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A note on subdifferentials of pointwise suprema

A.D. Ioffe

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Abstract The paper suggests a new approach to calculation of subdifferentials of suprema of convex functions without any qualification conditions which essentially relies on the Hirriart-Urruty–Phelps formula for subdifferentials of sums of convex l.s.c. functions (also supplied with a simple new proof). The approach in particular provides for a simpler way to (a certain generalization of) the most recent and so far most general formulas of Hantoute–López–Zalinescu and López–Volle.

Keywords Convex function · Subdifferential calculus · ε-subdifferential

Mathematics Subject Classification 49J52 · 52A41 · 90C25

1 Introduction

Let X be a locally convex space, let T be an arbitrary set, and let for any $t \in T$ a function φ_t on X be given. We are interested in calculating the subdifferential of the function

$$\varphi = \sup_{t \in T} \varphi_t.$$

This problem has a long history starting from the Dubovitzkii–Milyutin formula for the simplest case of the maximum of two continuous convex functions (Dubovitzkii and Milyutin 1965). We refer to Hantoute et al. (2008) and Tikhomirov (1987) for detailed description of the developments up to 2008. Suprema of infinite families of extended-real-valued convex functions (not necessarily continuous) seems

This paper is dedicated to Prof. Marco López on the occasion of his 60th birthday.

A.D. Ioffe (✉)
Department of Mathematics, Technion, Haifa 32000, Israel
e-mail: alexander.ioffe38@gmail.com

to be the most difficult for the analysis of subdifferentials among other major convexity preserving functional operations. This fact is underscored by the notable absence of such functions in the first study of the "unconditional" subdifferential calculus by Hiriart-Urruty and Phelps (1993) and the subsequent survey (Hiriart-Urruty et al. 1995) and also by the lengthy calculations in the most recent studies in Hantoute et al. (2008) and López and Volle (2010) where so far the most general results were obtained.

Two major points relating to the results obtained in Hantoute et al. (2008) and López and Volle (2010) should be emphasized: the first is that no restriction is imposed either on the nature of T or on the dependence of φ_t on t and the second is that only subdifferentials at the reference point are taken into account. Also mentioned should be the generality of the classes of chosen functions: the absence of the lower semicontinuity assumption in Hantoute et al. (2008) and even of the convexity assumption in López and Volle (2010). This, however, does not seem to be very essential: the condition that

$$\varphi^{**} = \sup_{t\in T} \varphi_t^{**} \tag{1}$$

imposed in both papers effectively reduces the situation to the case of convex lower semicontinuous and even affine functions (see the last section of this note).

The main purpose of this note is to demonstrate that there is a simpler way to derive the results of Hantoute et al. (2008) and López and Volle (2010) (including the description of sets of minimizers of second conjugates given in López and Volle 2010) with the help of the formula for the of sum of two convex lower semicontinuous functions established in Hiriart-Urruty and Phelps (1993):

$$\partial(f+g)(x) = \bigcap_{\varepsilon>0} \mathrm{cl}^*\big(\partial_\varepsilon f(x) + \partial_\varepsilon g(x)\big) \tag{2}$$

which itself admits a fairly simple proof (see e.g. the "second proof" of the formula in Hiriart-Urruty et al. (1995) and even a simpler proof in Sect. 3 below).

Following the proof of the Hiriart-Urruty–Phelps formula for the subdifferential of a sum in Sect. 3, we consider suprema of families of affine functions (Sect. 4), then suprema of families of convex lower semicontinuous functions (Sect. 5) and of families of still more general classes of functions in Sect. 6. The concluding result of the paper in Sect. 6 contains and somewhat generalizes the mentioned results of Hantoute et al. (2008) for convex but not necessarily lower semicontinuous functions, and of López and Volle (2010) for non-convex functions. It is to be again emphasized that the first step, involving affine functions, is the key element of the developments.

Despite its small size, the paper is basically self contained. We prove everything except the most elementary and standard facts of infinite dimensional convex analysis that can be found e.g. in Zalinescu (2002). And the proofs are indeed very simple. We even do not use separation in any of the proofs, and the only facts of convex analysis to which we refer and which do need separation are (a) that the subdifferential at zero of a function on X which is the support function of a certain non-empty set $Q \subset X^*$ coincides with the weak*-closure of the convex hull of Q and (b) that (for a convex l.s.c. f) $f'_\varepsilon(x;\cdot)$ is the support function of $\partial_\varepsilon f(x)$ (see e.g. Zalinescu 2002, Theorem 2.4.11). Most of the proofs seem to be new, with the exception of two nice observations which we borrow (along with their short proofs) from Hantoute et al. (2008) and López and Volle (2010).

2 Preliminaries

In what follows X is a locally convex topological vector space, X^* is its topological dual endowed with the weak*-topology and $\langle\cdot,\cdot\rangle$ is the canonical bilinear form on $X \times X^*$. By $\mathbb{R}_+$ we denote the collection of nonnegative real numbers.

