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ORIGINAL PAPER

On Hölder calmness of solution mappings in parametric equilibrium problems

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Abstract We consider parametric equilibrium problems in metric spaces. Sufficient conditions for the Hölder calmness of solutions are established. We also study the Hölder well-posedness for equilibrium problems in metric spaces.

Keywords Equilibrium problem · Calmness · Well-posedness

Mathematics Subject Classification 49K40 · 90C31 · 91B50

1 Introduction

Optimization is one of the most fertile areas of mathematics. Its conclusions and recommendations play a very important role in both theoretical and applied mathematics. Equilibrium problems were first considered in Blum and Oettli (1994) and since then have been studied by many researchers all over the world. The equilibrium problem model incorporates many other important problems in optimization and other areas such as: variational inequalities, fixed point problems, complementarity, etc. There have been many studies of existence of solutions to equilibrium problems (see Sadequi and Alizadeh 2011; Bazán 2001; Bianchi and Schaible 1996; Hai and Khanh 2007a, 2007b; Hai et al. 2009) and their stability,

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e.g., semi-continuity in the sense of Berge and Hausdorff (see Anh and Khanh 2004, 2007a, 2007b, 2010; Huang et al. 2006; Khanh and Luu 2007) or Hölder (Lipschitzian) continuity (see Anh and Khanh 2006, 2008b, 2009; Bianchi and Pini 2003; Li et al. 2009; Li and Li 2011a, 2011b; Mansour and Riahi 2005).

This paper extends Anh et al. (2011) and studies $(l.\alpha)$-*Hölder calmness* of solutions to parametric equilibrium problems. When $\alpha = 1$, this is a kind of calmness property which is in general stronger than the property of the same name usually used in variational analysis. Calmness property of multi-valued mappings has been examined by many authors (see Cánovas et al. 2009; Chuong et al. 2011; Henrion et al. 2002; Ioffe and Outrata 2008; Levy 2000; Ng and Zheng 2009) in which subdifferentials and coderivatives play the main role. As applications we investigate conditions for Hölder calmness of solutions to optimization problems and well-posedness in the Hölder sense. The last subject is intimately related to the stability property and plays a very important role in studying optimization and variational problems.

The structure of the paper is as follows. Section 2 presents the equilibrium problem model and materials used in the rest of this paper. We establish in Sect. 3 a sufficient condition for the Hölder calmness of the solution mapping to parametric equilibrium problems. The Hölder well-posedness of equilibrium problems is studied in Sect. 4.

Throughout the paper, if not explicitly stated otherwise, X, Λ, M are metric spaces and $\mathbb{R}$ is the set of all real numbers while $\mathbb{R}_+$ is the set of all positive numbers. We use $d(\cdot,\cdot)$ for all metrics.

2 Preliminaries

Given a subset $K \subseteq X$ and a function $f : X \times X \to \mathbb{R}$, a standard *equilibrium problem* is defined as follows:

(EP) Find $\bar{x} \in K$ such that $f(\bar{x}, y) \geq 0$ for all $y \in K$.

The set of solutions to this problem is denoted by S.

In this paper, we consider several extensions of (EP).

The constraint set K and objective function f can be perturbed by parameters $\lambda \in \Lambda$ and $\mu \in M$, respectively. Given a multi-valued mapping $K : \Lambda \rightrightarrows X$, a function $f : X \times X \times M \to \mathbb{R}$, and a pair $(\lambda, \mu) \in \Lambda \times M$, one can consider a parameterized equilibrium problem:

$(\mathrm{EP})_{\lambda,\mu}$ Find $\bar{x} \in K(\lambda)$ such that $f(\bar{x}, y, \mu) \geq 0$ for all $y \in K(\lambda)$.

The set of solutions to problem $(\mathrm{EP})_{\lambda,\mu}$ is denoted by $S(\lambda, \mu)$.

The approximate version of this problem can be of interest: for each $(\lambda, \mu) \in \Lambda \times M$ and $\varepsilon > 0$,

$(\widetilde{\mathrm{EP}})_{\varepsilon,\lambda,\mu}$ Find $\bar{x} \in K(\lambda)$ such that $f(\bar{x}, y, \mu) + \varepsilon \geq 0$ for all $y \in K(\lambda)$.

We denote by $\widetilde{S}(\varepsilon, \lambda, \mu)$ the solution set of $(\widetilde{\mathrm{EP}})_{\varepsilon,\lambda,\mu}$.

