sigma_A}{a^2}\right), \mu_A, v_A\right\rangle, a \neq 0$.

(5) $AB = \left\langle\left(ab, ab\sqrt{\frac{\sigma_A^2}{a^2} + \frac{\sigma_B^2}{b^2}}\right), \mu_A\mu_B, v_A + v_B - v_Av_B\right\rangle$.

(6) $A^\lambda = \left\langle\left(a^\lambda, \lambda^{\frac{1}{2}}a^{\lambda-1}\sigma_A\right), \mu_A^\lambda, 1 - (1 - v_A)^\lambda\right\rangle (\lambda \geq 0)$.

Example 1 Let $A = \langle(6, 0.5), 0.4, 0.2\rangle$, $B = \langle(4, 0.3), 0.6, 0.1\rangle$ and $\lambda = 2$. Then, the followings are true:

(1) $A+B = \left\langle(6 + 4, 0.5 + 0.3), \frac{|6|0.4+|4|0.6}{|6|+|4|}, \frac{|6|0.2+|4|0.1}{|6|+|4|}\right\rangle = \langle(10, 0.8), 0.48, 0.16\rangle$;

(2) $A - B = \left\langle(6 - 4, 0.5 + 0.3), \frac{|6|0.4+|4|0.6}{|6|+|4|}, \frac{|6|0.2+|4|0.1}{|6|+|4|}\right\rangle = \langle(2, 0.8), 0.48, 0.16\rangle$;

(3) $2A = \langle(2 \times 6, 2 \times 0.5), 0.4, 0.2\rangle = \langle(12, 1), 0.4, 0.2\rangle$;

(4) $\frac{1}{A} = \left\langle\left(\frac{1}{6}, \frac{0.5}{6^2}\right), 0.4, 0.2\right\rangle = \left\langle\left(\frac{1}{6}, \frac{1}{72}\right), 0.4, 0.2\right\rangle$;

(5) $AB = \left\langle\left(6 \times 4, 6 \times 4\sqrt{\frac{0.5^2}{6^2} + \frac{0.3^2}{4^2}}\right), 0.4 \times 0.6, 0.2 + 0.1 - 0.2 \times 0.1\right\rangle$
$= \langle(24, 2.69), 0.24, 28\rangle$;

(6) $A^2 = \left\langle\left(6^2, 2^{\frac{1}{2}} \times 6^{2-1} \times 0.5\right), 0.5^2, 1-(1 - 0.2)^2\right\rangle = \langle(36, 4.24), 0.25, 0.36\rangle$.

Proposition 1 *Let* $A = \langle(a, \sigma_A), \mu_A, v_A\rangle$, $B = \langle(b, \sigma_B), \mu_B, v_B\rangle$ *and* $C = \langle(c, \sigma_C), \mu_C, v_C\rangle$ *be three NINFs, and* $ab \geq 0, bc \geq 0, ac \geq 0$. *The following equations are true:*

(1) $A + B = B + A$;
(2) $(A + B) + C = A + (B + C)$;
(3) $AB = BA$;
(4) $(AB)C = A(BC)$;
(5) $\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A, \lambda_1, \lambda_2 \geq 0$;
(6) $\lambda(A + B) = \lambda A + \lambda B, \lambda \geq 0$.

Remark 1 NIFNs introduced in this paper are similar to TIFNs or TrIFNs, but the operations are not analogs of TIFNs' or TrIFNs' operations. However, TrIFNs' operations are similar to TIFNs', we just take the additive and multiplicative operations of TIFNs for example.

Given two TIFNs, $A = \langle(a_1, a_2, a_3), \mu_a, v_a\rangle$ and $B = \langle(b_1, b_2, b_3), \mu_b, v_b\rangle$, the additive and multiplicative operations between A and B are defined as follows (Li 2008):

(1) $A + B = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3), \min\{\mu_a, \mu_b\}, \max\{\nu_a, \nu_b\}\rangle$;
(2) $AB = \langle (a_1 b_1, a_2 b_2, a_3 b_3), \min\{\mu_a, \mu_b\}, \max\{\nu_a, \nu_b\}\rangle$.

Obviously, there are some deficiencies for the operations above. For example, when $A = \langle (a_1, a_2, a_3), 0, 1\rangle$ and $B = \langle (a_1, a_2, a_3), 1, 0\rangle$, $A + B = \langle (2a_1, 2a_2, 2a_3), 0, 1\rangle$ and $AB = \langle (a_1^2, a_2^2, a_3^2), 0, 1\rangle$. Intuitively, if one team unanimously approved an alternative while another unanimously opposed it, then the overall approval rating of this alternative should not be zero, so the additive operation of TIFNs is not reasonable enough. Meanwhile, the TIFNs' multiplicative operation is exclusive to the case of $a_1, a_2, a_3, b_1, b_2, b_3 > 0$. For two NIFNs $A = \langle (a, \sigma), 0, 1\rangle$ and $B = \langle (a, \sigma), 1, 0\rangle$, if the additive and multiplicative operations of NIFNs are applied to them, there will be $A + B = \langle (2a, 2\sigma), \frac{1}{2}, \frac{1}{2}\rangle$ and $AB = \langle (a^2, \sqrt{2}\sigma^2), 0, 1\rangle$. Here, the deficiencies of TIFNs' operations are not inherited in NIFNs' operations.

Definition 6 (Wang and Li 2012, 2013) Let $A = \langle (a, \sigma_A), \mu_A, v_A\rangle$ be an NIFN, then its score function is

$$s_1(A) = a(\mu_A - v_A), s_2(A) = \sigma_A(\mu_A - v_A), \tag{4}$$

and its accuracy function is

$$h_1(A) = a(\mu_A + v_A), h_2(A) = \sigma_A(\mu_A + v_A). \tag{5}$$

Definition 7 (Wang and Li 2012, 2013) Let $A = \langle (a, \sigma_A), \mu_A, v_A\rangle$ and $B = \langle (b, \sigma_B), \mu_B, v_B\rangle$ be two NIFNs, the score functions of A and B be $s_1(A), s_2(A)$ and $s_1(B), s_2(B)$, and the accuracy functions of A and B be $h_1(A), h_2(A)$ and $h_1(B), h_2(B)$, respectively. Then, there will be:

(1) when $s_1(A) > s_1(B)$, $A > B$;
(2) when $s_1(A) = s_1(B)$ and $h_1(A) > h_1(B)$, $A > B$;
(3) when $s_1(A) = s_1(B)$ and $h_1(A) = h_1(B)$;
(a) when $s_2(A) < s_2(B)$, $A > B$;
(b) when $s_2(A) = s_2(B)$ and $h_2(A) < h_2(B)$, $A > B$;
(c) when $s_2(A) = s_2(B)$ and $h_2(A) = h_2(B)$, $A = B$.

