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A minimum dimensional class of simple games

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Abstract This paper provides several extensions of the notion of dimension of a simple game and proves the existence of a minimum subclass of weighted games with the property that every simple game can be expressed as their intersection. Some further generalizations lead to the new concept of codimension which is obtained by considering the union instead of the intersection as the basic operation.

Keywords Simple games · Hypergraphs · Boolean algebra · Dimension · Codimension

Mathematics Subject Classification (2000) 05C65 · 91A12 · 94C10

1 Introduction

The first goal of this paper is to extend the concept of dimension, originally defined for weighted games, to other subclasses of simple games. It then becomes of interest to study whether there exists a "smallest" subclass of simple games for which the

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associated notion of dimension exists. We will prove that, in fact, such a smallest class exists. A natural question to suggest itself is whether it is feasible to get an analogous concept of dimension, namely codimension, by using the *union* instead of the intersection as the basic operation. Indeed, we will see that codimension is well defined for several subclasses of simple games. Moreover, we will prove that there exists a smallest subclass of simple games for which the associated notion of codimension exists.

The paper is organized as follows. After some preliminaries we extend in Sect. 2 the concept of dimension to several subclasses of simple games and introduce the notion of codimension of a simple game. Section 3 establishes some relationships between dimension and codimension. Section 4 proves the existence of subclasses of simple games being (co)dimensionally minimum.

A *simple* game is a pair (N, W) where $N = \{1, 2, \dots, n\}$ and W is an arbitrary collection of subsets of N. It is *monotonic* if, moreover, from $S \in W$ and $S \subseteq T$ it follows that $T \in W$. From now on we only deal with monotonic simple games with the two additional assumptions $\emptyset \notin W$ and $N \in W$. Simple games can be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo. Members in N are called *players* or *voters*, and the sets in W are called *winning coalitions*. A subset of N that is not in W is called a *losing coalition*, and the collection of losing coalitions is denoted by L. If each proper subcoalition of a winning coalition is losing, this winning coalition is called *minimal*. The set of minimal winning coalitions is denoted by W^m. If each proper supra-coalition of a losing coalition is winning, this losing coalition is called *maximal*. The set of maximal losing coalitions is denoted by L^M. A player $i \in N$ is called *null* in (N, W) if $i \notin S$ for every $S \in W^m$. A player $i \in N$ is a *winner* if $\{i\} \in W$. Hereafter $|D|$ denotes the cardinality of a finite set D.

A simple game (N, W) is a *weighted game* if it admits a representation by means of the $n + 1$ nonnegative real numbers $[q; w_1, \dots, w_n]$ such that $S \in W$ *iff* $w(S) \geq q$, where $w(S) = \sum_{i \in S} w_i$ for each coalition $S \subseteq N$. The number q is called the *quota* of the game, and w_i the *weight* of player i.

As far as we know, a single notion of dimension has been considered in the theory of simple games. Following Taylor and Zwicker (1999), a simple game W is said to be of *dimension k* if it can be represented as the intersection of k weighted games but cannot be represented as the intersection of $k - 1$ weighted games. Equivalently, the *dimension* of W is the least k such that there exist weighted simple games $W_1, \dots, W_k$ such that $W = W_1 \cap \dots \cap W_k$.

Probably the interest for this concept of dimension has been motivated by the observation that most naturally occurring voting systems in use are of small dimension. Indeed, the Electoral College of the United States is a weighted game and hence has dimension 1; also the United Nations Security Council (regarded as a simple game) has dimension 1. Interesting examples of dimension 2 are the United States federal system, the procedure for amending the Canadian Constitution (see Taylor 1995, Kilgour 1983, and Levesque and Kilgour 1984), and the current European Union Council of Ministers for motions not coming from the European Commission. There exist several real-world examples in use today where laws are passed by a method known as "count and account," which are examples of dimension 1 or 2; see some examples in Peleg (1992) and Carreras and Freixas (2004). Two conspicuous examples of dimension 3 can be found in the Union Council of Ministers after the enlargement to 27 members agreed in Nice in December 2000 (Freixas 2004). We do not know about the existence of some real-world voting system with dimension greater than 3.