Definition 2.1 For a function $f : X \to \mathbb{R}$ and positive numbers l, α,

(i) f is $(l.\alpha)$-Hölder continuous on a subset $U \subset X$ if

$$\left|f(x_1) - f(x_2)\right| \leq l d^\alpha(x_1, x_2) \quad \text{for all } x_1, x_2 \in U;$$

(ii) f is $(l.\alpha)$-Hölder calm at $\overline{x}$ on a neighborhood U of $\overline{x}$ if

$$\left|f(x) - f(\overline{x})\right| \leq l d^\alpha(x, \overline{x}) \quad \text{for all } x \in U.$$

We say that f satisfies a certain property on a subset $A \subseteq X$ if it is satisfied at every point of A.

From this definition, it is obvious that Hölder continuity is stronger than Hölder calmness.

To define extensions of these properties for multi-valued mappings we recall the definitions of point-to-set and set-to-set distances. For subsets A, B of X and a point $a \in X$,

$$d(a, B) = \inf_{b \in B} d(a, b);$$

$$H^*(A, B) = \sup_{a \in A} d(a, B);$$

$$H(A, B) = \max\left\{H^*(A, B), H^*(B, A)\right\};$$

$$\rho(A, B) = \sup_{a \in A, b \in B} d(a, b).$$

Note that H and ρ can take infinite values (if A or B is unbounded). It is also obvious that $H(A, B) \leq \rho(A, B)$ for any subsets A and B, and the inequality can be strict.

Definition 2.2 For a multi-valued mapping $K : \Lambda \rightrightarrows X$ and positive numbers l, α,

(i) K is $(l.\alpha)$-Hölder continuous on a subset $U \subset X$ if

$$H\big(K(\lambda_1), K(\lambda_2)\big) \leq l d^\alpha(\lambda_1, \lambda_2) \quad \text{for all } \lambda_1, \lambda_2 \in U;$$

(ii) K is $(l.\alpha)$-Hölder calm at $\overline{\lambda}$ on a neighborhood U of $\overline{\lambda}$ if

$$H\big(K(\lambda), K(\overline{\lambda})\big) \leq l d^\alpha(\lambda, \overline{\lambda}) \quad \text{for all } \lambda \in U. \tag{1}$$

We will also consider the versions of the properties in Definition 2.2 with H replaced by ρ. In this case, we will talk about the corresponding properties *with respect to ρ*.

Remark 2.1 The calmness in the above definition (when $\alpha = 1$) is a stronger property than the one usually considered in variational analysis. The latter corresponds to replacing H in (1) by H^* (see, e.g., Rockafellar and Wets 1998). Respectively, (l, α)-calmness is stronger than "calmness [α]" in Kummer (2009).

We next define *uniform Hölder calmness* as the natural counterpart of the relative Hölder continuity in Anh and Khanh (2007b).

Definition 2.3 For positive numbers m, β, θ, a function $f : X \times X \times M \to \mathbb{R}$ is $(m.\beta)$-Hölder calm at $\overline{\mu}$ on a neighborhood V of $\overline{\mu}$, θ-uniformly over a subset $S \subseteq X$ if

$$\left|f(x,y,\overline{\mu}) - f(x,y,\mu)\right| \le md^{\beta}(\overline{\mu},\mu)d^{\theta}(x,y), \quad \forall \mu \in V,\ \forall x, y \in S,\ x \ne y.$$

If $\theta = 0$, we say that f is $(m.\beta)$-Hölder calm at $\overline{\mu}$ on V, uniformly over S.

We next discuss several monotonicity properties some of which are going to play a crucial role in examining the Hölder calmness of the solution mapping of the equilibrium problems $(\mathrm{EP})_{\lambda,\mu}$.

Given a function $f : X \times X \to \mathbb{R}$, positive numbers h, β, and a subset $S \subseteq X$, consider the following properties.

(M_1) For all $x, y \in S, x \ne y$,

$$f(x,y) + f(y,x) + hd^{\beta}(x,y) \le 0. \tag{2}$$

(M_2) For all $x, y \in S$,

$$hd^{\beta}(x,y) \le d\big(f(x,y),\mathbb{R}_+\big) + d\big(f(y,x),\mathbb{R}_+\big). \tag{3}$$

(M_3) For all $x, y \in S, x \ne y$,

$$\big[f(x,y) \ge 0 \Rightarrow f(y,x) + hd^{\beta}(x,y) \le 0\big].$$

(M_4) For all $x, y \in S, x \ne y$,

$$\big[f(x,y) < 0 \Rightarrow f(y,x) \ge 0\big].$$

If any of the above properties is fulfilled, we say that f satisfies the corresponding condition on S with constants h and β (if applicable).

