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ORIGINAL PAPER

Optimal value bounds in nonlinear programming with interval data

Milan Hladík

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Abstract We consider nonlinear programming problems the input data of which are not fixed, but vary in some real compact intervals. The aim of this paper is to determine bounds of the optimal values. We propose a general framework for solving such problems. Under some assumption, the exact lower and upper bounds are computable by using two non-interval optimization problems. While these two optimization problems are hard to solve in general, we show that for some particular subclasses they can be reduced to easy problems. Subclasses that are considered are convex quadratic programming and posynomial geometric programming.

Keywords Interval systems · Nonlinear programming · Optimal value range · Interval matrix · Dependence

Mathematics Subject Classification (2000) 90C30 · 90C31 · 90C70

1 Introduction

Many practical problems are modeled and solved by mathematical programming. In real world applications, however, input data are not always known exactly and are subject to diverse uncertainties. Various approaches were developed to deal with uncertainties. Within this paper we assume that we are given interval estimates of the problem quantities.

We study nonlinear programming problems under interval uncertainty and our aim is to compute the range of optimal values for all instances of the interval quantities.

M. Hladík (✉)
Faculty of Mathematics and Physics, Department of Applied Mathematics, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic
e-mail: milan.hladik@matfyz.cz

Such knowledge provides a decision maker with useful information for making more appropriate decisions.

In the last years, many papers studied the problem of computing the range of optimal values of linear programming problems with data varying inside intervals (Chinneck and Ramadan 2000; Fiedler et al. 2006; Hladík 2007; Mráz 1998) among others. Nevertheless, few people were involved in a generalization to nonlinear programming: Liu and Wang (2007) considered convex quadratic programming, Liu (2006b, 2008) posynomial geometric programming, and Wu et al. (2006) a specific nonlinear programming problem with linear constraints. Another approach to nonlinear programming under interval uncertainty concerns, e.g., the work of Hu and Wang (2006a, 2006b), and Levin (1999). Recent applications of interval nonlinear programming involve, for instance, planning of waste management activities (Huang et al. 1997; Wu et al. 2006), water management modelling (Huang 1998), and inventory management (Liu 2006a).

In this paper, we generalize our result (Hladík 2007) on the optimal value range of interval linear programming, which is based on linear programming duality and several theorems on linear interval systems. Our general approach is applicable for various classes of nonlinear programs with interval data, however, better approximations are obtained in the case when there is an appropriate dual problem with zero duality gap and when dependencies between quantities do not occur. We discuss also some special cases later in this paper.

First, we introduce some notation from interval analysis (Alefeld and Herzberger 1983). An interval matrix is defined as a family of matrices

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{A \in \mathbb{R}^{m\times n} \mid \underline{A} \le A \le \overline{A}\right\},$$

where $\underline{A} \le \overline{A}$ are fixed matrices; n-dimensional interval vectors can be regarded as interval matrices n-by-1. By

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}), \qquad A_\Delta = \frac{1}{2}(\overline{A} - \underline{A})$$

we denote the midpoint and the radius of $\boldsymbol{A}$, respectively. By convention, we associate a degenerate interval $[a, a]$ with the real number a.

We now mention some important theorems on solvability of interval linear systems. The statements differ depending on the use of quantifiers and whether we consider equality or inequality systems and non-negative variables. All the theorems and corresponding proofs can be found in Fiedler et al. (2006).

Theorem 1 (Gerlach 1981) *A vector x solves the system $Ax \le b$ for* some $A \in \boldsymbol{A}$ *and $b \in \boldsymbol{b}$ iff it solves the system*

$$A_c x - A_\Delta |x| \le \overline{b}.$$

Proof See Fiedler et al. (2006) Theorem 2.19 or Gerlach (1981). □

Theorem 2 (Rohn and Kreslová 1994) *The system* $Ax \le b$ *is solvable for* all $A \in \boldsymbol{A}$ *and* $b \in \boldsymbol{b}$ *iff the system*

$$\overline{A}x^1 - \underline{A}x^2 \le \underline{b}, \quad x^1, x^2 \ge 0$$

is solvable.

Proof See Fiedler et al. (2006) Theorem 2.23 or Rohn and Kreslová (1994). □

Theorem 3 *A vector* x *solves the system* $Ax = b,\ x \ge 0$ *for* some $A \in \boldsymbol{A}$ *and* $b \in \boldsymbol{b}$ *iff it solves the system*

$$\overline{A}x \ge \underline{b}, \qquad \underline{A}x \le \overline{b}, \quad x \ge 0.$$

Proof See, e.g., Fiedler et al. (2006) Theorem 2.13. □

2 General approach

In this section, we present a general approach for computing the range of optimal values in interval nonlinear programming. By an interval nonlinear program we mean the family of nonlinear programs

$$f(A, c) = \inf\ f_c(x) \quad \text{subject to} \quad F_A(x) \le 0, \tag{1}$$

where $f_c : \mathbb{R}^n \mapsto \mathbb{R}$ is a real function with input data c varying inside an interval vector $\boldsymbol{c}$, and $F_A : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a vector function with input data A varying inside an interval matrix $\boldsymbol{A}$. We suppose that $f_c(x)$ and $F_A(x)$ do not have an interval parameter in common.