Whenever the dimension of a simple game is not a huge number, an efficient decomposition of this simple game as the intersection of weighted games can be used to compute power indices with generating functions as was illustrated in Algaba et al. (2001) for the Banzhaf power index for the European Union after the enlargement to 27 countries. However, in Deĭneko and Woeginger (2006) it is proven that computing the dimension of a simple game is an NP-hard problem and hence computationally intractable.

Theorem 1.1 (i) *Every simple game has a finite dimension* (Taylor and Zwicker 1999, Theorem 1.7.2).

(ii) *For every* $m \geq 1$, *there is a simple game* W *of dimension* m (Taylor and Zwicker 1999, Theorem 1.7.4).

The proof in Taylor and Zwicker (1999) of Theorem 1.1(i) uses the following argument. For a simple game W on N, let $L^M = \{T_1, \ldots, T_k\}$. Clearly, $W = W_1 \cap \cdots \cap W_k$, where each W_i admits the weighted representation $[1; w_1^i, \ldots, w_n^i]$ with

$$w_j^i = \begin{cases} 0 & \text{if } j \in T_i; \\ 1 & \text{otherwise.} \end{cases}$$

Hence, W is the intersection of $|L^M|$ weighted games, and, therefore it has a dimension. As a consequence of Theorem 1.1(i), the dimension of a simple game is bounded above by the number of maximal losing coalitions. The game W of dimension $m \geq 1$ defined on a set $M = \{1, \ldots, 2m\}$ considered in Theorem 1.1(ii) is given by the collection of sets: $S \in W$ *iff* $S \cap \{2i-1, 2i\} \neq \emptyset$ for all $i = 1, \ldots, m$.

2 Dimension and codimension

Let $\mathcal{S}$ be the class of simple games on N. With every simple game $W \in \mathcal{S}$, we can associate the *dual* game $W^* = \{S \subseteq N : N \setminus S \notin W\}$ whose elements are the *blocking* coalitions in W, i.e., those that can prevent an issue from being passed. It is straightforward to check that the dual game is idempotent, i.e., $W^{**} = W$, and satisfies the two Morgan's laws (see, e.g., Proposition 1.4.3 in Taylor and Zwicker 1999):

$$\begin{aligned} (W_1 \cup W_2)^* &= W_1^* \cap W_2^*, \\ (W_1 \cap W_2)^* &= W_1^* \cup W_2^*. \end{aligned} \tag{1}$$

Let $\mathcal{C} \subseteq \mathcal{S}$ be a subset of simple games on N, and let $\mathcal{C}^* = \{W^* \in \mathcal{S} : W \in \mathcal{C}\}$. We say that $\mathcal{C}$ is *closed* under duality if $\mathcal{C} = \mathcal{C}^*$. Particularly, $\mathcal{S}$ is itself closed under duality. An instance of a proper subset of $\mathcal{S}$ which is closed under duality is the class of complete games (i.e., simple games for which the *desirability relation* is total) briefly denoted by $\mathcal{L}$. A proper subset of $\mathcal{L}$ which is closed under duality is the class of weighted games, briefly denoted by $\mathcal{M}$, because if $W \in \mathcal{M}$ admits $[q; w_1, \ldots, w_n]$ as a weighted representation, then $[w - q + 1; w_1, \ldots, w_n]$ is a weighted representation for $W^* \in \mathcal{M}$, where w denotes hereafter the total sum of weights $\sum_{i=1}^{n} w_i$.

Finally, we say that a game $W \in \mathcal{S}$ is *unitary* if it admits a weighted representation $[q; w_1, \ldots, w_n]$, where $q = 1$, $w \geq 1$, and w_i is either 0 or 1 for all $i = 1, \ldots, n$. If $w_i = 1$, then i is a winner in W, whereas $w_i = 0$ implies that i is null in W. Equivalently, W is unitary iff W^m is uniquely formed by singletons (the winners of the game). Let $\mathcal{U}_1$ denote the class of unitary games; then $|\mathcal{U}_1| = 2^n - 1$.