Example 2 If $A = \langle (6, 0.5), 0.4, 0.2\rangle$ and $B = \langle (4, 0.3), 0.6, 0.1\rangle$, then $s_1(A) = 6(0.4 - 0.2) = 1.2$, $s_2(A) = 0.5(0.4 - 0.2) = 0.1$, $s_1(B) = 4(0.6 - 0.1) = 2$, and $s_2(B) = 0.3(0.6 - 0.1) = 0.15$, so $s_1(A) < s_1(B)$ and $A < B$.

Definition 8 (Wang and Li 2012) Let $A = \langle (a, \sigma_A), \mu_A, v_A\rangle$ be an NIFN, and its stability factor is

$$C_i = \frac{\sigma_i}{a_i}. \tag{6}$$

3 Normal Intuitionistic fuzzy aggregation operators

As we have mentioned, NIFNs can better reflect the evaluation information and offer wonderful solutions to MCDM problems. However, solving MCDM problems on the basis of NIFNs is not an easy task. After decision makers give the criterion values for each alternative in the form of NIFNs, an aggregation step must be performed for a collective overall evaluation. Here, some normal intuitionistic fuzzy aggregation operators are proposed for the information fusion.

Definition 9 Let $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ be a set of NIFNs, and $\mathrm{NIFWAA} : \mathrm{NIFNS}^n \to \mathrm{NIFNS}$ be a mapping:

$$\mathrm{NIFWAA}(A_1, A_2, \ldots, A_n) = \sum_{i=1}^{n} \omega_i A_i, \tag{7}$$

where $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ is the weight vector of $A_i (i = 1, 2, \ldots, n)$, satisfying $\omega_i \in [0, 1]$ $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. Then, NIFWAA is called normal intuitionistic fuzzy weighted arithmetic averaging operator. Particularly, when $\omega = (1/n, 1/n, \ldots, 1/n)$, the NIFWAA operator will degenerate into normal intuitionistic fuzzy arithmetic averaging (NIFAA) operator:

$$\mathrm{NIFAA}(A_1, A_2, \ldots, A_n) = \frac{1}{n} \sum_{i=1}^{n} A_i. \tag{8}$$

Theorem 1 *Let $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ be a set of NIFNs, and $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ be the weight vector of $A_i (i = 1, 2, \ldots, n)$ satisfying $\omega_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. Then the result obtained using Eq. (7) is still an NIFN and*

$$\begin{aligned} &\mathrm{NIFWAA}(A_1, A_2, \ldots, A_n) \\ &= \left\langle \left(\sum_{i=1}^{n} \omega_i a_i, \sum_{j=1}^{n} \omega_i \sigma_i \right), \frac{\sum_{i=1}^{n} \omega_i |a_i| \mu_i}{\sum_{i=1}^{n} \omega_i |a_i|}, \frac{\sum_{i=1}^{n} \omega_i |a_i| \nu_i}{\sum_{i=1}^{n} \omega_i |a_i|} \right\rangle. \end{aligned} \tag{9}$$

Theorem 1 can be demonstrated by the mathematical induction.

Example 3 If $A_1 = \langle (4, 0.2), 0.7, 0.2 \rangle$, $A_2 = \langle (3, 0.5), 0.5, 0.4 \rangle$, $A_3 = \langle (6, 0.4), 0.6, 0.1 \rangle$ and $\omega = (0.4, 0.2, 0.4)$, then

$$\begin{aligned} &\mathrm{NIFWAA}(A_1, A_2, A_3) \\ &= \Big\langle (4 \times 0.4 + 3 \times 0.2 + 6 \times 0.4, 0.2 \times 0.4 + 0.5 \times 0.2 + 0.4 \times 0.4), \\ &\quad \frac{4 \times 0.4 \times 0.7 + 3 \times 0.2 \times 0.5 + 6 \times 0.4 \times 0.6}{4 \times 0.4 + 3 \times 0.2 + 6 \times 0.4}, \\ &\quad \frac{4 \times 0.4 \times 0.2 + 3 \times 0.2 \times 0.4 + 6 \times 0.4 \times 0.1}{4 \times 0.4 + 3 \times 0.2 + 6 \times 0.4} \Big\rangle \\ &= \langle (4.6, 0.34), 0.6, 0.174 \rangle. \end{aligned}$$

Furthermore, it can be easily proved that the NIFWAA operator is idempotent, monotonic and bounded, and these are presented as below.

Proposition 2 (Idempotency) *Let* $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ *be a set of NIFNs. If* $A_i = A_0 \quad (i = 1, 2, \ldots, n)$*, then*

$$\text{NIFWAA}(A_1, A_2, \ldots, A_n) = A_0. \tag{10}$$

Proposition 3 (Monotonicity) *Let* $(A_1, A_2, \ldots, A_n)$ *and* $(B_1, B_2, \ldots, B_n)$ *be two sets of NIFNs satisfying* $A_i > B_i \ (i = 1, 2, \ldots n)$*, then*

$$\text{NIFWAA}(A_1, A_2, \ldots, A_n) > \text{NIFWAA}(B_1, B_2, \ldots, B_n). \tag{11}$$