Our aim is to compute lower and upper bound of the optimal value function $f(A, c)$. They are respectively defined as

$$\underline{f} = \inf f(A, c) \quad \text{subject to} \quad A \in \boldsymbol{A}, c \in \boldsymbol{c},$$

$$\overline{f} = \sup f(A, c) \quad \text{subject to} \quad A \in \boldsymbol{A}, c \in \boldsymbol{c}.$$

Using interval arithmetic directly for the problem (1) we obtain an enclosure of the optimal value range $[\underline{f}, \overline{f}]$. This enclosure is usually very overestimated, particularly when the input intervals are not extremely narrow. We want to develop a method for calculating the exact bounds (limited by using floating point arithmetic).

Our approach for computing the upper bound $\overline{f}$ is based on duality theory. In nonlinear programming, there are diverse ways for constructing dual problems. We consider any dual problem to (1) having the form of

$$g(A, c) = \sup\ g_{A,c}(y) \quad \text{subject to} \quad G_{A,c}(y) \le 0, \tag{2}$$

and satisfying the weak duality. Herein, $g_{A,c} : \mathbb{R}^k \mapsto \mathbb{R}$, and $G_{A,c} : \mathbb{R}^k \mapsto \mathbb{R}^l$ are functions depending on $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$. The weak duality property is defined as

$$f(A, c) \ge g(A, c) \quad \forall c \in \boldsymbol{c}, A \in \boldsymbol{A}. \tag{3}$$

Naturally, better results are obtained when stronger duality conditions hold. Strong duality means that a finite optimal value to one problem ensures the existence of an optimal solution to the other and that their optimal objective values equals. The most restrictive notion of zero duality gap refers to the equation

$$f(A,c) = g(A,c) \quad \forall c \in \boldsymbol{c},\, A \in \boldsymbol{A}. \tag{4}$$

Next, we introduce the functions

$$f(x) = \inf f_c(x) \quad \text{subject to} \quad c \in \boldsymbol{c},$$
$$g(y) = \sup g_{A,c}(y) \quad \text{subject to} \quad A \in \boldsymbol{A},\, c \in \boldsymbol{c},$$

and the sets

$$M = \left\{x \in \mathbb{R}^n \mid F_A(x) \le 0,\, A \in \boldsymbol{A}\right\},$$
$$N = \left\{y \in \mathbb{R}^k \mid G_{A,c}(y) \le 0,\, A \in \boldsymbol{A},\, c \in \boldsymbol{c}\right\}.$$

We are now ready to formulate our main result on how to compute $\underline{f}$ and $\overline{f}$.

Theorem 4 *We have*

$$\underline{f} = \inf f(x) \quad \textit{subject to} \quad x \in M. \tag{5}$$

If the zero duality gap is guaranteed, then

$$\overline{f} \le \sup g(y) \quad \textit{subject to} \quad y \in N. \tag{6}$$

If the functions $G_{A,c}(y)$ and $g_{A,c}(y)$ have no interval parameter in common, then

$$\overline{f} \ge \sup g(y) \quad \textit{subject to} \quad y \in N. \tag{7}$$

Proof 1. (Lower bound) From the definition of $\underline{f}$ we have that

$$\underline{f} = \inf_{A \in \boldsymbol{A}, c \in \boldsymbol{c}} \left(\inf_{x : F_A(x) \le 0} f_c(x) \right) = \inf_{c \in \boldsymbol{c}} \left(\inf_{A \in \boldsymbol{A}, x : F_A(x) \le 0} f_c(x) \right)$$
$$= \inf_{c \in \boldsymbol{c}} \left(\inf_{x \in M} f_c(x) \right) = \inf_{x \in M} \left(\inf_{c \in \boldsymbol{c}} f_c(x) \right) = \inf_{x \in M} f(x).$$

2. (Upper bound) Under the zero duality gap assumption (4) we have that for every $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$

$$f(A,c) = g(A,c).$$

We maximize both sides of the equation over $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$ and rearrange the right hand side

$$\sup_{A \in \boldsymbol{A}, c \in \boldsymbol{c}} f(A,c) = \sup_{A \in \boldsymbol{A}, c \in \boldsymbol{c}} g(A,c)$$

$$
\begin{aligned}
&= \sup_{A\in\boldsymbol{A},c\in\boldsymbol{c},y:G_{A,c}(y)\le 0} g_{A,c}(y)\\
&\le \sup_{A\in\boldsymbol{A},c\in\boldsymbol{c},y:G_{A,c}(y)\le 0} \Big(\sup_{A\in\boldsymbol{A},c\in\boldsymbol{c}} g_{A,c}(y)\Big)\\
&= \sup_{y\in N} g(y).
\end{aligned}
$$