We shall consider extended-real-valued functions f and denote by $\Gamma(X)$ the class of proper (everywhere greater than $-\infty$) convex lower semicontinuous functions on X. If f is a function on X, then f^* is its Fenchel conjugate and f^{**} is the second conjugate. As well known f^* is convex and weak*-lower semicontinuous and $f^{**} = f$ if and only if $f \in \Gamma(X)$. Given a convex function f, we as usual denote by $\operatorname{dom} f = \{x : f(x) < \infty\}$ its effective domain.

If $x \in \operatorname{dom} f$, then $\partial f(x) = \{x^* \in X^* : f(x+h) - f(x) \geq \langle x^*, h\rangle,\ \forall h \in X\}$ is the subdifferential of f at x. The set

$$\partial_\varepsilon f(x) = \big\{x^* \in X^* : f(x+h) - f(x) \geq \langle x^*, h\rangle - \varepsilon,\ \forall h \in X\big\},$$

where $\varepsilon > 0$, is called the ε-subdifferential of f at x. Both $\partial f(x)$ and $\partial_\varepsilon f(x)$ are convex weak*-closed sets and the intersection of $\partial_\varepsilon f(x)$ over $\varepsilon > 0$ coincides with $\partial f(x)$. If $f \in \Gamma(X)$ and $x \in \operatorname{dom} f$, then $\partial_\varepsilon f(x) \neq \emptyset$.

A function f is called sublinear if it is convex, homogeneous ($f(\lambda x) = \lambda f(x)$ if $\lambda > 0$) and $f(0) = 0$. If $Q \subset X^*$, then the function $s_Q(x) = \sup\{\langle x^*, h\rangle : x^* \in Q\}$ is called the support function of Q. This is a sublinear and lower semicontinuous function. As we have mentioned in the introduction the subdifferential of s_Q at zero coincides with the weak*-closure of $\operatorname{conv} Q$. We also mention that for any sublinear function the subdifferential at zero and the ε-subdifferential at zero coincide.

If $f \in \Gamma(X)$, then $\partial f(x) = \{x^* : f'(x;h) \geq \langle x^*, h\rangle,\ \forall h \in X\}$ and $\partial_\varepsilon f(x) = \{x^* : f'_\varepsilon(x;h) \geq \langle x^*, h\rangle,\ \forall h \in X\}$, where

$$f'(x;h) = \lim_{\lambda \to +0} \frac{f(x+\lambda h) - f(x)}{\lambda}; \qquad f'_\varepsilon(x;h) = \inf_{\lambda > 0} \frac{f(x+\lambda h) - f(x) + \varepsilon}{\lambda}.$$

Both functions (as functions of h) are sublinear. Moreover, if $f \in \Gamma(X)$, then $f'_\varepsilon(x;\cdot)$ is the support function of its subdifferential at zero (if $\varepsilon > 0$). We also note that the ratio $\lambda^{-1}(f(x+\lambda h) - f(x))$ is a non-decreasing function of λ on $(0,\infty)$.

Recall also the corresponding geometric concepts. If $Q \subset X$ and $x \in Q$, then $N(Q,x) = \{x^* : \langle x^*, u - x\rangle \leq 0,\ \forall u \in Q\}$ is the normal cone to Q at x and $N_\varepsilon(Q,x) = \{x^* : \langle x^*, u - x\rangle \leq \varepsilon,\ \forall u \in Q\}$ is the collection of ε-normals. If Q is itself a cone, then $N_\varepsilon(Q,0) = N(Q,0)$ for all ε. The normal cone to Q at x coincides with the subdifferential at x of the indicator of Q which is a function i_Q equal to zero on Q and infinity outside of Q. The same relation connects the sets of ε-normals and ε-subdifferentials.

3 Subdifferential of a sum

The starting point of our discussions will be the formula for the subdifferential of the sum of finitely many convex lower semicontinuous functions.

Theorem 3.1 (Hiriart-Urruty and Phelps 1993) *Let* $f_i \in \Gamma(X)$, $i = 1, \dots, k$, *set* $f = f_1 + \dots + f_k$, *and let* $\overline{x} \in \operatorname{dom} f$. *Then*

$$\partial f(\overline{x}) = \bigcap_{\varepsilon > 0} \operatorname{cl}^*\big(\partial_\varepsilon f_1(\overline{x}) + \dots + \partial_\varepsilon f_k(\overline{x})\big).$$