Remark 2.2 Properties (M_1), (M_3) and (M_4) were considered in Anh and Khanh (2006, 2007b, 2008a) where they were called *Hölder strong monotonicity*, *Hölder strong pseudo-monotonicity* and *quasi-monotonicity*, respectively. Property (M_2) is a particular case of the corresponding monotonicity property introduced by Anh and Khanh (see Anh and Khanh 2007b) for multi-valued mappings. This property has been employed to investigate the Hölder continuity of solution mappings in many articles (see Anh and Khanh 2008a; Li and Li 2011b; Anh et al. 2011).

The next proposition gives the relationships between these monotonicity properties.

Proposition 2.1

(i) $(M_1) \Rightarrow (M_2) \Rightarrow (M_3)$;
(ii) $[(M_3)\ \&\ (M_4)] \Rightarrow (M_2)$.

Proof The following simple observation is used in the proof:

$$d(a, \mathbb{R}_+) = \max\{-a, 0\} \geq -a.$$

$(M_1) \Rightarrow (M_2)$. If (2) holds for some $x \neq y$, then

$$hd^\beta(x, y) \leq -f(x, y) - f(y, x) \leq d\big(f(x, y), \mathbb{R}_+\big) + d\big(f(y, x), \mathbb{R}_+\big),$$

i.e., (3) holds. When $x = y$, (3) holds automatically.

$(M_2) \Rightarrow (M_3)$. If (3) holds for some $x \neq y$ and $f(x, y) \geq 0$, then $d(f(x, y), \mathbb{R}_+) = 0$ and (3) takes the form

$$hd^\beta(x, y) \leq d\big(f(y, x), \mathbb{R}_+\big).$$

It follows from the last inequality that $d(f(y, x), \mathbb{R}_+) > 0$ and consequently $d(f(y, x), \mathbb{R}_+) = -f(y, x)$. Hence, (M_3) holds true.

$[(M_3)\ \&\ (M_4)] \Rightarrow (M_2)$. Let (M_3) and (M_4) hold true. We only need to prove (3) when $x \neq y$. If $f(x, y) \geq 0$, then $d(f(x, y), \mathbb{R}_+) = 0$ and (M_3) implies

$$0 < hd^\beta(x, y) \leq -f(y, x) = d\big(f(y, x), \mathbb{R}_+\big).$$

Hence, (3) is true. If $f(x, y) < 0$, then (M_4) implies $f(y, x) \geq 0$, and we can apply (M_3) again to show that

$$0 < hd^\beta(x, y) \leq -f(x, y) = d\big(f(x, y), \mathbb{R}_+\big).$$

Taking into account that $d(f(y, x), \mathbb{R}_+) = 0$, we conclude that (3) is true in this case too. □

We now give examples showing that implications in Proposition 2.1 can be strict.

Example 2.1 The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = x - y$ satisfies (M_2) with $h = \beta = 1$. Indeed,

$$\begin{aligned} d\big(f(x, y), \mathbb{R}_+\big) + d\big(f(y, x), \mathbb{R}_+\big) &= d(x - y, \mathbb{R}_+) + d(y - x, \mathbb{R}_+) \\ &= |x - y| = d(x, y). \end{aligned}$$

At the same time, $f(x, y) + f(y, x) = 0$ and (2) is violated for any $x \neq y$. f does not satisfy (M_1). It is also obvious that f satisfies both (M_3) and (M_4).

Example 2.2 The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $f(x, y) = -\frac{1}{4}(|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}})$ satisfies (M_3) with $h = \sqrt{2}$ and $\beta = \frac{1}{2}$ as $f(x, y) \geq 0$ if and only if $x = y = 0$, it does not satisfy (M_2). Indeed, for any $y = -x \neq 0$, we have

$$d\big(f(x, y), \mathbb{R}_+\big) + d\big(f(y, x), \mathbb{R}_+\big) = \frac{1}{2}\big(|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}}\big) = |x|^{\frac{1}{2}} < 2|x|^{\frac{1}{2}} = \sqrt{2}d^{\frac{1}{2}}(x, y).$$

We can see that the combination of (M_3) and (M_4) implies (M_2), but they are not equivalent by considering the function $f(x, y) = -(|x| + |y|)$. This function satisfies (M_2) with $h = \beta = 1$, but breaks (M_4).

3 The Hölder calmness of the solution mapping

The next theorem gives a sufficient condition for the Hölder calmness of the solution mapping of the problem $(\mathrm{EP})_{\lambda,\mu}$. It improves Theorem 2.1 in Anh et al. (2011). We always assume that solution sets $S(\lambda,\mu)$ are nonempty for all (λ,μ) in a neighborhood of the considered point $(\overline{\lambda},\overline{\mu})$.