The proof of the inequality (7) is a slight modification of the previous way. The weak duality (3) says that for every $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$

$$f(A, c) \ge g(A, c).$$

Maximizing both sides of the equation over $A \in \boldsymbol{A}$ and $c \in \boldsymbol{c}$ and rearranging the right hand side, we get

$$
\begin{aligned}
\sup_{A\in\boldsymbol{A},c\in\boldsymbol{c}} f(A,c) \ge \sup_{A\in\boldsymbol{A},c\in\boldsymbol{c}} g(A,c)
&= \sup_{A\in\boldsymbol{A},c\in\boldsymbol{c},y:G_{A,c}(y)\le 0} g_{A,c}(y)\\
&= \sup_{A\in\boldsymbol{A},c\in\boldsymbol{c},y:G_{A,c}(y)\le 0} \Big(\sup_{A\in\boldsymbol{A},c\in\boldsymbol{c}} g_{A,c}(y)\Big)\\
&= \sup_{y\in N} g(y).
\end{aligned}
$$

□

It is a simple consequence that if the duality gap is zero and the functions $G_{A,c}(x)$ and $g_{A,c}(x)$ have no interval parameter in common, then

$$\overline{f} = \sup g(y) \quad \text{subject to} \quad y \in N. \tag{8}$$

Theorem 4 gives formulae for determining the exact lower and upper bounds of the optimal value function (under some assumption). These formulae represent optimization problems with point (non-interval) data. In the real world, we can hardly achieve their exact optima, but using interval arithmetic or some verification software we are able to get guaranteed bounds. We do not discuss this issue in detail here since the main purpose of the paper is to reduce interval-valued problems into non-interval ones.

Applicability of Theorem 4 depends on how efficiently we are able to compute the functions $f(x)$ and $g(y)$ and the sets M and N. In general, it is a great challenge, but for some special nonlinear problems the mentioned functions and sets can be quite easily simplified.

In the following sections, we apply the main result to two particular cases: convex quadratic programming and posynomial geometric programming. Some results for these cases were developed by Liu and Wang (2007) and by Liu (2006b, 2008), respectively. We discuss their approach in the corresponding sections.

3 Convex quadratic programming

3.1 Simple case

Consider a convex quadratic programming problem

$$\min x^{\mathrm{T}}Cx + d^{\mathrm{T}}x \quad \text{subject to} \quad Ax \le b, \quad x \ge 0, \tag{9}$$

where C, A, b, and d vary in given interval matrices $\boldsymbol{C}$, $\boldsymbol{A}$ and interval vectors $\boldsymbol{b}$ and $\boldsymbol{d}$. Suppose that C is positive semidefinite for all $C \in \boldsymbol{C}$. The aim is to determine the range $[\underline{f}, \overline{f}]$ in which the optimal value of (9) varies. This case was considered by Liu and Wang (2007). Our approach developed in the previous section is applicable, however, this case is so simple that we can determine lower and upper bounds directly.

For every feasible point x the following is true (due to non-negativity of x):

$$x^{\mathrm{T}}\underline{C}x + \underline{d}^{\mathrm{T}}x \le x^{\mathrm{T}}Cx + d^{\mathrm{T}}x \le x^{\mathrm{T}}\overline{C}x + \overline{d}^{\mathrm{T}}x.$$

Thus the lower bound $\underline{f}$ will be achieved for $C = \underline{C}$ and $d = \underline{d}$, whereas the upper bound $\overline{f}$ will be achieved for $C = \overline{C}$ and $d = \overline{d}$.

In a similar manner, we can also fix the values of A and b. Every solution of a system

$$Ax \le b, \quad x \ge 0$$

for any $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$ is also a solution of

$$\underline{A}x \le \overline{b}, \quad x \ge 0$$

(since $\underline{A}x \le Ax \le b \le \overline{b}$). For $A = \underline{A}$ and $b = \overline{b}$, the feasible set of (9) contains all the other feasible sets, and hence the minimal optimal value $\underline{f}$ will be achieved in this setting. Analogously, every solution of a system

$$\overline{A}x \le \underline{b}, \quad x \ge 0$$

is also a solution of

$$Ax \le b, \quad x \ge 0$$

for all $A \in \boldsymbol{A}$ and $b \in \boldsymbol{b}$. By fixing $A = \overline{A}$ and $b = \underline{b}$, the feasible set of (9) is the smallest possible (indeed, the intersection of all the others) and the maximal optimal value $\overline{f}$ must be achieved in this setting.

The following statement makes a summary for this case.

Proposition 5 *We have*

$$\underline{f} = \inf x^{\mathrm{T}}\underline{C}x + \underline{d}^{\mathrm{T}}x \quad \textit{subject to} \quad \underline{A}x \le \overline{b}, \quad x \ge 0,$$
$$\overline{f} = \inf x^{\mathrm{T}}\overline{C}x + \overline{d}^{\mathrm{T}}x \quad \textit{subject to} \quad \overline{A}x \le \underline{b}, \quad x \ge 0.$$

3.2 General case

Consider a general convex quadratic programming problem

$$\min\ x^{\mathrm{T}}Cx + d^{\mathrm{T}}x \quad \text{subject to} \quad Ax \le b, \qquad Ex = h, \quad x \ge 0, \tag{10}$$

where $C \in \boldsymbol{C}$, $A \in \boldsymbol{A}$, $b \in \boldsymbol{b}$, $d \in \boldsymbol{d}$, $E \in \boldsymbol{E}$, and $h \in \boldsymbol{h}$. Suppose that C is positive semidefinite for all $C \in \boldsymbol{C}$. This case is more complicated than the previous one, because the feasible set is described by both the equations and inequalities. Due to the equations the previous approach is not applicable. Note that generally it is not possible to transform the problem (10) to (9)—such a transformation causes dependencies between interval quantities.