Proposition 4 (Boundedness) *Let* $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ *be a set of NIFNs,* $A^* = \max(A_i)$*, and* $A_* = \min(A_i)$*, and then*

$$A_* \leq \text{NIFWAA}(A_1, A_2, \ldots, A_n) \leq A^*. \tag{12}$$

Definition 10 Let $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ be a set of NIFNs, and $\text{NIFWGA} : \text{NIFNS}^n \rightarrow \text{NIFNS}$ be a mapping:

$$\text{NIFWGA}(A_1, A_2, \ldots, A_n) = \prod_{i=1}^{n} A_i^{\omega_i}, \tag{13}$$

where $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ is the weight vector of $A_i (i = 1, 2, \ldots, n)$, satisfying $\omega_i \in [0, 1] \ (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$. Then, NIFWGA is called normal intuitionistic fuzzy weighted geometric averaging operator. Particularly, when $\omega = (1/n, \ldots, 1/n)$, the NIFWGA operator will degenerate into normal intuitionistic fuzzy geometric averaging (NIFGA) operator:

$$\text{NIFGA}(A_1, A_2, \ldots, A_n) = \prod_{i=1}^{n} A_i^{1/n}. \tag{14}$$

Theorem 2 *Let* $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ *be a set of NIFNs, and* $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ *be the weight vector of* $A_i (i = 1, 2, \ldots, n)$ *satisfying* $\omega_i \in [0, 1] (i = 1, 2, \ldots, n)$ *and* $\sum_{i=1}^{n} \omega_i = 1$*. Then the result obtained using Eq. (13) is still an NIFN and*

$$\begin{aligned} &\text{NIFWGA}(A_1, A_2, \ldots, A_n) \\ &= \left\langle \left(\prod_{i=1}^{n} a_i^{\omega_i}, \prod_{i=1}^{n} a_i^{\omega_i} \sqrt{\sum\nolimits_{i=1}^{n} \frac{\omega_i \sigma_i^2}{a_i^2}} \right), \prod_{i=1}^{n} \mu_i^{\omega_i}, 1 - \prod_{i=1}^{n} (1 - v_i)^{\omega_i} \right\rangle. \end{aligned} \tag{15}$$

Example 4 If $A_1 = \langle (4, 0.2), 0.7, 0.2 \rangle$, $A_2 = \langle (3, 0.5), 0.5, 0.4 \rangle$, $A_3 = \langle (6, 0.4), 0.6, 0.1 \rangle$ and $\omega = (0.4, 0.2, 0.4)$, then

$$\text{NIFWGA}(A_1, A_2, A_3)$$
$$= \left\langle \left(4^{0.4} \times 3^{0.2} \times 6^{0.4}, 4^{0.4} \times 3^{0.2} \times 6^{0.4} \sqrt{\frac{0.4 \times 0.2^2}{4^2} + \frac{0.2 \times 0.5^2}{3^2} + \frac{0.4 \times 0.4^2}{6^2}} \right), \right.$$
$$\left. 0.7^{0.4} \times 0.5^{0.2} \times 0.6^{0.4}, 1 - (1-0.2)^{0.4} \times (1-0.4)^{0.2} \times (1-0.1)^{0.4} \right\rangle$$
$$= \langle (4.4412, 0.0294), 0.6153, 0.2083 \rangle.$$

Similarly, it can be easily proved that the NIFWGA operator is idempotent, monotonic and bounded.

Definition 11 Let $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ be a set of NIFNs, and $\text{NIFIOWA} : \text{NIFNS}^n \to \text{NIFNS}$ be a mapping:

$$\text{NIFIOWA } ((C_1, A_1), (C_2, A_2), \ldots, (C_n, A_n)) = \sum_{i=1}^{n} \omega_i \beta_i, \tag{16}$$

where $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ is the weight vector correlative with theNIFIOWA operator satisfying $\omega_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$; β_i is the value of A_i in the pair (C_i, A_i) having the i-th smallest C_i, and C_i in (C_i, A_i) is the order inducing variable. Then, NIFIOWA is called normal intuitionistic fuzzy-induced ordered weighted averaging operator.

Theorem 3 *Let $A_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle (i = 1, 2, \ldots, n)$ be a set of NIFNs, $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ be the weight vector correlative with the NIFIOWA operator satisfying $\omega_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$, $\beta_i = \langle (a_i', \sigma_i'), \mu_i', v_i' \rangle$ be the value of A_i in the pair (C_i, A_i) having the i-th smallest C_i, and C_i in (C_i, A_i) be the order inducing variable. Then, the result obtained using Eq. (16) is still an NIFN and*

$$\text{NIFIOWA}((C_1, A_1), (C_2, A_2), \ldots, (C_n, A_n))$$
$$= \left\langle \sum_{i=1}^{n} \omega_i a_i', \sum_{j=1}^{n} \omega_i \sigma_i', \frac{\sum_{i=1}^{n} \omega_i |a_i'| \mu_i'}{\sum_{i=1}^{n} \omega_i |a_i'|}, \frac{\sum_{i=1}^{n} \omega_i |a_i'| v_i'}{\sum_{i=1}^{n} \omega_i |a_i'|} \right\rangle. \tag{17}$$

Example 5 If $A_1 = \langle (4, 0.2), 0.7, 0.2 \rangle$, $A_2 = \langle (3, 0.5), 0.5, 0.4 \rangle$, $A_3 = \langle (6, 0.4), 0.6, 0.1 \rangle$ and $\omega = (0.4, 0.2, 0.4)$, then

$$C_1 = 0.05, C_2 = 0.1667, C_3 = 0.0667, C_1 < C_3 < C_2,$$
$$\text{NIFIOWA}(A_1, A_2, A_3)$$
$$= \left\langle (4 \times 0.4 + 6 \times 0.2 + 3 \times 0.4, 0.2 \times 0.4 + 0.4 \times 0.2 + 0.5 \times 0.4), \right.$$
$$\frac{4 \times 0.4 \times 0.7 + 6 \times 0.2 \times 0.6 + 3 \times 0.4 \times 0.5}{4 \times 0.4 + 6 \times 0.2 + 3 \times 0.4},$$

$$\frac{4\times0.4\times0.2+6\times0.2\times0.1+3\times0.4\times0.4}{4\times0.4+6\times0.2+3\times0.4}\Big\rangle$$
$$=\langle(4, 0.36), 0.61, 0.23\rangle.$$

Furthermore, it can be easily proved that the NIFIOWA operator is idempotent, monotonic and bounded. Meanwhile, the NIFIOWA operator is commutative.