Theorem 3.1 *Consider equilibrium problem* $(\mathrm{EP})_{\lambda,\mu}$ *and suppose the following conditions hold.*

- (i) *There exist neighborhoods $U(\overline{\lambda})$ of $\overline{\lambda}$ and $V(\overline{\mu})$ of $\overline{\mu}$ and positive numbers n_1, δ_1 and θ such that f is $(n_1.\delta_1)$-Hölder calm at $\overline{\mu}$ on $V(\bar{\mu})$, θ-uniformly over $K(U(\overline{\lambda}))$.*
- (ii) *There exist positive numbers n_2 and δ_2 such that, for all $x\in K(U(\overline{\lambda}))$ and $\mu\in V(\overline{\mu})$, the function $f(x,\cdot,\mu)$ is $(n_2.\delta_2)$-Hölder continuous on $K(U(\overline{\lambda}))$.*
- (iii) *$f(\cdot,\cdot,\overline{\mu})$ satisfies condition (M_2) on $K(U(\bar{\lambda}))$ with constants $h>0$ and $\beta>\theta$.*
- (iv) *K is $(l.\alpha)$-Hölder calm at $\overline{\lambda}$ on $U(\bar{\lambda})$ with some positive l and α.*

Then solutions to $(\mathrm{EP})_{\lambda,\mu}$ *satisfy the condition of Hölder calmness with respect to ρ: there exist constants $k_1,k_2>0$ such that*

$$\rho\big(S(\overline{\lambda},\overline{\mu}),S(\lambda,\mu)\big)\le k_1 d^{\alpha\delta_2/\beta}(\overline{\lambda},\lambda)+k_2 d^{\delta_1/(\beta-\theta)}(\overline{\mu},\mu),$$

for all (λ,μ) in a neighborhood of $(\overline{\lambda},\overline{\mu})$.

Proof Take $\lambda\in U(\overline{\lambda})$ and $\mu\in V(\overline{\mu})$.

Step 1. We prove that for each $x(\lambda,\overline{\mu})\in S(\lambda,\overline{\mu})$ and $x(\lambda,\mu)\in S(\lambda,\mu)$,

$$d_1:=d\big(x(\lambda,\overline{\mu}),x(\lambda,\mu)\big)\le\left(\frac{n_1}{h}\right)^{1/(\beta-\theta)} d^{\delta_1/(\beta-\theta)}(\overline{\mu},\mu). \tag{4}$$

Suppose $x(\lambda,\overline{\mu})\ne x(\lambda,\mu)$ (if the equality holds, then (4) holds trivially). Because both $x(\lambda,\overline{\mu})$ and $x(\lambda,\mu)$ belong to $K(\lambda)$ and are solutions of $(\mathrm{EP})_{\lambda,\mu}$, one has

$$f\big(x(\lambda,\overline{\mu}),x(\lambda,\mu),\overline{\mu}\big)\ge 0; \tag{5}$$

$$f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\mu\big)\ge 0. \tag{6}$$

At the same time, (iii) implies

$$d\big(f\big(x(\lambda,\overline{\mu}),x(\lambda,\mu),\overline{\mu}\big),\mathbb{R}_+\big)+d\big(f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\overline{\mu}\big),\mathbb{R}_+\big)\ge hd_1^{\beta}.$$

Combining this inequality with (5) and (6), we get

$$d\big(f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\overline{\mu}\big),f\big(x(\lambda,\mu),x(\lambda,\overline{\mu}),\mu\big)\big)\ge hd_1^{\beta}.$$

Because f is $(n_1.\delta_1)$-Hölder calm at $\overline{\mu}$, θ-uniformly over $K(U(\overline{\lambda}))$ by (i), the above relationship implies

$$n_1 d_1^{\theta} d^{\delta_1}(\overline{\mu},\mu)\ge hd_1^{\beta}.$$

This is equivalent to $d_1^{\beta-\theta} \leq \frac{n_1}{h} d^{\delta_1}(\overline{\mu}, \mu)$ from which we get (4) proved.