The approach of the previous section is applicable only partially. For the similar reasons, the lower bound $\underline{f}$ is achieved for $C = \underline{C}$, $d = \underline{d}$, $A = \underline{A}$ and $b = \overline{b}$, and the upper $\overline{f}$ bound for $C = \overline{C}$ and $d = \overline{d}$, $A = \overline{A}$ and $b = \underline{b}$. The remaining quantities E and h cannot be fixed.

We now use Theorem 4 to determine $\underline{f}$ and $\overline{f}$. First, we consider the lower bound $\underline{f}$, and from the previous line of thought we know that we can set and fix $C = \underline{C}$, $d = \underline{d}$, $A = \underline{A}$ and $b = \overline{b}$. Clearly, $f(x) = x^{\mathrm{T}}\underline{C}x + \underline{d}^{\mathrm{T}}x$. From Theorem 3, we get a characterization of the set M as

$$\underline{A}x \le \overline{b}, \qquad \overline{E}x \ge \underline{h}, \qquad \underline{E}x \le \overline{h}, \quad x \ge 0.$$

Now, we are ready to propose a formula for computing the lower bound $\underline{f}$.

Proposition 6 *We have*

$$\underline{f} = \inf\ x^{\mathrm{T}}\underline{C}x + \underline{d}^{\mathrm{T}}x \quad \textit{subject to} \quad \underline{A}x \le \overline{b}, \qquad \overline{E}x \ge \underline{h}, \qquad \underline{E}x \le \overline{h}, \quad x \ge 0.$$

To determine the upper bound $\overline{f}$ we have to consider a dual problem to (10). The Dorn dual problem (Bazaraa et al. 1993; Dorn 1960; Martos 1975) is

$$\max\ -x^{\mathrm{T}}Cx - b^{\mathrm{T}}u - h^{\mathrm{T}}v \quad \text{subject to} \quad 2Cx + A^{\mathrm{T}}u + E^{\mathrm{T}}v + d \ge 0, \quad u \ge 0. \tag{11}$$

The following lemma gives a description of the function $g(y) = g(x, u, v)$. W.l.o.g. we set and fix $C = \overline{C}$, $d = \overline{d}$, $A = \overline{A}$ and $b = \underline{b}$.

Lemma 7 *We have* $g(x, u, v) = -x^{\mathrm{T}}\overline{C}x - \underline{b}^{\mathrm{T}}u - h_c^{\mathrm{T}}v + h_{\Delta}^{\mathrm{T}}|v|$.

Proof The function $g(x, u, v)$ is defined as

$$g(x, u, v) = \sup\ g_h(x, u, v) \quad \text{subject to} \quad h \in \boldsymbol{h},$$

where $g_h(x, u, v) = -x^{\mathrm{T}}\overline{C}x - \underline{b}^{\mathrm{T}}u - h^{\mathrm{T}}v$. Clearly,

$$g(x, u, v) = -x^{\mathrm{T}}\overline{C}x - \underline{b}^{\mathrm{T}}u - \inf_{h \in \boldsymbol{h}} h^{\mathrm{T}}v. \tag{12}$$

Define the vector $h^* \in \boldsymbol{h}$ componentwise as

$$h_i^* = \begin{cases} \underline{h}_i, & v_i \geq 0, \\ \overline{h}_i, & v_i < 0. \end{cases}$$

The vector h^* is the optimal solution of (12), as for every $h \in \boldsymbol{h}$ one has

$$h^{\mathrm{T}} v = \sum_{i: v_i \geq 0} h_i v_i + \sum_{i: v_i < 0} h_i v_i \geq \sum_{i: v_i \geq 0} \underline{h}_i v_i + \sum_{i: v_i < 0} \overline{h}_i v_i = h^{*T} v.$$

Eventually, we simplify the expression $h^{*\mathrm{T}} v$:

$$\begin{aligned} h^{*\mathrm{T}} v &= \sum_{i: v_i \geq 0} \underline{h}_i v_i + \sum_{i: v_i < 0} \overline{h}_i v_i = \sum_{i: v_i \geq 0} (h_c - h_\Delta)_i v_i + \sum_{i: v_i < 0} (h_c + h_\Delta)_i v_i \\ &= h_c^{\mathrm{T}} v - \sum_{i: v_i \geq 0} (h_\Delta)_i |v_i| - \sum_{i: v_i < 0} (h_\Delta)_i |v_i| = h_c^{\mathrm{T}} v - h_\Delta^{\mathrm{T}} |v|. \end{aligned}$$

□

To describe the set N we call Theorem 1, which yields

$$2\overline{C} x + \overline{A}^{\mathrm{T}} u + E_c^{\mathrm{T}} v + E_\Delta^{\mathrm{T}} |v| + \overline{d} \geq 0, \quad u \geq 0.$$