The dual representation of a unitary game $[1; w_1, \ldots, w_n]$ is $[w; w_1, \ldots, w_n]$, which is a weighted representation of the *unanimity* game W of coalition $\{k \in N : w_k = 1\}$; equivalently, W^m is formed by a single coalition whenever the game is unanimous. Let $\mathcal{U}_2$ denote the class of unanimity games; then $\mathcal{U}_1 \neq \mathcal{U}_2$, $\mathcal{U}_1{}^* = \mathcal{U}_2$, and by idempotency $\mathcal{U}_1 = \mathcal{U}_2{}^*$. Of course, the set $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ is closed under duality, although neither $\mathcal{U}_1$ nor $\mathcal{U}_2$ are. It is clear that $|\mathcal{U}_2| = 2^n - 1$ and $|\mathcal{U}| = 2^{n+1} - (n+2)$.

Let $\mathcal{C} \subsetneq \mathcal{S}$ be a proper subset of simple games on N. $W \in \mathcal{S}$ is said to be of *$\mathcal{C}$-dimension k* if it can be represented as the intersection of k games in $\mathcal{C}$ but cannot be represented as the intersection of $k - 1$ games in $\mathcal{C}$. Equivalently, the *$\mathcal{C}$-dimension* of W is the least k such that there exist games $W_1, \ldots, W_k$ in $\mathcal{C}$ such that $W = W_1 \cap \cdots \cap W_k$. Briefly, we will denote $\dim_{\mathcal{C}}(W) = k$. Notice that $W \in \mathcal{C}$ *iff* $\dim_{\mathcal{C}}(W) = 1$.

$W \in \mathcal{S}$ is said to be of *$\mathcal{C}$-codimension k* if it can be represented as the *union* of k games in $\mathcal{C}$ but cannot be represented as the union of $k - 1$ games in $\mathcal{C}$. Equivalently, the *$\mathcal{C}$-codimension* of W is the least k such that there exist games $W_1, \ldots, W_k$ in $\mathcal{C}$ such that $W = W_1 \cup \cdots \cup W_k$. Briefly, we will denote $\mathrm{codim}_{\mathcal{C}}(W) = k$. Notice that $W \in \mathcal{C}$ *iff* $\mathrm{codim}_{\mathcal{C}}(W) = 1$. The $\mathcal{M}$-codimension is simply called *codimension*.

Both notions, of dimension and codimension, have a practical meaning. An interpretation of the notion of codimension is obtained by considering the new game "*blocking the law to pass*" for each of the games with dimension 2 or 3 mentioned in the introduction. Indeed, these new games are the respective dual games of the original ones, and thus they decompose as the union of two or three weighted games (which are at the same time the duals of the weighted games appearing in the original decomposition as intersections of them). As we shall see in Theorem 3.2(ii), the dimension of a game coincides with the codimension of its dual game. Thus, *every real-world* example of a voting system with dimension p provides a real-world example of a voting system with codimension p, and conversely.

A subset $\mathcal{C} \subsetneq \mathcal{S}$ is *dimensionally minimal* if it has $\mathcal{C}$-dimension for all $W \in \mathcal{S}$, but for all $\mathcal{B} \subsetneq \mathcal{C}$, there exists a game $W \in \mathcal{S}$ without $\mathcal{B}$-dimension. Analogously, a subset $\mathcal{C} \subsetneq \mathcal{S}$ is *codimensionally minimal* if it has $\mathcal{C}$-codimension for all $W \in \mathcal{S}$, but for all $\mathcal{B} \subsetneq \mathcal{C}$, there exists a $W \in \mathcal{S}$ without $\mathcal{B}$-codimension.

A subset $\mathcal{C} \subsetneq \mathcal{S}$ is *dimensionally minimum* if it is dimensionally minimal and $\mathcal{C} \subseteq \mathcal{D}$ for all dimensionally minimal sets $\mathcal{D} \subseteq \mathcal{S}$. Analogously, a subset $\mathcal{C} \subsetneq \mathcal{S}$ is *codimensionally minimum* if it is codimensionally minimal and $\mathcal{C} \subseteq \mathcal{D}$ for all codimensionally minimal sets $\mathcal{D} \subseteq \mathcal{S}$. It is an important issue to determine whether there exists a set of games $\mathcal{C}$ being (co)dimensionally minimum because $\mathcal{C}$ would be the *smallest* class of simple games with (co)dimension.