Step 2. We prove that for each $x(\overline{\lambda}, \overline{\mu}) \in S(\overline{\lambda}, \overline{\mu})$ and $x(\lambda, \overline{\mu}) \in S(\lambda, \overline{\mu})$,

$$d_2 := d\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \overline{\mu})\big) \leq \left(\frac{2n_2 l^{\delta_2}}{h}\right)^{1/\beta} d^{\alpha\delta_2/\beta}(\overline{\lambda}, \lambda). \tag{7}$$

Suppose $x(\overline{\lambda}, \overline{\mu}) \neq x(\lambda, \overline{\mu})$. (iv) implies that there exist $\overline{x} \in K(\overline{\lambda})$ and $x \in K(\lambda)$ such that

$$d\big(x(\overline{\lambda}, \overline{\mu}), x\big) \leq l d^{\alpha}(\overline{\lambda}, \lambda); \tag{8}$$

$$d\big(x(\lambda, \overline{\mu}), \overline{x}\big) \leq l d^{\alpha}(\overline{\lambda}, \lambda). \tag{9}$$

We get from the definition of $(EP)_{\lambda,\mu}$,

$$f\big(x(\overline{\lambda}, \overline{\mu}), \overline{x}, \overline{\mu}\big) \geq 0; \tag{10}$$

$$f\big(x(\lambda, \overline{\mu}), x, \overline{\mu}\big) \geq 0. \tag{11}$$

At the same time, (iii) implies

$$d\big(f\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \overline{\mu}), \overline{\mu}\big), \mathbb{R}_+\big) + d\big(f\big(x(\lambda, \overline{\mu}), x(\overline{\lambda}, \overline{\mu}), \overline{\mu}\big), \mathbb{R}_+\big) \geq h d_2^{\beta}.$$

Combining this inequality with (10) and (11), we get

$$\begin{aligned} &d\big(f\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \overline{\mu}), \overline{\mu}\big), f\big(x(\overline{\lambda}, \overline{\mu}), \overline{x}, \overline{\mu}\big)\big) \\ &\quad + d\big(f\big(x(\lambda, \overline{\mu}), x(\overline{\lambda}, \overline{\mu}), \overline{\mu}\big), f\big(x(\lambda, \overline{\mu}), x, \overline{\mu}\big)\big) \geq h d_2^{\beta}. \end{aligned}$$

Because f is $(n_2\delta_2)$-Hölder continuous with respect to the second component in $K(U(\overline{\lambda}))$ by (ii), the last inequality implies that

$$n_2 d^{\delta_2}\big(x(\lambda, \overline{\mu})\big), \overline{x}\big) + n_2 d^{\delta_2}\big(x(\overline{\lambda}, \overline{\mu}), x\big) \geq h d_2^{\beta}.$$

We combine this with (8) and (9) and get

$$n_2 l^{\delta_2} d^{\alpha\delta_2}(\overline{\lambda}, \lambda) + n_2 l^{\delta_2} d^{\alpha\delta_2}(\overline{\lambda}, \lambda) \geq h d_2^{\beta},$$

or equivalently $d_2^{\beta} \leq \frac{2n_2 l^{\delta_2}}{h} d^{\alpha\delta_2}(\overline{\lambda}, \lambda)$. We have (7) proved.

Step 3. For all $x(\overline{\lambda}, \overline{\mu}) \in S(\overline{\lambda}, \overline{\mu})$ and $x(\lambda, \mu) \in S(\lambda, \mu)$, we always have

$$d\big(x(\overline{\lambda}, \overline{\mu}), x(\lambda, \mu)\big) \leq d_1 + d_2.$$

From (4) and (7), by taking $k_1 = (\frac{2n_2 l^{\delta_2}}{h})^{1/\beta}$ and $k_2 = (\frac{n_1}{h})^{1/(\beta-\theta)}$, we get

$$\rho\big(S(\overline{\lambda}, \overline{\mu}), S(\lambda, \mu)\big) \leq k_1 d^{\alpha\delta_2/\beta}(\overline{\lambda}, \lambda) + k_2 d^{\delta_1/(\beta-\theta)}(\overline{\mu}, \mu).$$

Therefore, Theorem 3.1 has been proved. □

By using the technique similar to the one in the proof of Theorem 2.1 in Anh and Khanh (2007b), we can show that, under assumption (iii), the solution to $(\mathrm{EP})_{\bar{\lambda},\bar{\mu}}$ is unique. However, when $(\lambda, \mu) \neq (\bar{\lambda}, \bar{\mu})$, the solutions to $(\mathrm{EP})_{\lambda,\mu}$ do not have to be unique as demonstrated by the following example.

Example 3.1 Let $X = \mathbb{R}$, $\Lambda \equiv M = [0; 1]$, $K(\lambda) = [0; 1]$, $f(x, y, \lambda) = y - x + \lambda$ for all $\lambda \in \Lambda$, and $\bar{\lambda} = 0$.