Proof The inclusion $\supset$ is a trivial consequence of definitions and the fact that $\partial f(x)$ coincides with the intersection of $\partial_\varepsilon f(x)$. Indeed, if $x_i^* \in \partial_\varepsilon f_i(\overline{x})$, then by the definition $f(\overline{x}+h) - f(\overline{x}) = \sum (f_i(\overline{x}+h) - f_i(\overline{x})) \geq \langle \sum x_i^*, h\rangle - k\varepsilon$, so that $x_1^* + \dots + x_k^* \in \partial_{k\varepsilon} f(\overline{x})$. Thus (as $\partial_\varepsilon f(x)$ is weak*-closed),

$$\partial_\varepsilon f_1(\overline{x}) + \dots + \partial_\varepsilon f_k(\overline{x}) \subset \operatorname{cl}^*\big(\partial_\varepsilon f_1(\overline{x}) + \dots + \partial_\varepsilon f_k(\overline{x})\big) \subset \partial_{k\varepsilon} f(\overline{x}),$$

whence the inclusion.

To prove the opposite inclusion, note first that $\lim_{\lambda \to +0} \lambda^{-1}(f(x + \lambda h) - f(x) + \varepsilon) = \infty$ if both x and $x + h$ are in $\operatorname{dom} f$ and $\varepsilon > 0$. Applying this for every f_i we conclude that for any $h \in \operatorname{dom} f - \overline{x} = \bigcap_i (\operatorname{dom} f_i - \overline{x})$ and any $\varepsilon > 0$ there is a $\bar{\lambda} > 0$ such that

$$f'_{i\varepsilon}(\overline{x}; h) = \inf_{\lambda > 0} \frac{f_i(\overline{x}+\lambda h) - f_i(\overline{x}) + \varepsilon}{\lambda} = \inf_{\lambda > \bar{\lambda}} \frac{f_i(\overline{x}+\lambda h) - f_i(\overline{x}) + \varepsilon}{\lambda}.$$

Now let $x^* \in \partial_\varepsilon f(\overline{x})$. Take a $\delta > 0$ and choose $\lambda_i = \lambda_i(\delta) > \bar{\lambda}$ to make sure that for any i we would have $f'_{i\varepsilon}(\overline{x}; h) + \delta \geq \lambda_i^{-1}(f_i(\overline{x} + \lambda_i h) - f_i(\overline{x}) + \varepsilon)$. Then, taking a $\lambda < \bar{\lambda}$, we get

$$\begin{aligned} f'(\overline{x}; h) &\leq \frac{f(\overline{x}+\lambda h) - f(\overline{x})}{\lambda} = \sum_i \frac{f_i(\overline{x}+\lambda h) - f_i(\overline{x})}{\lambda} \\ &\leq \sum_i \frac{f_i(\overline{x}+\lambda_i h) - f_i(\overline{x})}{\lambda_i} \leq \sum_i \frac{f_i(\overline{x}+\lambda_i h) - f_i(\overline{x}) + \varepsilon}{\lambda_i} \\ &\leq \sum_i f'_{i\varepsilon}(\overline{x}; h) + k\delta. \end{aligned}$$

Thus $f'(\overline{x}; h) \leq \sum_i f'_{i\varepsilon}(\overline{x}; h)$ as δ can be chosen arbitrarily small. As each of $f'_{i\varepsilon}(\overline{x}; \cdot)$ is the support function of its subdifferential at zero, their sum is the support function of the sum of the subdifferentials. So we have

$$\partial f(\overline{x}) \subset \partial\left(\sum_{i=1}^k f'_{i\varepsilon}(\overline{x}; \cdot)\right)(0) = \operatorname{cl}^*\left(\sum_{i=1}^k \partial f'_{i\varepsilon}(\overline{x}; \cdot)(0)\right) = \operatorname{cl}^*\left(\sum_{i=1}^k \partial_\varepsilon f_i(\overline{x})\right).$$

This completes the proof. □

4 The case of affine functions

Throughout the paper we fix an $\overline{x} \in \operatorname{dom} \varphi$ and for any $\varepsilon > 0$ set

$$T_\varepsilon = \{t \in T : \varphi_t(\overline{x}) > \varphi(\overline{x}) - \varepsilon\}.$$

We begin to study subdifferentials of suprema of families (φ_t) with the simplest case when all functions φ_t are affine. Namely we assume that there are $x_t^* \in X^*$ and $a_t \in \mathbb{R}$ such that $\varphi_t(x) = \langle x_t^*, x\rangle + a_t$. It turns out that the most essential features of the result are already present in this simple situation and can be translated to more general classes of functions without serious problems.

For any $t \in T$ we define $f_t(x) = \langle x_t^*, x - \overline{x}\rangle + \varphi(\overline{x})$, so that $f_t(\overline{x}) = \varphi(\overline{x})$ for all t, and set

$$f^\varepsilon(x) = \sup_{t \in T_\varepsilon} f_t(x), \qquad S = \overline{x} + \mathbb{R}_+(\operatorname{dom} \varphi - \overline{x}).$$

We have for any t and any x

$$f_t(x) - \varphi_t(x) = \langle x_t^*, x - \overline{x}\rangle + \varphi(\overline{x}) - \big(\langle x_t^*, x\rangle + a_t\big) = \varphi(\overline{x}) - \varphi_t(\overline{x}) \ge 0. \tag{3}$$

The following lemma is crucial for the further discussions.