3 Relationship between dimension and codimension

First, we state a result which easily follows from the previous definitions.

Lemma 3.1 *Let $\mathcal{B} \subsetneq \mathcal{C} \subsetneq \mathcal{S}$, and let $W \in \mathcal{S}$ with a finite $\mathcal{B}$-dimension ($\mathcal{B}$-codimension). Then*:

(i) *W has a finite $\mathcal{C}$-dimension ($\mathcal{C}$-codimension), and*
(ii) $\dim_{\mathcal{C}}(W) \leq \dim_{\mathcal{B}}(W)$ $(\operatorname{codim}_{\mathcal{C}}(W) \leq \operatorname{codim}_{\mathcal{B}}(W))$.

Theorem 3.2 *Let $\mathcal{C} \subseteq S$, and let $\mathcal{C}^*$ be its dual. Let $W \in \mathcal{S}$. Then*:

(i) *W has a finite $\mathcal{C}$-dimension* iff *W^* has a finite $\mathcal{C}^*$-codimension, and*
(ii) $\operatorname{codim}_{\mathcal{C}^*}(W^*) = \dim_{\mathcal{C}}(W)$ *if W has a finite $\mathcal{C}$-dimension.*

Proof (i) ($\Rightarrow$) If $W \in \mathcal{C}$, then $W^* \in \mathcal{C}^*$, so that $\dim_{\mathcal{C}}(W) = 1$ and $\operatorname{codim}_{\mathcal{C}}(W^*) = 1$. If $W \notin \mathcal{C}$ has $\mathcal{C}$-dimension, then $W = W_1 \cap \cdots \cap W_k$ for some integer $k > 1$ and $W_i \in \mathcal{C}$ for all $i = 1, \ldots, k$, where the right-hand side is defined associatively, so W can be written as $U \cap V$, where $U = W_1$ and $V = W_2 \cap \cdots \cap W_k$. Apply the second Morgan's law in (1) to U and V to get $W^* = (U \cap V)^* = U^* \cup V^*$. Then take $U = W_2$ and $V = W_3 \cap \cdots \cap W_k$ and apply the second Morgan's law in (1) to U and V, and so on to finally get $W^* = W_1^* \cup \cdots \cup W_k^*$. Hence, W^* has $\mathcal{C}^*$-codimension.

($\Leftarrow$) It follows the guidelines of the previous implication.

(ii) Assume that the positive integer k in the right implication of (i) is minimum; then $\dim_{\mathcal{C}}(W) = k$ and $\operatorname{codim}_{\mathcal{C}^*}(W^*) \leq k$ because W^* decomposes as the union of k games in $\mathcal{C}^*$. Let $\operatorname{codim}_{\mathcal{C}^*}(W^*) = k' \leq k$; then the left implication in (i) yields $\dim_{\mathcal{C}}(W) \leq k'$, but $\dim_{\mathcal{C}}(W) = k$. Thus, $k' = k$, and the equality in (ii) arises. □

Several significant consequences of the previous results can be deduced.

Remark 3.3 (i) The games appearing in the proof of Theorem 1.1(i) are unitary. Hence, it follows that each simple game has a finite $\mathcal{U}_1$-dimension.

(ii) Analogously, taking the game considered in the proof of Theorem 1.1(i) and applying Theorem 3.2, it follows that each simple game has a finite $\mathcal{U}_2$-codimension. Moreover, if we apply Morgan's laws (1) to the unitary game W considered in the proof of Theorem 1.1(i), we obtain that $W^* = U_{S_1} \cup \cdots \cup U_{S_k}$, where every U_S is the unanimity game of coalition S, which is weighted and admits the representation $[w; w_1, \ldots, w_n]$, where

$$w_i = \begin{cases} 1 & \text{if } i \in S; \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the $\mathcal{U}_2$-codimension is bounded above by the number of minimal winning coalitions.