Proposition 5 (Commutativity) *Let $A_i=\langle(a_i,\sigma_i),\mu_i,v_i\rangle(i=1,2,\ldots,n)$ be a set of NIFNs. If $(A'_1,A'_2,\ldots,A'_n)$ is any permutation of $(A_1,A_2,\ldots,A_n)$, then*

$$\begin{aligned}&\text{NIFIOWA}((C_1,A_1),(C_2,A_2),\ldots,(C_n,A_n))\\&\quad=\text{NIFIOWA}((C_1,A'_1),(C_2,A'_2),\ldots,(C_n,A'_n)).\end{aligned}\tag{18}$$

Definition 12 Let $A_i=\langle(a_i,\sigma_i),\mu_i,v_i\rangle(i=1,2,\ldots,n)$ be a set of NIFNs, and $\text{NIFIOWGA}:\ \text{NIFNS}^n\rightarrow\text{NIFNS}$ be a mapping:

$$\text{NIFIOWGA}((C_1,A_1),(C_2,A_2),\ldots,(C_n,A_n))=\sum_{i=1}^{n}\beta_i^{\omega_i},\tag{19}$$

where $\omega=(\omega_1,\omega_2,\ldots\omega_n)$ is the weight vector correlative with the NIFIOWGA operator satisfying $\omega_i\in[0,1](i=1,2,\ldots,n)$ and $\sum_{i=1}^{n}\omega_i=1$, β_i is the value of A_i in the pair (C_i,A_i) having the i-th smallest C_i, and C_i in (C_i,A_i) is the order inducing variable. Then, NIFIOWGA is called normal intuitionistic fuzzy-induced ordered weighted geometric averaging operator.

Theorem 4 *Let $A_i=\langle(a_i,\sigma_i),\mu_i,v_i\rangle(i=1,2,\ldots,n)$ be a set of NIFNs, $\omega=(\omega_1,\omega_2,\ldots\omega_n)$ be the weight vector correlative with the NIFIOWGA operator satisfying $\omega_i\in[0,1](i=1,2,\ldots,n)$ and $\sum_{i=1}^{n}\omega_i=1$, $\beta_i=\langle(a'_i,\sigma'_i),\mu'_i,v'_i\rangle$ be the value of A_i in the pair (C_i,A_i) having the i-th smallest C_i, and C_i in (C_i,A_i) be the order inducing variable. Then, the result obtained using Eq. (19) is still an NIFN and*

$$\begin{aligned}&\text{NIFIOWGA}((C_1,A_1),(C_2,A_2),\ldots,(C_n,A_n))\\&=\left\langle\left(\prod_{i=1}^{n}a_i'^{\omega_i},\prod_{i=1}^{n}a_i'^{\omega_i}\sqrt{\sum\nolimits_{i=1}^{n}\frac{\omega_i\sigma_i'^2}{a_i'^2}}\right),\prod_{i=1}^{n}\mu_i'^{\omega_i},1-\prod_{i=1}^{n}(1-v'_i)^{\omega_i}\right\rangle.\end{aligned}\tag{20}$$

Example 6 If $A_1=\langle(4,0.2),0.7,0.2\rangle$, $A_2=\langle(3,0.5),0.5,0.4\rangle$, $A_3=\langle(6,0.4),0.6,0.1\rangle$ and $\omega=(0.4,0.2,0.4)$, then

$$\begin{aligned}&C_1=0.05, C_2=0.1667, C_3=0.0667, C_1<C_3<C_2,\\&\text{NIFIOWGA}(A_1,A_2,A_3)\\&=\Bigg\langle\Bigg(4^{0.4}\times6^{0.2}\times3^{0.4},4^{0.4}\times6^{0.2}\times3^{0.4}\sqrt{\frac{0.4\times0.2^2}{4^2}+\frac{0.2\times0.4^2}{6^2}+\frac{0.4\times0.5^2}{3^2}}\Bigg),\end{aligned}$$

$$0.7^{0.4} \times 0.6^{0.2} \times 0.5^{0.4}, 1-(1-0.2)^{0.4} \times (1-0.1)^{0.2} \times (1-0.4)^{0.4}\big\rangle$$
$$= \langle (3.8663, 0.0378), 0.5933, 0.2700\rangle.$$

Furthermore, it can be easily proved that the NIFIOWGA operator is commutative, monotonic, bounded, and idempotent.

Definition 13 Let $A_i = \langle (a_i, \sigma_i), \mu_i, v_i\rangle (i = 1, 2, \ldots, n)$ be a set of NIFNs, and $\text{NIFIGOWA} : \text{NIFNS}^n \to \text{NIFNS}$ be a mapping:

$$\text{NIFIGOWA}((C_1, A_1), (C_2, A_2), \ldots, (C_n, A_n)) = \left(\sum_{i=1}^{n} \omega_i \beta_i^{\lambda}\right)^{1/\lambda}, \tag{21}$$

where $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ is the weight vector correlative with the NIFIGOWA operator satisfying $\omega_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$, β_i is the value of A_i in the pair (C_i, A_i) having the i-th smallest C_i, C_i in (C_i, A_i) is the order inducing variable, and $\lambda \in (0, +\infty)$ is determined according to the characteristics needed in real-life situations. Then, NIFIGOWAis called normal intuitionistic fuzzy-induced generalized ordered weighted averaging operator.