Lemma 4.1 *Set $g^\varepsilon = f^\varepsilon + i_S$. Then $\partial\varphi(\overline{x}) \subset \partial g^\varepsilon(\overline{x})$.*

Proof Without loss of generality we may assume that $\overline{x} = 0$ and $\varphi(\overline{x}) = 0$. Then $f_t(x) = \langle x_t^*, x\rangle$ so that f^ε becomes a sublinear function and so is g^ε as S is a cone. Let $x^* \in \partial\varphi(0)$, that is $\varphi'(0; x) \ge \langle x^*, x\rangle$ for all x. If $(f^\varepsilon)(x) \ge \varphi'(0; x)$ for some x, then

$$g^\varepsilon(x) = f^\varepsilon(x) \ge \langle x^*, x\rangle.$$

If $\varphi'(0; x) = \infty$, then $x \notin S$ and so $g^\varepsilon(x) = \infty$ as well. Thus again $g^\varepsilon(x) \ge \langle x^*, x\rangle$.

Assume finally that $\infty > \varphi'(0; x) > f^\varepsilon(x)$. This means that there is a t such that $\varphi_t(x) > f^\varepsilon(x)$. Clearly, $t \notin T_\varepsilon$ since $\varphi_t(x) \le f^\varepsilon(x)$ for $t \in T_\varepsilon$ as follows from the definition of f^ε and (3). For $t \notin T_\varepsilon$ we have

$$\varphi_t(\lambda x) \le \lambda\varphi_t(x) + (1 - \lambda)\varphi_t(0) \le \lambda\varphi_t(x) - (1 - \lambda)\varepsilon.$$

Thus, the inequality $\lambda f^\varepsilon(x) = f^\varepsilon(\lambda x) < \varphi_t(\lambda x)$ may hold only if

$$\lambda > \bar{\lambda} = \frac{\varepsilon}{\varphi(x) - f^\varepsilon(x) + \varepsilon} > 0$$

and for $\lambda \le \bar{\lambda}$ we again have $f^\varepsilon(\lambda x) \ge \varphi(\lambda x)$. This means that $f^\varepsilon(x) \ge \varphi'(0; x)$ contrary to our assumption. Thus $g^\varepsilon(x) \ge \varphi'(0; x)$ for all x. □

We also need the following.

Lemma 4.2 (Hantoute et al. 2008, Lemma 1) *If f is a convex function on X and $Q \subset X$ is a convex set with* $(\operatorname{ri} Q) \cap \operatorname{dom} f \neq \emptyset$, *then*

$$\inf_{Q} f(x) = \inf_{\operatorname{cl} Q} f(x).$$

Proof Let $x \in (\operatorname{cl} Q) \cap \operatorname{dom} f$, and let $w \in (\operatorname{ri} Q) \cap \operatorname{dom} f$. Then $(1-\lambda)x + \lambda w \in Q \cap \operatorname{dom} f$ for $\lambda \in (0,1)$ and $f(x) \geq \lim_{\lambda \to +0} f((1-\lambda)x + \lambda w)$. □

Combining these two lemmas with Theorem 3.1 we arrive at

Theorem 4.3 *Suppose all functions φ_t are affine: $\varphi_t(x) = \langle x_t^*, x\rangle + a_t$ for some $x_t^* \in X^*$ and $a_t \in \mathbb{R}$. Let further a convex set $Q \subset S$ containing $\overline{x}$ be given. Then the following two statements hold true.*

(a) *if Q is closed, then for any $\varepsilon > 0$*

$$\partial\varphi(\overline{x}) \subset \operatorname{cl}^*\big(\operatorname{conv}\{x_t^* : t \in T_\varepsilon\} + N_\varepsilon(Q, \overline{x})\big);$$

in particular, if the intersection of $\operatorname{dom}\varphi$ *with a closed neighborhood of $\overline{x}$ is closed, then*

$$\partial\varphi(\overline{x}) \subset \operatorname{cl}^*\big(\operatorname{conv}\{x_t^* : t \in T_\varepsilon\} + N_\varepsilon(\operatorname{dom}\varphi, \overline{x})\big).$$