To give a resulting formula for computing $\overline{f}$, we must still specify a condition when the duality gap is zero. In the non-interval case, the duality gap is zero as long as either the primal or the dual problem is feasible (it has a feasible solution). It is not an easy problem to check whether this is true when the data may vary inside intervals. It seems better to investigate feasibility of only one of them. The primal problem is not convenient, as it combines equations and inequalities, however, the dual problem can be used. Theorem 2 implies that if the linear system

$$2\overline{C} x + \overline{A}^{\mathrm{T}} u + \underline{E}^{\mathrm{T}} v^1 - \overline{E}^{\mathrm{T}} v^2 + \overline{d} \geq 0, \quad u \geq 0, v^1 \geq 0, v^2 \geq 0 \tag{13}$$

has a solution, then the corresponding dual problem is feasible for all $E \in \boldsymbol{E}$, and hence the zero duality gap is ensured. This is particularly true if $\overline{C}$ is positive definite.

Proposition 8 *We have*

$$\begin{aligned} \overline{f} \geq \sup\ & -x^{\mathrm{T}} \overline{C} x - \underline{b}^{\mathrm{T}} u - h_c^{\mathrm{T}} v + h_\Delta^{\mathrm{T}} |v| \\ \text{subject to}\ & 2\overline{C} x + \overline{A}^{\mathrm{T}} u + E_c^{\mathrm{T}} v + E_\Delta^{\mathrm{T}} |v| + \overline{d} \geq 0, \quad u \geq 0, \end{aligned} \tag{14}$$

and equality holds if the system (13) *is solvable.*

The optimization problem (14) is nonconvex and nonsmooth, and therefore not easy to solve. Beyond nonsmooth optimization techniques, we can solve this problem also by using a global optimization approach (Hansen and Walster 2004; Neumaier 2004). Nevertheless, as long as the dimension of v (i.e., the number of equations in (10)) is small, the simplest way to solve this problem is to decompose it into a series of convex quadratic programming problems. The number of these convex quadratic programming problems is 2^k, where k denotes the dimension of v. The decomposition is based on partitioning the space into 2^k orthants according to the signs of the v_i. In this way, the optimization problem (14) can equivalently be formulated as

$$\sup f_z \quad \text{subject to} \quad z \in \{\pm 1\}^k,$$

where

$$\begin{aligned} f_z = \sup -x^{\mathrm{T}}\overline{C}x - \overline{b}^{\mathrm{T}}u - \left(h_c^{\mathrm{T}} - h_\Delta^{\mathrm{T}} \operatorname{diag}(z)\right)v \\ \text{subject to} \quad 2\overline{C}x + \overline{A}^{\mathrm{T}}u + \left(E_c^{\mathrm{T}} + E_\Delta^{\mathrm{T}} \operatorname{diag}(z)\right)v + \overline{d} \geq 0, \quad u \geq 0, \operatorname{diag}(z)v \geq 0, \end{aligned}$$

and $\operatorname{diag}(z)$ denotes the diagonal matrix with entries $z_1, \ldots, z_k$.

Example 9 Consider the interval convex quadratic programming problem

$$\begin{aligned} &\inf [2,3]x_1^2 + 2x_2^2 - 2x_1x_2 + [-5,-3]x_1 + [1,2]x_2 \\ &\text{subject to} \quad [1,2]x_1 + x_2 \leq [2,4], \\ &\qquad [2,3]x_1 + [-1,-0.5]x_2 \leq [3,4], \\ &\qquad [4,5]x_1 + [-8,-7]x_2 = [1,1.5], \\ &\qquad x_1, x_2 \geq 0. \end{aligned}$$

Thus the corresponding interval matrices and vectors are

$$\boldsymbol{C} = \begin{pmatrix} [2,3] & -1 \\ -1 & 2 \end{pmatrix}, \quad \boldsymbol{d} = \begin{pmatrix} [-5,-3] \\ [1,2] \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} [1,2] & 1 \\ [2,3] & [-1,-0.5] \end{pmatrix},$$
$$\boldsymbol{b} = \begin{pmatrix} [2,4] \\ [3,4] \end{pmatrix}, \quad \boldsymbol{E} = \begin{pmatrix} [4,5] & [-8,-7] \end{pmatrix}, \quad \boldsymbol{h} = \begin{pmatrix} [1,1.5] \end{pmatrix}.$$

Using Proposition 6, we determine the lower bound of the optimal value function by computing the convex quadratic program

$$\begin{aligned} \underline{f} = \inf\ & 2x_1^2 + 2x_2^2 - 2x_1x_2 - 5x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 4, \qquad 2x_1 - x_2 \leq 4, \\ & 5x_1 - 7x_2 \geq 1, \qquad 4x_1 - 8x_2 \leq 1.5, \quad x_1, x_2 \geq 0. \end{aligned}$$

The optimal solution is $x^* = (0.875, 0.25)^{\mathrm{T}}$ with optimal value -2.90625.