Theorem 5 *Let $A_i = \langle (a_i, \sigma_i), \mu_i, v_i\rangle (i = 1, 2, \ldots, n)$ be a set of NIFNs, $\omega = (\omega_1, \omega_2, \ldots \omega_n)$ be the weight vector correlative with the NIFIGOWA operator satisfying $\omega_i \in [0, 1] (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} \omega_i = 1$, $\beta_i = (a_i', \sigma_i'), \mu_i', v_i'\rangle$ be the value of A_i in the pair (C_i, A_i) having the i-th smallest C_i, and C_i in (C_i, A_i) be the order inducing variable. Then, the result obtained using Eq. (21) is still an NIFN and*

$$\begin{aligned}
&\mathit{NIFIGOWA}_{\omega}((C_1, A_1), (C_2, A_2), \ldots, (C_n, A_n)) \\
&= \left\{ \left\langle \sum_{i=1}^{n} \omega_i a_i'^{\lambda}, \lambda^{1/2} \left(\sum_{j=1}^{n} \omega_i a_i'^{\lambda-1} \sigma_i'^{\lambda} \right), \right.\right. \\
&\left.\left. \frac{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda} \mu_i'^{\lambda}}{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda}}, \frac{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda} [1-(1-v_i')^{\lambda}]}{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda}} \right\rangle \right\}^{1/\lambda} \\
&= \left\langle \left(\sum_{i=1}^{n} \omega_i a_i'^{\lambda} \right)^{1/\lambda}, \lambda^{1/(2\lambda)} \left(\sum_{j=1}^{n} \omega_i a_i^{\lambda-1} \sigma_i'^{\lambda} \right)^{1/\lambda}, \left(\frac{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda} \mu_i'^{\lambda}}{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda}} \right)^{1/\lambda}, \right. \\
&\left. 1 - \left(1 - \frac{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda} [1-(1-v_i')^{\lambda}]}{\sum_{i=1}^{n} \omega_i |a_i'|^{\lambda}} \right)^{1/\lambda} \right\rangle.
\end{aligned} \tag{22}$$

Obviously, there are some properties for the NIFIGOWA operator as follows.

(1) *When $\lambda \to 0$, NIFIGOWA $((C_1, A_1), (C_2, A_2), \ldots, (C_n, A_n)) = (\sum_{i=1}^{n} \omega_i \beta_i^{\lambda})^{1/\lambda} = \prod_{i=1}^{n} \beta_i^{\omega_i}$, and the NIFIGOWA operator is reduced to the NIFIOWGA operator.*
(2) *When λ=1, NIFIGOWA $((C_1, A_1), (C_2, A_2), \ldots, (C_n, A_n))$=$(\sum_{i=1}^{n} \omega_i \beta_i^{\lambda})^{1/\lambda} = \sum_{i=1}^{n} \omega_i \beta_i$, and the NIFIGOWA operator is reduced to the NIFIOWA operator.*

Therefore, the NIFIOWGA operator and the NIFIOWA operator are two particular cases of the NIFIGOWA operator, and the NIFIGOWA operator is the generalized form of the NIFIOWGA operator and the NIFIOWA operator.

4 Multi-criteria decision-making approach based on normal intuitionistic fuzzy-induced generalized aggregation operator

Atanassov et al. (2005) have studied the intuitionistic fuzzy interpretations of multi-person MCDM method and the multi-measurement tool MCDM method, where the experts' weights and the measurement tools' weights are corresponded to their own reliability scores, and have presented the way to get the experts' and measurement tools' reliability scores. This paper is committed to studying a one-person one-measurement MCDM method. It is a very partial case of the study done by Atanassov et al. (2005), both the experts' weights and the measurement tools' weights are out of considerations and the weight vector of criteria is known completely and in the form of real numbers. The problem is described as follows.

For an MCDM selecting or ranking problem, let $A = \{a_1, a_2, \ldots, a_m\}$ be a set of alternatives, $C = \{c_1, c_2, \ldots, c_l\}$ be a set of criteria, and $\omega = (\omega_1, \omega_2, \ldots, \omega_l)$ be a set of fixed criteria weights satisfying $\omega_j \in [0, 1]$ and $\sum_{j=1}^{l} \omega_j = 1$. Suppose the criteria are independent of each other, and that the evaluation of the alternative a_i with respect to the criterion c_j is an NIFN represented by $r_{ij} = \langle (a_{ij}, \sigma_{ij}), \mu_{ij}, v_{ij} \rangle$.

To obtain the best alternative(s), the decision-making procedure is given as follows.

Step 1 The normalization of the decision matrix

By virtue of the following formulae, the decision matrix $D = (r_{ij})_{m\times l}$ can be changed into the standardized decision matrix $\bar{D} = (\bar{r}_{ij})_{m\times l} (\bar{r}_{ij} = \langle (\bar{a}_{ij}, \bar{\sigma}_{ij}), \bar{\mu}_{ij}, \bar{v}_{ij} \rangle)$.

$$\text{For benefit criteria: } \bar{a}_{ij} = \frac{a_{ij}}{\max_i(a_{ij})}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\max_i(\sigma_{ij})} \frac{\sigma_{ij}}{a_{ij}}, \quad \bar{\mu}_{ij} = u_{ij}, \bar{v}_{ij} = v_{ij}. \tag{23}$$

$$\text{For cost criteria: } \bar{a}_{ij} = \frac{\min_i(a_{ij})}{a_{ij}}, \quad \bar{\sigma}_{ij} = \frac{\sigma_{ij}}{\max_i(\sigma_{ij})} \frac{\sigma_{ij}}{a_{ij}}, \quad \bar{\mu}_{ij} = u_{ij}, \bar{v}_{ij} = v_{ij}. \tag{24}$$

Step 2 Aggregate the criteria values of each alternative.

By applying Eq. (22), we can obtain the comprehensive evaluation for the alternatives:

$$R_i = \langle (a_i, \sigma_i), \mu_i, v_i \rangle \; (i = 1, 2, \ldots, l).$$

Step 3 Determine the ranking of the alternatives.

By virtue of Definitions 6 and 7, we can get the ranking of all alternatives.

5 Numerical example

An engine part manufacturing company wants to select a supplier, and there are four suppliers as alternatives to choose, denoted by a_1, a_2, a_3, a_4. The suppliers include the following four criteria that influence the choice of the company: (1) c_1 expresses supply capacity; (2) c_2 pertains to delivery capability; (3) c_3 denotes quality of service; (4) c_4 represents research strength. It is easy to know that the four criteria above are all benefit criteria. The criteria are independent of each other and their weights comprise a vector of $\omega = (0.15, 0.25, 0.32, 0.28)$. The four considered alternatives are evaluated according to the normal intuitionistic fuzzy information by the expert using the criteria mentioned above, and the evaluation outcomes are presented in Table 1.