(b) *If either* $\operatorname{ri} Q \neq \emptyset$ *or $R_+(Q - \overline{x})$ is a closed set, then for any $\varepsilon > 0$*

$$\partial\varphi(\overline{x}) \subset \operatorname{cl}^*\big(\operatorname{conv}\{x_t^* : t \in T_\varepsilon\} + N(Q, \overline{x})\big),$$

in particular if either $\operatorname{ri}(\operatorname{dom}\varphi) \neq \emptyset$ *or* $R_+(\operatorname{dom}\varphi - \overline{x})$ *is a closed set, then*

$$\partial\varphi(\overline{x}) \subset \operatorname{cl}^*\big(\operatorname{conv}\{x_t^* : t \in T_\varepsilon\} + N(\operatorname{dom}\varphi, \overline{x})\big).$$

Proof As in the proof of Lemma 4.1 we may assume that $\overline{x} = 0$ and $\varphi(\overline{x}) = 0$. Then $x^* \in \partial f^\varepsilon(0)$ means that $\sup_{t \in T_\varepsilon}\langle x_t^*, x\rangle \geq \langle x^*, x\rangle$ for all x, which is the same as $x^* \in \operatorname{cl}^*(\operatorname{conv}\{x_t^* : t \in T_\varepsilon\})$, and as f^ε is a sublinear function, $\partial_\delta f^\varepsilon(0) = \partial f^\varepsilon(0)$ for all $\delta > 0$.

Furthermore, as $Q \subset S$ by the assumption, $f^\varepsilon(x) + i_Q(x) \geq g^\varepsilon(x)$ for all x. By Lemma 4.1 this implies that $\partial\varphi(0) \subset \partial(f^\varepsilon + i_Q)(0)$. If Q is closed, we get (a) from Theorem 3.1.

Likewise, if $\overline{x} + R_+(Q - \overline{x}) \subset S$ and $R_+(Q - \overline{x})$ is a closed set, we can replace Q by $\overline{x} + R_+(Q - \overline{x})$ in the above argument. As $R_+(Q - \overline{x})$ is a cone, then for any $\varepsilon > 0$

$$N_\varepsilon\big(\overline{x} + R_+(Q - \overline{x}), \overline{x}\big) = N_\varepsilon\big(R_+(Q - \overline{x}), 0\big) = N\big(R_+(Q - \overline{x}), 0\big) = N(Q, \overline{x}).$$

Suppose now that $\operatorname{ri} Q \neq \emptyset$. Then the same is true for the cone generated by Q: $\operatorname{cone} Q = \mathbb{R}_+ Q$. Let $x^* \in \partial(f^\varepsilon + i_Q)(0) = \partial(f^\varepsilon + i_{\operatorname{cone} Q})(0)$. Then by Lemma 4.2 the lower bounds of $f^\varepsilon(x) - \langle x^*, x\rangle$ on $\operatorname{cone} Q$ and $\operatorname{cl}(\operatorname{cone} Q)$ coincide which means that $x^* \in \partial(f^\varepsilon + i_{\operatorname{cl}(\operatorname{cone} Q)})(0)$. Applying again Theorem 3.1, we get (b) since $N_\varepsilon(\operatorname{cone} Q, 0) = N(\operatorname{cone} Q, 0) = N(Q, 0)$. □

5 The case of convex l.s.c. functions

Here we turn to a more general case of arbitrary $\varphi_t \in \Gamma(X)$. The reduction to the case of the supremum of a family of affine functions is simple. Let $\mathcal{T}$ stand for the collection of triples $\tau = (t, x^*, a) \in T \times X^* \times \mathbb{R}$ such that $\langle x^*, x\rangle + a \le \varphi_t(x)$ for all x. For every $\tau = (t, x^*, a)$ we set $\psi_\tau(x) = \langle x^*, x\rangle + a$. Then $\sup_{\tau\in\mathcal{T}} \psi_\tau(x) = \varphi(x)$ as every φ_t is the supremum of the collection of its affine minorants. Set $\mathcal{T}_\varepsilon = \{\tau \in \mathcal{T} : \psi_\tau(\overline{x}) \ge \varphi(\overline{x}) - \varepsilon\}$. Then $\tau \in \mathcal{T}_\varepsilon$ means that $\langle x^*, \overline{x}\rangle + a > \varphi(\overline{x}) - \varepsilon \ge \varphi_t(\overline{x}) - \varepsilon$. Together with the fact that $\varphi_t(x) \ge \langle x^*, x\rangle + a$ for all x, this shows that $x^* \in \partial_\varepsilon\varphi_t(\overline{x})$ and, as an immediate corollary of Theorem 4.3 (applied to the family $(\psi_\tau)_{\tau\in\mathcal{T}}$ of affine functions), we get

Theorem 5.1 *Let $\varphi_t \in \Gamma(X)$ for all $t \in T$. Let $Q \subset \overline{x} + R_+(\mathrm{dom}\,\varphi - \overline{x})$ be a convex set containing $\overline{x}$. Then*