Then $|f(x, y, \lambda) - f(x, y, \bar{\lambda})| = |\lambda|$. Hence, f is (1.1)-Hölder calm at $\bar{\lambda}$ uniformly over $[0; 1]$. We have $|f(x, y, \lambda) - f(x, z, \lambda)| = |y - z|$ for all $y, z \in [0; 1]$. So $f(x, \cdot, \lambda)$ is (1.1)-Hölder continuous on $[0; 1]$. Therefore, assumptions (i) and (ii) hold. It is clear that condition (iv) also holds. Assumption (iii) is fulfilled as shown in Example 2.1. Hence, Theorem 3.1 derives the Holder calmness of $S(\cdot)$ at $\bar{\lambda}$. It is not difficult to check that $S(0) = \{0\}$ and $S(\lambda) = [0, \lambda]$ for all $\lambda \in (0; 1]$.

Normally, to receive a property of solution mappings, the problem's hypotheses are also required at the level corresponding to that property. We can see from the preceding theorem that all the hypotheses are related to Hölder continuity and Hölder calmness, except (iii), which is about monotonicity.

The next example indicates the essential role of assumption (iii) in Theorem 3.1.

Example 3.2 Take $X = \mathbb{R}$, $M \equiv \Lambda = [0; 1]$, $K(\lambda) = [-1; 1]$ for all $\lambda \in [0; 1]$. For each $\lambda \in [0; 1]$, consider the function f defined by $f(x, y, \lambda) = \lambda(x + y)$. Take $\bar{\lambda} = 0$.

We have $|f(x, y, \lambda) - f(x, y, \bar{\lambda})| = |x + y| \cdot |\lambda - \bar{\lambda}| \leq 2|\lambda - \bar{\lambda}|$ for all $x, y \in [-1; 1]$. So f is (2.1)-Hölder calm at $\bar{\lambda}$ on $[0; 1]$ uniformly over $[-1; 1]$. At the same time, $|f(x, y, \lambda) - f(x, z, \lambda)| = |\lambda| \cdot |y - z| \leq |y - z|$ for all $y, z \in [-1; 1]$. This means that $f(x, \cdot, \lambda)$ is (1.1)-Hölder continuous on $[-1; 1]$. Hence, conditions (i) and (ii) are fulfilled.

Condition (iv) is also true straightforwardly. However, we have

$$S(0) = [-1; 1], \qquad S(\lambda) = \{1\}, \quad \forall \lambda \in (0; 1].$$

So $\rho(S(\lambda), S(0)) = 2$ for any $\lambda \in (0; 1]$.

Therefore, the solution mapping S is not Hölder calm at $\overline{\mu} = 0$. The reason here is that f breaks condition (M_2). Indeed,

$$d\big(f(1, 0, 0), \mathbb{R}_+\big) + d\big(f(0, 1, 0), \mathbb{R}_+\big) = 0 < h|1 - 0|^\beta = h, \quad \forall h,\ \beta > 0.$$

Condition (M_2) in Theorem 3.1 is indispensable.

Remark 3.1 It follows from Proposition 2.1 that the conclusion of Theorem 3.1 remains true if condition (iii) is replaced by either condition (M_1) or conditions (M_3) and (M_4).

The next proposition aims to illustrate application of Theorem 3.1. For each $(\lambda, \mu) \in \Lambda \times M$, we consider the minimization problem

(MP) Minimize $f(x, \mu)$ subject to $x \in K(\lambda)$, where $f : X \times M \to \mathbb{R}$ and $K : \Lambda \rightrightarrows X$. We denote $S(\lambda, \mu) = \{\overline{x} \in K(\lambda) : f(\overline{x}, \mu) = \min_{x \in K(\lambda)} f(x, \mu)\}$ and assume that $S(\lambda, \mu) \neq \emptyset$ for all (λ, μ) near the considered point $(\overline{\lambda}, \overline{\mu})$.

Proposition 3.2 *Consider (MP) and suppose the following conditions hold.*

(i) *There exist neighborhoods $V(\overline{\mu})$ of $\overline{\mu}$ and $U(\overline{\lambda})$ of $\overline{\lambda}$ and numbers $n_1 > 0$ and $\delta_1 > 0$ such that f is $(n_1.\delta_1)$-Hölder calm at $\overline{\mu}$ on $V(\bar{\mu})$ uniformly over $K(U(\overline{\lambda}))$, i.e.,*

$$\left|f(x, \mu) - f(x, \overline{\mu})\right| \leq n_1 d^{\delta_1}(\mu, \overline{\mu})$$

for all $x \in K(U(\overline{\lambda}))$ and $\mu \in V(\overline{\mu})$.

(ii) *There exist numbers $n_2 > 0$ and $\delta_2 > 0$ such that f is $(n_2.\delta_2)$-Hölder continuous in x on $K(U(\overline{\lambda}))$ uniformly over $\mu \in V(\overline{\mu})$, i.e.,*

$$\left|f(x, \mu) - f(y, \mu)\right| \leq n_2 d^{\delta_2}(x, y) \tag{12}$$

for all $\mu \in V(\overline{\mu})$ and $x, y \in K(U(\overline{\lambda}))$, and (12) holds as an equality when $\mu = \overline{\mu}$.