5.1 Procedures of decision-making based on the NIFIGOWA operator

Step 1 The normalization of the decision matrix
The four criteria are all the benefit-type criteria, and by applying Eq. (23), we can obtain the standardized decision matrix $\bar{D}$ as shown in Table 2.
Step 2 Calculate the comprehensive evaluations for the alternatives
By applying Eq. (22), the comprehensive evaluation for four alternatives can be shown in Table 3.
Step 3 Determine the ranking of alternatives
By applying Definitions 6 and 7, we can obtain the ranking of all alternatives as shown in Table 4.
From Tables 3 and 4, it is obvious that when the value of λ changes, the rankings are different, and the corresponding best alternatives are also different. Furthermore, if Eq. (9) or (15) is used in Step 2, the ranking of the alternatives is $a_4 \succ a_2 \succ a_1 \succ a_3$, and the best alternative is a_4.

Table 1 The expert's evaluations for the alternatives

	c_1	c_2	c_3	c_4
a_1	⟨(3, 0.4), 0.7, 0.2⟩	⟨(7, 0.6), 0.6, 0.3⟩	⟨(5, 0.4), 0.6, 0.2⟩	⟨(7, 0.6), 0.6, 0.3⟩
a_2	⟨(4, 0.2), 0.6, 0.3⟩	⟨(8, 0.4), 0.8, 0.1⟩	⟨(6, 0.7), 0.8, 0.2⟩	⟨(5, 0.3), 0.7, 0.3⟩
a_3	⟨(3.5, 0.3), 0.6, 0.4⟩	⟨(6, 0.2), 0.6, 0.3⟩	⟨(5.5, 0.6), 0.5, 0.5⟩	⟨(6, 0.4), 0.8, 0.1⟩
a_4	⟨(5, 0.5), 0.8, 0.2⟩	⟨(7, 0.5), 0.6, 0.2⟩	⟨(4.5, 0.5), 0.8, 0.2⟩	⟨(7, 0.2), 0.7, 0.1⟩

Table 2 The standardized decision matrix $\bar{D}$

	c_1	c_2	c_3	c_4
a_1	⟨(0.6, 0.1067), 0.7, 0.2⟩	⟨(0.875, 0.0857), 0.6, 0.3⟩	⟨(0.8333, 0.0457), 0.6, 0.2⟩	⟨(1, 0.08571), 0.6, 0.3⟩
a_2	⟨(0.8, 0.02), 0.6, 0.3⟩	⟨(1, 0.0333), 0.8, 0.1⟩	⟨(1, 0.1167), 0.8, 0.2⟩	⟨(0.7143, 0.03), 0.7, 0.3⟩
a_3	⟨(0.7, 0.1371), 0.6, 0.4⟩	⟨(0.75, 0.0111), 0.6, 0.3⟩	⟨(0.9167, 0.0935), 0.5, 0.5⟩	⟨(0.8571, 0.0444), 0.8, 0.1⟩
a_4	⟨(1, 0.1), 0.8, 0.2⟩	⟨(0.875, 0.0595), 0.6, 0.2⟩	⟨(0.75, 0.0794), 0.8, 0.2⟩	⟨(1, 0.0095), 0.7, 0.1⟩

Table 3 The comprehensive evaluations for four alternatives

λ	a_1	a_2	a_3	a_4
0	$\langle(0.8081, 0.0099), 0.6265, 0.2586\rangle$	$\langle(0.8684, 0.0029), 0.7342, 0.2262\rangle$	$\langle(0.8111, 0.0076), 0.6082, 0.3590\rangle$	$\langle(0.8923, 0.0050), 0.7297, 0.1857\rangle$
1	$\langle(0.8230, 0.0856), 0.6204, 0.2644\rangle$	$\langle(0.8786, 0.0536), 0.7467, 0.2112\rangle$	$\langle(0.8161, 0.0811), 0.6441, 0.3158\rangle$	$\langle(0.8988, 0.0705), 0.7346, 0.1833\rangle$
2	$\langle(0.8366, 0.0926), 0.6154, 0.2693\rangle$	$\langle(0.8884, 0.0784), 0.7582, 0.1971\rangle$	$\langle(0.8211, 0.0972), 0.6538, 0.2964\rangle$	$\langle(0.9049, 0.0889), 0.7396, 0.1808\rangle$
5	$\langle(0.8692, 0.0893), 0.6060, 0.2793\rangle$	$\langle(0.9142, 0.1063), 0.7826, 0.1660\rangle$	$\langle(0.8348, 0.1130), 0.6817, 0.2450\rangle$	$\langle(0.9216, 0.0975), 0.7538, 0.1729\rangle$
10	$\langle(0.9029, 0.0871), 0.6010, 0.2870\rangle$	$\langle(0.9432, 0.1153), 0.7965, 0.1438\rangle$	$\langle(0.8530, 0.1179), 0.7120, 0.2005\rangle$	$\langle(0.9422, 0.1005), 0.7705, 0.1609\rangle$
20	$\langle(0.9377, 0.0866), 0.6000, 0.2939\rangle$	$\langle(0.9689, 0.1180), 0.7998, 0.1290\rangle$	$\langle(0.8744, 0.1221), 0.7331, 0.1752\rangle$	$\langle(0.9648, 0.1018), 0.7846, 0.1433\rangle$
50	$\langle(0.9727, 0.0867), 0.6000, 0.2993\rangle$	$\langle(0.9874, 0.1183), 0.8, 0.1134\rangle$	$\langle(0.8965, 0.1249), 0.7439, 0.1631\rangle$	$\langle(0.9850, 0.1016), 0.7938, 0.1202\rangle$

Table 4 The ranking of all alternatives

λ	Ranking	The best alternative(s)
0	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
1	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
2	$a_4 \succ a_2 \succ a_3 \succ a_1$	a_4
5	$a_2 \succ a_4 \succ a_3 \succ a_1$	a_2
10	$a_2 \succ a_4 \succ a_3 \succ a_1$	a_2
20	$a_2 \succ a_4 \succ a_3 \succ a_1$	a_2
50	$a_2 \succ a_4 \succ a_3 \succ a_1$	a_2