As the matrix $\overline{C} = \left(\begin{smallmatrix} 3 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ is positive definite, the duality gap is zero, and we determine according to Proposition 8 the accurate upper bound by solving the optimization program

$$\overline{f} = \sup -3x_1^2 - 2x_2^2 + 2x_1x_2 - 2u_1 - 3u_2 - 1.25v + 0.25|v|$$

$$\begin{aligned} \text{subject to} \quad & 6x_1 - 2x_2 + 2u_1 + 3u_2 + 4.5v + 0.5|v| - 3 \geq 0, \\ & -2x_1 + 4x_2 + u_1 - 0.5u_2 - 7.5v + 0.5|v| + 2 \geq 0, \\ & u_1, u_2 \geq 0. \end{aligned}$$

The vector v is one-dimensional, so the most convenient method to solve this problem is to decompose it into two convex quadratic programming problems:

$$\begin{aligned} f_1 = \sup\ & -3x_1^2 - 2x_2^2 + 2x_1x_2 - 2u_1 - 3u_2 - 1.25v + 0.25v \\ \text{subject to} \quad & 6x_1 - 2x_2 + 2u_1 + 3u_2 + 4.5v + 0.5v - 3 \geq 0, \\ & -2x_1 + 4x_2 + u_1 - 0.5u_2 - 7.5v + 0.5v + 2 \geq 0, \\ & u_1, u_2, v \geq 0, \end{aligned}$$

and

$$\begin{aligned} f_2 = \sup\ & -3x_1^2 - 2x_2^2 + 2x_1x_2 - 2u_1 - 3u_2 - 1.25v - 0.25v \\ \text{subject to} \quad & 6x_1 - 2x_2 + 2u_1 + 3u_2 + 4.5v - 0.5v - 3 \geq 0, \\ & -2x_1 + 4x_2 + u_1 - 0.5u_2 - 7.5v - 0.5v + 2 \geq 0, \\ & u_1, u_2, -v \geq 0. \end{aligned}$$

The former has an optimal value $f_1 \simeq -0.52165$ and the latter $f_2 = -0.75$. The resulting upper bound is the maximum of both, that is, $\overline{f} \simeq -0.52165$.

We conclude that the optimal values of the given interval convex quadratic programming problem vary inside the range $[-2.90625, -0.52165]$ and its bounds are the best possible.

4 Posynomial geometric programming

A posynomial geometric program (Bazaraa et al. 1993; Liu 2006b, 2008) is a problem of the form

$$\begin{aligned} \inf\ & \sum_{i \in I_0} c_i \prod_{j=1}^{n} x_j^{a_{ij}} \\ \text{subject to} \quad & \sum_{i \in I_k} c_i \prod_{j=1}^{n} x_j^{a_{ij}} \leq 1, \quad k = 1, \ldots, m, \\ & x_j > 0, \quad j = 1, \ldots, n. \end{aligned} \tag{15}$$

The index sets enumerate the terms sequentially, that is, they are of the form $I_0 = \{1, \ldots, p_0\}$, $I_1 = \{p_0 + 1, \ldots, p_1\}, \ldots, I_m = \{p_{m-1} + 1, \ldots, p\}$. By the definition of posynomial, all the coefficients c_i, $i = 1, \ldots, p$, are positive.

The coefficients vary in given intervals as follows: $c_i \in \boldsymbol{c}_i$, $i = 1, \ldots, p$, $a_{ij} \in \boldsymbol{a}_{ij}$, $i = 1, \ldots, p$, $j = 1, \ldots, n$.

The corresponding dual problem (Bazaraa et al. 1993; Boţ et al. 2006; Liu 2006b, 2008) is

$$\begin{aligned}
&\sup \left(\prod_{i=1}^{p}\left(\frac{c_i}{y_i}\right)^{y_i}\right)\left(\prod_{k=1}^{m} z_k^{z_k}\right)\\
&\text{subject to} \quad \sum_{i\in I_0} y_i = 1,\\
&\qquad\qquad \sum_{i\in I_k} y_i = z_k, \quad k=1,\dots,m,\\
&\qquad\qquad \sum_{i=1}^{p} a_{ij} y_i = 0, \quad j=1,\dots,n,\\
&\qquad\qquad y_i, z_k \ge 0, \quad i=1,\dots,p,\ k=1,\dots,m.
\end{aligned}$$

The duality gap is zero as long as the primal problem has an interior point (Bazaraa et al. 1993; Duffin and Peterson 1966) for all realizations of interval values. In the following proposition, we give a sufficient condition for this property. Notice that it is an open question whether this condition is also necessary.

Proposition 10 *If the system*

$$\begin{aligned}
\sum_{i\in I_k} \overline{c}_i \prod_{j=1}^{n} y_j^{\overline{a}_{ij}} z_j^{-\underline{a}_{ij}} < 1, \quad k=1,\dots,m,\\
y_j, z_j \ge 1, \quad j=1,\dots,n
\end{aligned} \tag{16}$$

has a solution y^*, z^*, *then the vector* x^* *defined as* $x_j^* = \frac{y_j^*}{z_j^*}$ *is an interior point of the feasible set* (15) *for all* $c_i \in \boldsymbol{c}_i$ *and* $a_{ij} \in \boldsymbol{a}_{ij}$.