(a) *if Q is closed, then for any $\varepsilon > 0$*

$$\partial\varphi(\overline{x}) \subset \mathrm{cl}^*\left(\mathrm{conv} \bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N_\varepsilon(Q, \overline{x})\right),$$

in particular, if the intersection of $\mathrm{dom}\,\varphi$ *with a neighborhood of $\overline{x}$ is closed,*

$$\partial\varphi(\overline{x}) \subset \mathrm{cl}^*\left(\mathrm{conv} \bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N_\varepsilon(\mathrm{dom}\,\varphi, \overline{x})\right);$$

(b) *if either* $\mathrm{ri}\, Q \neq \emptyset$ *or $R_+(Q - \overline{x})$ is a closed set, then for any $\varepsilon > 0$*

$$\partial\varphi(\overline{x}) \subset \mathrm{cl}^*\left(\mathrm{conv} \bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N(Q, \overline{x})\right),$$

in particular, if $\mathrm{ri}(\mathrm{dom}\,\varphi) \neq \emptyset$ *or* $R_+(\mathrm{dom}\,\varphi - \overline{x})$ *is a closed set, then*

$$\partial\varphi(\overline{x}) \subset \mathrm{cl}^*\left(\mathrm{conv} \bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N(\mathrm{dom}\,\varphi, \overline{x})\right).$$

6 The general case

The passage to functions which are not elements of $\Gamma(X)$ is almost equally simple. But to state the final result containing precise formulas for the subdifferential we need the following lemma.

Lemma 6.1 *Let φ_t be arbitrary functions on X. Then*

$$\bigcap_{\substack{\varepsilon>0\\ x\in\mathrm{dom}\,\varphi}} \mathrm{cl}^*\left(\mathrm{conv}\left(\bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x})\right) + (x - \overline{x})^-_\varepsilon\right) \subset \partial\varphi(\overline{x}).$$

Here $(x - \overline{x})^-_\varepsilon = \{x^* : \langle x^*, x - \overline{x}\rangle \le \varepsilon\}$.

Proof The proof actually repeats almost word for word the proof of the first part of Theorem 4.1 in López and Volle (2010). The first and elementary fact to be mentioned is that $\partial_\varepsilon\varphi_t(\overline{x}) \subset \partial_{2\varepsilon}\varphi(\overline{x})$ if $t \in T_\varepsilon$. If now $x \in \operatorname{dom}\varphi$, $u^* \in \partial_{2\varepsilon}\varphi(\overline{x})$ and $v^* \in (x - \overline{x})_\varepsilon^-$, then $\langle u^* + v^*, x - \overline{x}\rangle \le \varphi(x) - \varphi(\overline{x}) + 3\varepsilon$ or equivalently

$$u^* + v^* \in P_\varepsilon(x) = \big\{w^* : \langle w^*, x\rangle - \varphi(x) \le \langle w^*, \overline{x}\rangle - \varphi(\overline{x}) + 3\varepsilon\big\}.$$

Thus for any $t \in T_\varepsilon$

$$\partial_\varepsilon\varphi_t + (x - \overline{x})_\varepsilon^- \subset \partial_{2\varepsilon}\varphi(\overline{x}) + (x - \overline{x})_\varepsilon^- \subset P_\varepsilon(x).$$

We have

$$\begin{aligned}\bigcap_{x\in\operatorname{dom}\varphi} P_\varepsilon(x) &= \Big\{w^* : \sup_x\big(\langle w^*, x\rangle - \varphi(x)\big) \le \langle w^*, \overline{x}\rangle - \varphi(\overline{x}) + 3\varepsilon\Big\}\\ &= \big\{w^* : \varphi^*(w^*) \le \langle w^*, \overline{x}\rangle - \varphi(\overline{x}) + 3\varepsilon\big\} = \partial_{3\varepsilon}\varphi(\overline{x})\end{aligned}$$

and therefore (as both P_ε and $\partial_{3\varepsilon}\varphi(\overline{x})$ are weak*-closed and convex)

$$\begin{aligned}&\bigcap_{x\in\operatorname{dom}\varphi} \operatorname{cl}^*\Bigg(\operatorname{conv}\Bigg(\bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + (x - \overline{x})_\varepsilon^-\Bigg)\Bigg)\\ &\quad\subset \bigcap_{x\in\operatorname{dom}\varphi} \operatorname{cl}^*\big(\operatorname{conv}\big(\partial_{2\varepsilon}\varphi(\overline{x}) + (x - \overline{x})_\varepsilon^-\big)\big) \subset \partial_{3\varepsilon}\varphi(\overline{x}).\end{aligned}$$

Taking the intersection over $\varepsilon > 0$, we get the result. □

Now we are ready to state and prove the main result of the paper.