Table 5 The expert's evaluations for alternatives expressed by TIFNs

	c_1	c_2	c_3	c_4
a_1	$\langle(1.8, 3, 4.2), 0.7, 0.2\rangle$	$\langle(5.2, 7, 8.8), 0.6, 0.3\rangle$	$\langle(3.8, 5, 6.2), 0.6, 0.2\rangle$	$\langle(5.2, 7, 8.8), 0.6, 0.3\rangle$
a_2	$\langle(3.4, 4, 4.6), 0.6, 0.3\rangle$	$\langle(6.8, 8, 9.2), 0.8, 0.1\rangle$	$\langle(3.9, 6, 8.1), 0.8, 0.2\rangle$	$\langle(4.1, 5, 5.9), 0.7, 0.3\rangle$
a_3	$\langle(2.6, 3.5, 4.4), 0.6, 0.4\rangle$	$\langle(5.4, 6, 6.6), 0.6, 0.3\rangle$	$\langle(3.7, 5.5, 7.3), 0.5, 0.5\rangle$	$\langle(4.8, 6, 7.2), 0.8, 0.1\rangle$
a_4	$\langle(3.5, 5, 6.5), 0.8, 0.2\rangle$	$\langle(5.5, 7, 8.5), 0.6, 0.2\rangle$	$\langle(3, 4.5, 6), 0.8, 0.2\rangle$	$\langle(6.4, 7, 7.6), 0.7, 0.1\rangle$

5.2 Comparison analysis and discussion

To verify the feasibility of the proposed method based on normal intuitionistic fuzzy-induced generalized aggregation operator, a comparison study is conducted between the proposed method and the method based on triangular intuitionistic fuzzy aggregation operator (Li 2010b) and the extended VIKOR method of TIFNs (Wan et al. 2013b).

(a) The criterion values in the example above are transformed from NIFNs into TIFNs, and the decision-making problem is solved by the MCDM method based on triangular intuitionistic fuzzy aggregation operator.

Given an NIFN $A = \langle(a, \sigma), \mu, v\rangle$, it can be transformed into a TIFN denoted by $\tilde{A} = \langle(\tilde{a}, \tilde{b}, \tilde{c}), \tilde{\mu}, \tilde{\nu}\rangle$, where $\tilde{b} = a$, $\tilde{a} = a - 3\sigma$, $\tilde{c} = a + 3\sigma$, $\tilde{\mu} = \mu$ and $\tilde{\nu} = \nu$. According to this, the expert's evaluations for alternatives in Table 1 can be transformed into TIFNs denoted by $\hat{r}_{ij} = \langle(\hat{a}_{ij}, \hat{b}_{ij}, \hat{c}_{ij}), \hat{\mu}_{ij}, \hat{\nu}_{ij}\rangle$, as shown in Table 5.

Here, the procedures of the MCDM method based on triangular intuitionistic fuzzy aggregation operator (Li 2010b) are shown as follows.

(1) Normalize the TIFN decision matrix.
The normalized decision matrix is denoted by $\tilde{R} = (\tilde{r}_{ij})_{m\times l}$, where $\tilde{r}_{ij} = \langle(\tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij}), \tilde{\mu}_{ij}, \tilde{\nu}_{ij}\rangle = \langle(\frac{\hat{a}_{ij}}{\hat{c}_j^+}, \frac{\hat{b}_{ij}}{\hat{c}_j^+}, \frac{\hat{c}_{ij}}{\hat{c}_j^+}), \hat{\mu}_{ij}, \hat{\nu}_{ij}\rangle (i = 1, 2, \ldots, m)$ for benefit criteria. Here, $\hat{c}_j^+ = \max\{\hat{c}_{ij} | i = 1, 2, \ldots, m\} \quad (j = 1, 2, \ldots, l)$.
By applying the formula above, the normalized decision matrix $\tilde{R}$ can be obtained and shown in Table 6.

Table 6 The normalized decision matrix $\tilde{R}$

	c_1	c_2	c_3	c_4
a_1	$\langle(0.28, 0.46, 0.65), 0.7, 0.2\rangle$	$\langle(0.57, 0.76, 0.96), 0.6, 0.3\rangle$	$\langle(0.47, 0.62, 0.77), 0.6, 0.2\rangle$	$\langle(0.59, 0.80, 1.00), 0.6, 0.3\rangle$
a_2	$\langle(0.52, 0.62, 0.71), 0.6, 0.3\rangle$	$\langle(0.74, 0.87, 1.00), 0.8, 0.1\rangle$	$\langle(0.48, 0.74, 1.00), 0.8, 0.2\rangle$	$\langle(0.47, 0.57, 0.67), 0.7, 0.3\rangle$
a_3	$\langle(0.40, 0.54, 0.68), 0.6, 0.4\rangle$	$\langle(0.59, 0.65, 0.72), 0.6, 0.3\rangle$	$\langle(0.46, 0.68, 0.90), 0.5, 0.5\rangle$	$\langle(0.55, 0.68, 0.82), 0.8, 0.1\rangle$
a_4	$\langle(0.54, 0.77, 1.00), 0.8, 0.2\rangle$	$\langle(0.60, 0.76, 0.92), 0.6, 0.2\rangle$	$\langle(0.37, 0.56, 0.74), 0.8, 0.2\rangle$	$\langle(0.73, 0.80, 0.86), 0.7, 0.1\rangle$

(2) Calculate the weighted comprehensive values $\tilde{S}_i$ for the alternative a_i.

$$\tilde{S}_i = \sum_{j=1}^{l} \omega_j \tilde{r}_{ij} = \left\langle \left(\sum_{j=1}^{l} \omega_j \tilde{a}_{ij}, \sum_{j=1}^{l} \omega_j \tilde{b}_{ij}, \sum_{j=1}^{l} \omega_j \tilde{c}_{ij} \right), \min_j\{\tilde{\mu}_{ij}\}, \max_j\{\tilde{\nu}_{ij}\} \right\rangle$$
$$= \langle (\tilde{a}_i, \tilde{b}_i, \tilde{c}_i), \tilde{\mu}_i, \tilde{\nu}_i \rangle.$$

Thus, the weighted comprehensive values of a_i $(i = 1, 2, 3, 4)$ are:

$$\tilde{S}_1 = \langle (0.50, 0.68, 0.86), 0.6, 0.3 \rangle; \quad \tilde{S}_2 = \langle (0.55, 0.71, 0.86), 0.6, 0.3 \rangle;$$
$$\tilde{S}_3 = \langle (0.51, 0.65, 0.80), 0.5, 0.5 \rangle; \quad \tilde{S}_4 = \langle (0.55, 0.71, 0.86), 0.6, 0.2 \rangle.$$