(iii) *K is $(l.\alpha)$-Hölder calm at $\overline{\lambda}$ on $U(\bar{\lambda})$ with some $l > 0$ and $\alpha > 0$.*

Then the mapping S is Hölder calm with respect to ρ, i.e., there exist constants $k_1, k_2 > 0$ such that

$$\rho\big(S(\overline{\lambda}, \overline{\mu}), S(\lambda, \mu)\big) \leq k_1 d^{\alpha}(\overline{\lambda}, \lambda) + k_2 d(\overline{\mu}, \mu) \tag{13}$$

for all (λ, μ) in a neighborhood of $(\overline{\lambda}, \overline{\mu})$.

Proof We define the function $g : X \times X \times M \to \mathbb{R}$ as follows

$$g(x, y, \mu) = f(y, \mu) - f(x, \mu).$$

We observe that $\overline{x} \in S(\lambda, \mu)$ if and only if $\overline{x} \in K(\lambda)$ and $g(\overline{x}, y, \mu) \geq 0, \forall y \in K(\lambda)$. So to prove the proposition, it suffices to check that g satisfies the conditions of Theorem 3.1.

We first check condition (i). For every $\mu \in V(\overline{\mu})$ and $x, y \in K(U(\overline{\lambda}))$ we have

$$\begin{aligned}
&\left|g(x, y, \mu) - g(x, y, \overline{\mu})\right| \\
&\quad = \left|f(y, \mu) - f(x, \mu) - f(y, \overline{\mu}) + f(x, \overline{\mu})\right| \\
&\quad \leq \left|f(x, \mu) - f(x, \overline{\mu})\right| + \left|f(y, \mu) - f(y, \overline{\mu})\right| \leq 2n_1 d^{\delta_1}(\mu, \overline{\mu}).
\end{aligned}$$

This means that g is $(2n_1.\delta_1)$-Hölder calm at $\overline{\mu}$ on $V(\overline{\mu})$ uniformly over $K(U(\overline{\lambda}))$.

We have at the same time

$$\left|g(x, y, \mu) - g(x, z, \mu)\right| = \left|f(y, \mu) - f(z, \mu)\right| \leq n_2 d^{\delta_2}(y, z),$$

i.e., g is $(n_2.\delta_2)$-Hölder continuous with respect to the second component. So conditions (i) and (ii) in Theorem 3.1 are fulfilled.

We now check condition (iii) in Theorem 3.1. For all $x, y \in K(U(\overline{\lambda}))$, we have

$$\begin{aligned} &d\big(g(x,y,\overline{\mu}),\mathbb{R}_+\big)+d\big(g(y,x,\overline{\mu}),\mathbb{R}_+\big)\\ &\quad = d\big(f(y,\overline{\mu})-f(x,\overline{\mu}),\mathbb{R}_+\big)+d\big(f(x,\overline{\mu})-f(y,\overline{\mu}),\mathbb{R}_+\big)\\ &\quad = \big|f(x,\overline{\mu})-f(y,\overline{\mu})\big| = n_2 d^{\delta_2}(x,y). \end{aligned}$$

So g satisfies condition (M_2), and (iii) in Theorem 3.1 is fulfilled. Therefore, it follows from Theorem 3.1 that (13) holds true with some $k_1, k_2 > 0$. □

4 The Hölder well-posedness of equilibrium problems

We will denote by $(\mathcal{EP})$ the family of problems $\{(\mathrm{EP})_{\lambda,\mu} : (\lambda,\mu) \in \Lambda \times M\}$ and extend the concept of Lipschitzian well-posedness for optimization problems introduced in Bednarczuk (2007) to equilibrium problems.

Definition 4.1 $(\mathcal{EP})$ is Hölder well-posed at $(\bar{\lambda}, \bar{\mu})$ if $\widetilde{S}(0, \bar{\lambda}, \bar{\mu})$ is a singleton and $\widetilde{S}$ is Hölder calm at $(0, \bar{\lambda}, \bar{\mu})$ on a neighborhood of $(0, \bar{\lambda}, \bar{\mu})$.

The next theorem gives a sufficient condition for the Hölder well-posedness of $(\mathcal{EP})$. It improves and modifies Theorem 3.1 in Anh et al. (2011).