Theorem 6.2 *Let* $\{\varphi_t,\ t \in T\}$ *be a collection of functions on* X *satisfying* (1). *Let* $\overline{x} \in \operatorname{dom}\varphi$, *and let* $\{Q_\alpha,\ \alpha \in \mathcal{A}\}$ (*where* $\mathcal{A}$ *is a certain index set*) *be a family of convex subsets of* $\operatorname{dom}\varphi^{**}$ *such that* $\overline{x} \in Q_\alpha$ *for every* $\alpha \in \mathcal{A}$ *and the union of* Q_α *contains* $\operatorname{dom}\varphi$.

(a) *If all* Q_α *are closed, then*

$$\partial\varphi(\overline{x}) = \bigcap_{\substack{\varepsilon>0\\ \alpha\in\mathcal{A}}} \operatorname{cl}^*\Bigg(\operatorname{conv}\Bigg(\bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N_\varepsilon(Q_\alpha, \overline{x})\Bigg)\Bigg).$$

In particular, if the intersection of $\operatorname{dom}\varphi^{**}$ *with a closed neighborhood of* $\overline{x}$ *is closed, then*

$$\partial\varphi(\overline{x}) = \bigcap_{\varepsilon>0} \operatorname{cl}^*\Bigg(\operatorname{conv}\Bigg(\bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N_\varepsilon(\operatorname{dom}\varphi, \overline{x})\Bigg)\Bigg).$$

(b) *If either every* Q_α *has nonempty relative interior or* $\mathbb{R}_+(Q_\alpha - \overline{x})$ *is a closed set for every* α, *then*

$$\partial\varphi(\overline{x}) = \bigcap_{\substack{\varepsilon>0\\ \alpha\in\mathcal{A}}} \mathrm{cl}^*\left(\mathrm{conv}\left(\bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N(Q_\alpha, \overline{x})\right)\right).$$

In particular if either $\mathrm{ri}(\mathrm{dom}\,\varphi^{**}) \neq \emptyset$ *or* $R_+(\mathrm{dom}\,\varphi^{**} - \overline{x})$ *is a closed set*, *then*

$$\partial\varphi(\overline{x}) = \bigcap_{\varepsilon>0} \mathrm{cl}^*\left(\mathrm{conv}\left(\bigcup_{t\in T_\varepsilon} \partial_\varepsilon\varphi_t(\overline{x}) + N(\mathrm{dom}\,\varphi, \overline{x})\right)\right).$$

Proof The inclusions $\supset$ follow from the last lemma because $N_\varepsilon(Q_\alpha \cap \mathrm{dom}\,\varphi, \overline{x}) \subset (x - \overline{x})^-_\varepsilon$ if $x \in Q_\alpha \cap \mathrm{dom}\,\varphi$ and, as follows from the assumption, we can find a suitable α for every $x \in \mathrm{dom}\,\varphi$.

To prove the opposite inclusion we fix an $x^* \in \partial\varphi(\overline{x})$, an $\varepsilon > 0$ and an $\alpha \in \mathcal{A}$. As $\partial\varphi(\overline{x}) \neq \emptyset$, we have

$$\varphi(\overline{x}) = \varphi^{**}(\overline{x}), \qquad \partial\varphi(\overline{x}) = \partial\varphi^{**}(\overline{x}). \tag{4}$$

Set further $T_\varepsilon^{**} = \{t \in T : \varphi_t^{**}(\overline{x}) \geq \varphi^{**}(\overline{x}) - \varepsilon\}$. Then for $t \in T_\varepsilon^{**}$

$$\varphi_t(\overline{x}) \geq \varphi_t^{**}(\overline{x}) \geq \varphi^{**}(\overline{x}) - \varepsilon = \varphi(\overline{x}) - \varepsilon,$$

that is

$$T_\varepsilon^{**} \subset T_\varepsilon. \tag{5}$$

If now $t \in T_\varepsilon^{**}$ and $u^* \in \partial_\varepsilon\varphi_t^{**}(\overline{x})$, then for any x we have

$$\varphi_t(x) - \varphi_t(\overline{x}) \geq \varphi_t^{**}(x) - \varphi(\overline{x}) \geq \varphi_t^{**}(x) - \varphi_t^{**}(\overline{x}) - \varepsilon \geq \langle u^*, x - \overline{x}\rangle - 2\varepsilon,$$

that is for $t \in T_\varepsilon^{**}$

$$\partial_\varepsilon\varphi_t^{**}(\overline{x}) \subset \partial_{2\varepsilon}\varphi_t(\overline{x}). \tag{6}$$

Since (1) holds, so does Theorem 5.1 with φ, φ_t and Q replaced, respectively, by φ^{**}, φ_t^{**} and Q_α. Along with (4)–(6) this gives the desired result.