(iii) Taking the game considered in the proof of Theorem 1.1(ii) and applying Theorem 3.2, we obtain that for every $m \geq 1$, there is a game of codimension m. Indeed, let $M = \{1, \ldots, 2m\}$ and W be the collection of sets formed by the union of the minimal winning coalitions $W_i = \{2i - 1, 2i\}$ for all $i = 1, \ldots, m$. Each W_i can be represented by a game that gives weight 1 to each voter in $\{2i-1, 2i\}$ and weight zero to everyone else. For each of these representations, let the quota q be 2. Note that $S \in W$ *iff* $\{2i-1, 2i\} \subseteq S$ for some i *iff* $w(S) \geq 2$ for some i *iff* $S \in W_i$ for some i *iff* $S \in \bigcup_{i=1}^{m} W_i$.

(iv) Theorem 4.3 in Freixas and Puente (2008) shows the existence of complete games with $\mathcal{M}$-dimension m for every $m \geq 1$. By considering the dual games of these complete games we obtain complete games of every codimension m. Further, Theorem 1.7.5 in Taylor and Zwicker (1999) and also Theorem 2.1 in Freixas and Puente (2001) show games with exponential $\mathcal{M}$-dimension (or simply, dimension). By Remark 3.3(i), these games have $\mathcal{U}_1$-dimension, and by Lemma 3.1(ii), $\dim_{\mathcal{M}}(W) \leq \dim_{\mathcal{U}_1}(W)$ for all $W \in \mathcal{S}$. Hence, those games have exponential $\mathcal{U}_1$-dimension. Taking the dual games of those games, we get also games with exponential $\mathcal{U}_2$-codimension.

(v) If we desire to compute a power index of a simple game that satisfies the transfer axiom and the game has a large number of players, a large $\mathcal{M}$-dimension, but a reduced $\mathcal{M}$-codimension, then we can compute the power index of the game by using generating functions (see, e.g., Brams and Affuso 1976) to each game appearing in the decomposition obtained by applying the union–exclusion principle of an efficient union of weighted games.

(vi) By applying Theorem 3.2(ii). The results in Deĭneko and Woeginger (2006) about $\mathcal{M}$-dimension complexity extend to $\mathcal{M}$-codimension, i.e., if given k weighted games, decide whether the dimension of their union exactly equals k' is NP-hard and hence computationally intractable.

The next result allows computing the $\mathcal{U}_1$-dimension and the $\mathcal{U}_2$-codimension.

Theorem 3.4 (i) *The $\mathcal{U}_1$-dimension of a simple game is the number of maximal losing coalitions.*

(ii) *The $\mathcal{U}_2$-codimension of a simple game is the number of minimal winning coalitions.*

Proof (i) Remark 3.3(i) guarantees that the $\mathcal{U}_1$-dimension is bounded by $|L^M|$. Assume, by contradiction, that W is the intersection of $|L^M|-1$ unitary games and let T_1, T_2 be two different maximal losing coalitions. By the pigeonhole principle (and rearranging that loses no generality), we can assume that we have a single unitary game, $[1; w_1, \ldots, w_n]$, $w \geq 1$, and $w_i = 0$ or 1 for all i, that makes both T_1 and T_2 losing coalitions with respect to that system, so that $T_i \subseteq \{k \in N : w_k = 0\}$ for $i = 1, 2$. If the inclusion for some i were strict, it would mean that $T_i \in L \setminus L^M$, which would be a contradiction. Hence, $T_1 = \{k \in N : w_k = 0\} = T_2$, but this contradicts that T_1 and T_2 are different maximal losing coalitions.