Theorem 4.1 *Assume $S(\bar{\lambda}, \bar{\mu}) \neq \emptyset$ and the following conditions hold.*

- (i) *There exist neighborhoods $U(\overline{\lambda})$ of $\overline{\lambda}$ and $V(\overline{\mu})$ of $\overline{\mu}$ and positive numbers n_1, δ_1 and θ such that f is $(n_1.\delta_1)$-Hölder calm at $\overline{\mu}$ on $V(\bar{\mu})$, θ-uniformly over $K(U(\overline{\lambda}))$.*
- (ii) *There exist positive numbers n_2 and δ_2 such that, for all $x \in K(U(\overline{\lambda}))$ and $\mu \in V(\overline{\mu})$, the function $f(x, \cdot, \mu)$ is $(n_2.\delta_2)$-Hölder continuous on $K(U(\overline{\lambda}))$.*
- (iii) *$f(\cdot, \cdot, \overline{\mu})$ satisfies condition (M_2) on $K(U(\bar{\lambda}))$ with constants $h > 0$ and $\beta > \theta$.*
- (iv) *K is $(l.\alpha)$-Hölder calm at $\overline{\lambda}$ on $U(\bar{\lambda})$ with some positive l and α.*

Then $(\mathcal{EP})$ is Hölder well-posed at $(\bar{\lambda}, \bar{\mu})$.

Proof Take $N = [0, +\infty) \times M$. For $\eta = (\varepsilon, \mu)$, $\eta' = (\varepsilon', \mu') \in N$, consider a function d_N defined by

$$d_N\big(\eta,\eta'\big) = \max\big\{\big|\varepsilon-\varepsilon'\big|, d\big(\mu,\mu'\big)\big\}.$$

Then, (N, d_N) is a metric space. We define a function $g : X \times X \times N \to \mathbb{R}$ as follows

$$g(x,y,\eta) = f(x,y,\mu) + \varepsilon.$$

To prove the theorem, it suffices to check that g satisfies the conditions of Theorem 3.1.

Take any neighborhood W of 0 in [0; 1]. Then for all $\eta = (\varepsilon, \mu) \in W \times V(\bar{\mu})$, $\bar{\eta} = (0, \bar{\mu})$, and $x, y \in K(U(\overline{\lambda}))$, one has

$$\big|g(x,y,\eta) - g(x,y,\overline{\eta})\big| = \big|f(x,y,\mu) - f(x,y,\bar{\mu}) + \varepsilon\big|$$

$$\le \varepsilon + \left|f(x,y,\mu) - f(x,y,\bar{\mu})\right| \le \varepsilon + n_1 d^{\delta_1} d(\mu,\bar{\mu})$$
$$\le \varepsilon^{\delta_1} + n_1 d^{\delta_1}(\mu,\bar{\mu}) \le 2\max\{1,n_1\} d_N^{\delta_1}(\eta,\bar{\eta})$$

since $\varepsilon \in V \subseteq [0,1]$ and the Hölder order $\delta_1 \le 1$. So g is $(2\max\{1,n_1\}.\delta_1)$-Hölder calm at $\bar{\eta}$ on $W \times V(\bar{\mu})$ uniformly over $K(U(\overline{\lambda}))$.

We have at the same time

$$\left|g(x,y,\eta) - g(x,z,\eta)\right| = \left|f(x,y,\mu) - f(x,z,\mu)\right| \le n_2 d^{\delta_2}(y,z),$$

or g is $(n_2.\delta_2)$-Hölder continuous with respect to the second component on $K(U(\overline{\lambda}))$. Conditions (i) and (ii) of Theorem 3.1 are fulfilled.

We now check condition (iii) of Theorem 3.1. For all $x, y \in K(U(\overline{\lambda}))$, we get

$$d\big(g(x,y,\bar{\eta}),\mathbb{R}_+\big) + d\big(g(y,x,\bar{\eta}),\mathbb{R}_+\big)$$
$$= d\big(f(x,y,\bar{\mu}),\mathbb{R}_+\big) + d\big(f(y,x,\bar{\mu}),\mathbb{R}_+\big) \ge h d^{\beta}(x,y).$$

This means that g satisfies condition (iii) of Theorem 3.1 and we have all its hypotheses satisfied. Therefore, the mapping of solutions to $(\mathcal{EP})$ is both Hölder calm and single-valued at $(0,\bar{\eta})$ which combined with Definition 4.1 gives the conclusion of the theorem. □

5 Conclusion

Assuming Hölder calmness and Hölder continuity in Hausdorff distance, we have established the Hölder calm property of the solution mapping with respect to ρ. This obviously implies the Hölder calm property in Hausdorff distance. We have established a sufficient condition for the Hölder well-posedness of equilibrium problems. These may be extended to many other classes of problems.

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