Proof Let $c_i \in \boldsymbol{c}_i$ and $a_{ij} \in \boldsymbol{a}_{ij}$. Then

$$\sum_{i\in I_k} c_i \prod_{j=1}^{n}\left(\frac{y_j^*}{z_j^*}\right)^{a_{ij}} \le \sum_{i\in I_k} \overline{c}_i \prod_{j=1}^{n} y_j^{\overline{a}_{ij}} z_j^{-\underline{a}_{ij}} < 1.$$

In other words, $x^* > 0$ is an interior feasible solution for all realizations of the interval data. □

Remark that solvability of the system (16) can be checked effectively by solving the following posynomial geometric program:

$$\begin{aligned}
&\inf \varepsilon\\
&\text{subject to} \quad \sum_{i\in I_k} \overline{c}_i \prod_{j=1}^{n} y_j^{\overline{a}_{ij}} z_j^{-\underline{a}_{ij}} \varepsilon^{-1} \le 1, \quad k=1,\dots,m,\\
&\qquad\qquad y_j^{-1} \le 1, \qquad z_j^{-1} \le 1, \quad j=1,\dots,n,\\
&\qquad\qquad y_j, z_j, \varepsilon > 0, \quad j=1,\dots,n.
\end{aligned}$$

If its optimal solution is less than one, then the system (16) is solvable, otherwise it is not solvable.

In the remainder of this section, we suppose that the duality gap is zero.

4.1 Lower bound

To use Theorem 4 we have to determine the function $f(x)$ and the set M. As the variables x are positive, we immediately have that every parameter c_i attains its lower value $\underline{c}_i$. The parameter a_{ij} attains its lower value $\underline{a}_{ij}$ if $x_j \geq 1$ and the upper value $\overline{a}_{ij}$ if $x_j < 1$. We formulate this as $a_{ij} = (a_{ij})_c - (a_{ij})_\Delta \operatorname{sgn}(\log(x_j))$. Therefore, the function $f(x)$ can be written as

$$f(x) = \sum_{i \in I_0} \underline{c}_i \prod_{j=1}^{n} x_j^{(a_{ij})_c - (a_{ij})_\Delta \operatorname{sgn}(\log(x_j))}. \tag{17}$$

The set M is described by the constraints

$$\begin{aligned} \sum_{i \in I_k} \underline{c}_i \prod_{j=1}^{n} x_j^{(a_{ij})_c - (a_{ij})_\Delta \operatorname{sgn}(\log(x_j))} &\leq 1, \quad k = 1, \dots, m, \\ x_j &> 0, \quad j = 1, \dots, n. \end{aligned} \tag{18}$$

It is an analogy of the Gerlach Theorem (Theorem 1) and holds for the following reason. If x satisfies (15) for some $c_i \in \boldsymbol{c}_i$ and $a_{ij} \in \boldsymbol{a}_{ij}$, then it also satisfies (18) due to the inequality

$$c_i \prod_{j=1}^{n} x_j^{a_{ij}} \geq \underline{c}_i \prod_{j=1}^{n} x_j^{(a_{ij})_c - (a_{ij})_\Delta \operatorname{sgn}(\log(x_j))}, \quad i = 1, \dots, p.$$

Conversely, if x satisfies (18), then it satisfies (15) for the choice $c_i = \underline{c}_i \in \boldsymbol{c}_i$ and $a_{ij} = (a_{ij})_c - (a_{ij})_\Delta \operatorname{sgn}(\log(x_j)) \in \boldsymbol{a}_{ij}$.

Proposition 11 *The lower bound of the optimal value function is computable as follows*

$$\underline{f} = \inf f(x) \quad \textit{subject to} \quad x \in M, \tag{19}$$

where $f(x)$ is described by (17) and M is described by (18).

The optimization problem (19) is not easy to solve in general. What we can do is, for instance, to decompose it into 2^n sub-problems according to the signs of the $\log(x_j)$. Each such sub-problem is a geometric program restricted on an orthant-like subset; cf. Sect. 3.2.

A simpler approach was introduced by Liu (2008), but it does not yield an exact lower bound in general. This is shown in the following example adopted from Liu (2008, Example 2):

$$\inf [1.5, 2] x_1^{[-0.13,-0.09]} x_2^{[1.2,1.5]} x_3^{-1} x_4^{[1.1,1.6]} + [3.5, 4] x_1^{[-1.2,-1]} x_2^{-1} x_3^{[-0.2,-0.1]} x_4^{-1}$$

$$\begin{aligned} \text{subject to} \quad & 2x_1^{-2}x_2^{-1}x_3^2x_4 + [1.2, 1.6]x_1x_3x_4^2 \le [3, 3.2], \\ & [1.7, 1.9]x_1^2x_2^{1.4}x_3x_4 + [2.6, 3.1]x_1^{2.2}x_4 \le [2, 2.4], \\ & x_1, x_2, x_3, x_4 > 0. \end{aligned}$$

According to Liu (2008), the optimal value range is $[4.9271, 8.5429]$. Nevertheless, this result is not correct, as the following instance

$$\begin{aligned} & \inf 1.5x_1^{-0.13}x_2^{1.2}x_3^{-1}x_4^{1.6} + 3.5x_1^{-1.2}x_2^{-1}x_3^{-0.1}x_4^{-1} \\ & \text{subject to} \quad 2x_1^{-2}x_2^{-1}x_3^2x_4 + 1.2x_1x_3x_4^2 \le 3.2, \\ & \qquad\qquad 1.7x_1^2x_2^{1.4}x_3x_4 + 2.6x_1^{2.2}x_4 \le 2.4, \\ & \qquad\qquad x_1, x_2, x_3, x_4 > 0 \end{aligned}$$

yields the optimal solution $x = (35.97, 529.46, 0.0012, 0.0001)^{\mathrm{T}}$ with the corresponding objective value 2.3411.