To prove the second part of (a) we can take $\mathcal{A} = \mathrm{dom}\,\varphi$ and for $x \in \mathrm{dom}\,\varphi$ set $Q_x = cl(\mathrm{conv}(V \cap \{x\}))$. where V is the neighborhood of $\overline{x}$ mentioned in the statement. To prove the second part of (b) we take $Q_\alpha = \mathrm{dom}\,\varphi^{**}$ with $\mathcal{A}$ being a one point set. □

Remark 6.3 It is possible to make some changes in the chosen sets of normals in both parts of the theorem that will not affect the result. For instance, we can write $N_\varepsilon(Q_\alpha \cap \mathrm{dom}\,\varphi, \overline{x})$ instead of $N_\varepsilon(Q_\alpha, \overline{x})$ (and the same without ε in (b)), or else we can take $N_\delta(Q_\alpha, \overline{x})$ with δ independent of ε and add the intersection over $\delta > 0$. This, however, will not allow one to eliminate δ at all and to write the real normal cone in (a).

We shall next show how the main results of Hantoute et al. (2008) and López and Volle (2010) (for functions of $\Gamma(X)$) can be obtained from Theorem 5.1.

Theorem 6.4 (López and Volle 2010, Theorem 4.1) *Assume* (1). *Then*

$$\partial\varphi(\overline{x}) = \bigcap_{\substack{\varepsilon>0\\ x\in\operatorname{dom}\varphi}} \operatorname{cl}^*\Big(\operatorname{conv}\bigcup_{t\in T_\varepsilon}\partial_\varepsilon\varphi_t(\overline{x}) + (x-\overline{x})^-\Big). \tag{7}$$

Proof Take $\mathcal{A} = \operatorname{dom}\varphi$ and let $Q_x,\ x\in\mathcal{A}$, be the line segment connecting x and $\overline{x}$. Then $\mathbb{R}_+(Q_x-\overline{x})$ is clearly a closed ray and $Q_x\subset\overline{x}+\mathbb{R}_+(Q_x-\overline{x})$. Apply part (b) of Theorem 6.2 with $\mathcal{A}=\operatorname{dom}\varphi$. □

Theorem 6.5 (Hantoute et al. 2008, Theorem 4) *Assume that φ_t are convex and* (1) *holds. Let $\mathcal{F}$ stand for the collection of finite dimensional subspaces of X. Then for any $\varepsilon>0$*

$$\partial\varphi(\overline{x}) = \bigcap_{\substack{\varepsilon>0\\ L\in\mathcal{F}}} \operatorname{cl}^*\Big(\operatorname{conv}\bigcup_{t\in T_\varepsilon}\partial_\varepsilon\varphi_t(\overline{x}) + N\big((L+\overline{x})\cap\operatorname{dom}\varphi,\overline{x}\big)\Big). \tag{8}$$

Proof Take an $L\in\mathcal{F}$ and set $Q_L=(\overline{x}+L)\cap\operatorname{dom}\varphi$. Then the relative interior of Q is nonempty. Apply part (b) of Theorem 6.2 with $\mathcal{A}=\mathcal{F}$. □

Observe that in López and Volle (2010) Theorem 6.4 was obtained as a consequence of a certain result about ε-minimizers of the second conjugate of a function. We conclude the paper by showing that this (actually main) result of López and Volle (2010) is in turn a direct consequence of Theorem 6.4 and actually of the simplest version of the theorem, corresponding to affine φ_t.

Theorem 6.6 (López and Volle 2010, Theorem 3.3) *Let Y be a locally convex topological space, and let f be a function on Y such that f^* is a proper function. Then*

$$\operatorname{argmin} f^{**} = \bigcap_{\substack{\varepsilon>0\\ y^*\in\operatorname{dom} f^*}} \operatorname{cl}^*\big(\pi(\varepsilon-\operatorname{argmin} f)+(y^*)^-\big).$$

Here f^{**} is a function on X^{**}, the conjugate to f^*, and π stands for the natural embedding $Y\to Y^{**}$.

Proof We shall apply Theorem 6.4 with $X=Y^*$, $T=\operatorname{dom} f$. Set $x^*_y=\pi(y)$, $a_y=-f(y)$, $\varphi_y(x)=\langle x,y\rangle+a_y$ and $\overline{x}=0$. Then $\varphi(x)=f^*(x)$, and therefore $\varphi^*(x^*)=f^{**}(x^*)$, and $T_\varepsilon=\{y:\ a_y\ge\varphi(0)-\varepsilon\}=\{y: f(y)+f^*(0)\le\varepsilon\}$.

We have $\operatorname{argmin}\varphi^*=\partial\varphi(0)$. Then in view of the said, the reference to Theorem 6.4 completes the proof. □

Remark 6.7 Additional information provided by Theorem main (compare to Theorems 6.4 and 6.5) is that there is a huge variety of sets that can be used to calculate the subdifferential, not just line segments or finite dimensional sections. It is not a priori clear whether this information is particularly useful but in principle it may serve to decrease the collection of sets involved in the intersection.

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