(3) Rank all alternatives.
Let $Z(\tilde{S}_i, \lambda) = \frac{V(\tilde{S}_i,\lambda)}{1+A(\tilde{S}_i,\lambda)}$, where $\lambda \in [0, 1]$ is a weight representing the decision maker's preference information. If $\lambda \in [0, \frac{1}{2})$, it indicates the decision maker prefers uncertainty or negative feeling; if $\lambda \in (\frac{1}{2}, 1]$, the decision maker prefers certainty or positive feeling; if $\lambda = \frac{1}{2}$, the decision maker is indifferent between positive feeling and negative feeling. Besides, $V(\tilde{S}_i, \lambda) = V_\mu(\tilde{S}_i) + \lambda(V_\nu(\tilde{S}_i) - V_\mu(\tilde{S}_i))$ and $A(\tilde{S}_i, \lambda) = A_\nu(\tilde{S}_i) - \lambda(A_\nu(\tilde{S}_i) - A_\mu(\tilde{S}_i))$, where $V_\mu(\tilde{S}_i) = \frac{(\tilde{a}_i+4\tilde{b}_i+\tilde{c}_i)\tilde{\mu}_i}{6}$, $V_\nu(\tilde{S}_i) = \frac{(\tilde{a}_i+4\tilde{b}_i+\tilde{c}_i)(1-\tilde{\nu}_i)}{6}$, $A_\mu(\tilde{S}_i) = \frac{(\tilde{c}_i-\tilde{a}_i)\tilde{\mu}_i}{3}$ and $A_\nu(\tilde{S}_i) = \frac{(\tilde{c}_i-\tilde{a}_i)(1-\tilde{\nu}_i)}{3}$.
Here, there is no indication of the decision maker's preference, so we assume $\lambda = \frac{1}{2}$.
Calculate the values of $Z(\tilde{S}_i, \frac{1}{2})$, and then we can obtain the weighted comprehensive values of the alternatives as follows:

$$Z(\tilde{S}_1, \frac{1}{2}) = 0.416, \quad Z(\tilde{S}_2, \frac{1}{2}) = 0.432, \quad Z(\tilde{S}_3, \frac{1}{2}) = 0.311, \quad Z(\tilde{S}_4, \frac{1}{2}) = 0.462.$$

Rank the TIFNs $\tilde{S}_i$ according to non-increasing order of the values of $Z(\tilde{S}_i, \frac{1}{2})$, and the maximum TIFN is the one with the largest ratio, i.e., $\tilde{S}_4 \succ \tilde{S}_2 \succ \tilde{S}_1 \succ \tilde{S}_3$. Hence, the ranking of the four alternatives should be: $a_4 \succ a_2 \succ a_1 \succ a_3$.

(b) After converting the NIFNs into the TIFNs and normalizing the TIFNs, a normalized decision-making matrix $\tilde{R}$ can be obtained (seen in Table 6), and then we apply the extended VIKOR method of TIFNs (Wan et al. 2013b) to the normalized decision matrix. The results are shown in Table 7.

The alternatives are ranked in accordance with the increasing order of $Q(a_i)$, where $Q(a_i) = \lambda \frac{S(a_i)-S^+}{S^- - S^+} + (1-\lambda) \frac{R(a_i)-R^+}{R^- - R^+}$ ($S^+ = \text{Min}_i\{S(a_i)\}$, $S^- = \text{Max}_i\{S(a_i)\}$, $R^+ = \text{Min}_i\{R(a_i)\}$, and $R^- = \text{Max}_i\{R(a_i)\}$). With the variation of λ, the rankings of the alternatives change, as shown in Table 8.

From the analysis above, we conclude that both the results of triangular intuitionistic fuzzy aggregation operator method and the extended VIKOR method are consistent with the proposed method, which fully illustrates the feasibility of the proposed

Table 7 The results of the extended VIKOR method

	Values of $S(a_i)$	Values of $R(a_i)$
a_1	0.413	0.141
a_2	0.284	0.126
a_3	0.428	0.151
a_4	0.307	0.104

Table 8 The results of the extended VIKOR method with TIFNs

λ	Ranking	The best alternative(s)
0	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.1	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.2	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.3	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.4	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.5	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.6	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.7	$a_4 \succ a_2 \succ a_1 \succ a_3$	a_4
0.8	$a_2 \succ a_4 \succ a_1 \succ a_3$	a_2
0.9	$a_2 \succ a_4 \succ a_1 \succ a_3$	a_2
1.0	$a_2 \succ a_4 \succ a_1 \succ a_3$	a_2

method. However, the benefits of the normal intuitionistic fuzzy MCDM method proposed in this paper are far more than that. From the perspective of the definition of NIFNs, it has been well known that the normal distribution is the most important probability distribution to address the approximation of large numbers of random phenomena (Li et al. 2009), and certainly, NIFNs are of much more realistic senses than TIFNs. From the perspective of the aggregation operators, the induced generalized aggregation operator allows the value of λ to be a variable rather than a fixed number, which makes the decision-making more flexible and reliable.

6 Conclusions

Normal distribution phenomena widely exist in practical life, and NIFNs can better express normal distribution phenomena than other intuitionistic fuzzy numbers. Applying NIFNs to MCDM problems can not only complete the fuzzy MCDM theoretic system, but also render the MCDM methods much closer to reality. In this paper, some new aggregation operators of NIFNs are studied, which lay the foundation for the proposed MCDM method with normal intuitionistic fuzzy information. Particularly, the NIFIGOWA operator is used to aggregate the evaluation information of alternatives, so that the decision makers can properly select the desirable alternative according to their interest and the actual need by changing the value of λ, which makes the decision-making results of the proposed method more flexible and reliable. Future research may extend the proposed approach to evaluating other practical cases of MCDM problems in the uncertain environment.

Acknowledgments The authors thank the editors and anonymous reviewers for their helpful comments and suggestions. This work was supported by the National Natural Science Foundation of China (Nos. 71271218, 71221061).

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