(ii) From Theorem 3.4(i) it follows that $\dim_{\mathcal{U}_1}(W^*) = |(L^*)^M|$. From Theorem 3.2(ii) it follows that $\dim_{\mathcal{C}}(W^*) = \operatorname{codim}_{\mathcal{C}^*}(W)$. Taking $\mathcal{C} = \mathcal{U}_1$ in the last equality, we obtain $|(L^*)^M| = \dim_{\mathcal{U}_1}(W^*) = \operatorname{codim}_{\mathcal{U}_2}(W)$. Hence, $\operatorname{codim}_{\mathcal{U}_2}(W) = |(L^*)^M|$, but $(L^*)^M = W^m$ for all simple games W, and thus $\operatorname{codim}_{\mathcal{U}_2}(W) = |W^m|$. □

Theorem 3.4, together with Sperner's theorem, allows providing upper bounds for the $\mathcal{U}_1$-dimension and $\mathcal{U}_2$-dimension. Indeed, $\dim_{\mathcal{U}_1} W \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and $\operatorname{codim}_{\mathcal{U}_2} W \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

4 Minimum (co)dimensionally class of simple games

It remains open the question about the existence of minimum dimensionally and codimensionally classes of simple games.

Theorem 4.1 (i) *The class $\mathcal{U}_1$ of unitary games is dimensionally minimum.*
(ii) *The class $\mathcal{U}_2$ of unanimity games is codimensionally minimum.*

Proof (i) By Remark 3.3(i) every game has $\mathcal{U}_1$-dimension, and by Theorem 3.4(i) $\dim_{\mathcal{U}_1}(W) = |L^M|$ for all simple games W.

Firstly, we will prove that $\mathcal{U}_1$ is dimensionally minimal. For $0 \leq a \leq n-1$, consider the unitary game W_a with weighted representation $[1; \underbrace{0, \ldots, 0}_{a}, \underbrace{1, \ldots, 1}_{n-a}]$. Without loss of generality assume that $\mathcal{B} \subsetneq \mathcal{U}_1$ is a class of games such that $W_a \notin \mathcal{B}$ but all simple games W have $\mathcal{B}$-dimension. Then $W_a = W_1 \cap \cdots \cap W_k$ with $W_i \in \mathcal{B}$ for all $i = 1, \ldots, k$. Let W_i be the game with representation $[1; w_1^i, \ldots, w_n^i]$ for all $i = 1, \ldots, k$. Then $w_j^i = 0$ for some i and all $j \leq a$ because the coalition $\{1, \ldots, a\}$ loses in W_a. However, $w_j^i = 1$ for all $j \geq a+1$ because coalition $\{j\}$ wins in W_a. Hence, $W_a \in \mathcal{B}$, which is a contradiction. The above proof does not depend on the position of the "a" null weights.

Secondly, we will prove that $\mathcal{U}_1$ is dimensionally minimum. Let $\mathcal{B}$ be an arbitrary class of simple games with minimal dimension. Such a class $\mathcal{B}$ exists because we can take $\mathcal{B} = \mathcal{U}_1$. Suppose that the unitary game W_a with weighted representation $[1; \underbrace{0, \ldots, 0}_{a}, \underbrace{1, \ldots, 1}_{n-a}]$ does not belong to $\mathcal{B}$. Then $W_a = W_1 \cap \cdots \cap W_k$ with $W_i \in \mathcal{B}$ for all $i = 1, \ldots, k$. Coalition $\{1, \ldots, a\}$ loses in W_a; then it must lose in some W_i, and each player j with $j \geq a+1$ must win in each game W_i. Consequently, there is some game W_i in the decomposition $W_1 \cap \cdots \cap W_k$ wherein all voters j with $j \leq a$ are nulls, and all voters j with $j \geq a+1$ are winners. Such a game W_i coincides with W_a, and therefore W_a must belong to every dimensionally minimal class $\mathcal{B}$.

Let W_i be the game with representation $[1; w_1^i, \ldots, w_n^i]$ for all $i = 1, \ldots, k$. Then $w_j^i = 0$ for some i and all $j \leq a$ because the coalition $\{1, \ldots, a\}$ loses in W_a. However, $w_j^i = 1$ for all $j \geq a+1$ because coalition $\{j\}$ wins in W_a. Hence, $W_a \in \mathcal{B}$, and so W_a belongs to every dimensionally class of games. As the above proof does not depend on either ''a'' or the allocation of the a null weights, we conclude that $\mathcal{U}_1 \subseteq \mathcal{B}$ for all dimensionally classes $\mathcal{B}$. Hence, $\mathcal{U}_1$ is dimensionally minimum.

(ii) It follows the guidelines of part (i). □

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