4.2 Upper bound

Surprisingly, the upper bound of the optimal value function is much easier to compute than the lower bound. As the variables x_j in the primal problem (15) are positive, the upper bound $\overline{f}$ will be achieved for $c_i = \overline{c}_i$. So we set and fix $c_i = \overline{c}_i$. By Theorem 4,

$$\overline{f} = \sup g(y, z) \quad \text{subject to} \quad (y, z) \in N.$$

The function $g(y, z)$ is computable as

$$g(y, z) = \left(\prod_{i=1}^{p}\left(\frac{\overline{c}_i}{y_i}\right)^{y_i}\right) l \left(\prod_{k=1}^{m} z_k^{z_k}\right).$$

Using Theorem 3, the set N is described by

$$\begin{aligned} & \sum_{i \in I_0} y_i = 1, \\ & \sum_{i \in I_k} y_i = z_k, \quad k = 1, \dots, m, \\ & \sum_{i=1}^{p} \overline{a}_{ij} y_i \ge 0, \quad j = 1, \dots, n, \\ & \sum_{i=1}^{p} \underline{a}_{ij} y_i \le 0, \quad j = 1, \dots, n, \\ & y_i, z_k \ge 0, \quad i = 1, \dots, p, k = 1, \dots, m. \end{aligned}$$

Thus, we obtain an optimization problem which is efficiently solvable. Moreover, this model uses fewer variables than the one proposed in Liu (2008).

References

Alefeld G, Herzberger J (1983) Introduction to interval computations. Academic Press, London
Bazaraa MS, Sherali HD, Shetty CM (1993) Nonlinear programming. Theory and algorithms, 2nd edn. Wiley, New York
Boţ RI, Grad SM, Wanka G (2006) Fenchel-Lagrange duality versus geometric duality in convex optimization. J Optim Theory Appl 129(1):33–54
Chinneck JW, Ramadan K (2000) Linear programming with interval coefficients. J Oper Res Soc 51(2):209–220
Dorn WS (1960) Duality in quadratic programming. Q Appl Math 18(2):155–162
Duffin RJ, Peterson EL (1966) Duality theory for geometric programming. SIAM J Appl Math 14(6):1307–1349
Fiedler M, Nedoma J, Ramik J, Rohn J, Zimmermann K (2006) Linear optimization problems with inexact data. Springer, New York
Gerlach W (1981) Zur Lösung linearer Ungleichungssysteme bei Störung der rechten Seite und der Koeffizientenmatrix. Math Oper Stat Ser Optim 12:41–43
Hansen E, Walster GW (2004) Global optimization using interval analysis, 2nd edn. Dekker, New York
Hladík M (2007) Optimal value range in interval linear programming. Fuzzy Optim Decis Mak (Submitted). Available as a technical report KAM-DIMATIA Series 2007-824, Dep of Appl Math, Prague
Hu B, Wang S (2006a) A novel approach in uncertain programming part I: new arithmetic and order relation for interval numbers. J Ind Manage Optim 2(4):351–371
Hu B, Wang S (2006b) A novel approach in uncertain programming. II: a class of constrained nonlinear programming problems with interval objective functions. J Ind Manage Optim 2(4):373–385
Huang GH (1998) A hybrid inexact-stochastic water management model. Eur J Oper Res 107(1):137–158
Huang GH, Baetz BW, Patry GG, Terluk V (1997) Capacity planning for an integrated waste management system under uncertainty: a North American case study. Waste Manage Res 15(5):523–546
Levin VI (1999) Nonlinear optimization under interval uncertainty. Cybern Syst Anal 35(2):297–306
Liu ST (2006a) Computational method for the profit bounds of inventory model with interval demand and unit cost. Appl Math Comput 183(1):499–507
Liu ST (2006b) Posynomial geometric programming with parametric uncertainty. Eur J Oper Res 168(2):345–353
Liu ST (2008) Posynomial geometric programming with interval exponents and coefficients. Eur J Oper Res 186(1):17–27
Liu ST, Wang RT (2007) A numerical solution method to interval quadratic programming. Appl Math Comput 189(2):1274–1281
Martos B (1975) Nonlinear programming. Theory and methods. Akadémiai Kiadó, Budapest
Mráz F (1998) Calculating the exact bounds of optimal values in LP with interval coefficients. Ann Oper Res 81:51–62
Neumaier A (2004) Complete search in continuous global optimization and constraint satisfaction. Acta Numer 13:271–369
Rohn J, Kreslová J (1994) Linear interval inequalities. Linear Multilinear Algebra 38(1/2):79–82
Wu XY, Huang GH, Liu L, Li JB (2006) An interval nonlinear program for the planning of waste management systems with economies-of-scale effects—a case study for the region of Hamilton, Ontario, Canada. Eur J Oper Res 171(2):349–372