Cooperation in dividing the cake

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Abstract This paper defines models of cooperation among players partitioning a completely divisible good (such as a cake or a piece of land). The novelty of our approach lies in the players' ability to form coalitions before the actual division of the good with the aim to maximize the average utility of the coalition. A social welfare function which takes into account coalitions drives the division. In addition, we derive a cooperative game which measures the performance of each coalition. This game is compared with the game in which players start cooperating only after the good has been portioned and has been allocated among the players. We show that a modified version of the game played before the division outperforms the game played after the division.

1 Introduction

The problem of dividing a non-homogeneous cake among children with subjective likes has gone a long way from the first pioneering works of Banach, Knaster, Stein-

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haus (cited in Brams and Taylor 1996) and Dubins and Spanier (1961) to reach the status of an independent field of research, named *fair division* theory. An overview of the advances in this topic can be found in Hill (1993), Brams and Taylor (1996), and Brams (2008).

The attention of most authors in the field has been focused on the design of simple procedures to achieve a satisfactory division, and the classification of the various and often conflicting optimality criteria. Less urgent, and therefore less developed—but by no means less important—is the analysis of the players' strategic concerns. Most approaches to fair division require the adherence of the players to a procedure, usually under the supervision of a referee. This leaves little freedom to the players, whose strategic behavior is usually limited to actions, such as cutting (a portion of) the cake according to a specified ratio in their own preferences, choosing a part of the cake among many, or instructing the referee about their likes and dislikes. All these actions deal with the revelation of the personal preferences by the players. Truthfulness is not usually guaranteed in many procedures, and the attention of some authors has been focused on designing strategy-proof procedures that encourage players in revealing their true preferences, or, conversely, on showing the impossibility of this effort (see Tadenuma and Thomson 1995; Brams and Taylor 1996).

Beyond the mere process of division, players may engage in other strategic actions: for instance, they may exchange parts of the slice they have been assigned, or they may compensate a player who gives up a part of her fair share for the sake of the whole coalition. Traditionally, this model has been associated with the division of a piece of land among heirs (instead of a cake among children) who have already received their share of inheritance and look for a better allocation of the whole lot of land, but the mathematical structure underlying the problem remains essentially unaltered. In Berliant (1985), in Berliant et al. (1992), and in Berliant and Dunz (2004), it is shown that heirs trade their endowments knowing that there exists an equilibrium allocation which belongs to the core of an NTU game, implying that such allocation cannot be improved upon by a further redistribution among groups of heirs. In these works, great care is devoted to the formal description of the preferences, which departs from the classical measure-theoretic settings in fair division to take into account non-additivity and the preference for shape and location of the plots.

Legut (1990) defines a model of cooperation within the classical framework of fair division. Similarly to the model for land division, children attending the division of the cake may redistribute the total amount of cake within a group in order to maximize the joint utility of the group. This time side payments are allowed and a TU game is defined. The same model is also considered by Legut et al. (1994) who characterize the TU game and its core in great detail.

The models reviewed so far introduce cooperation by exchanges within the context of fair division. As a matter of fact, however, this results in the juxtaposition of two stages of activity which are separate both in time and in goals. At first, each child on his own (with no hindsight for future cooperation) attends the division of the cake. Then, once the division is completed, they may turn to the other players in search for a mutually better arrangement. As it is presented, this is a division procedure, followed by a model of exchange economy where the slices of the cake (or the plots of land) represent the agents' initial endowments. There is no interaction of sorts between the

two stages, so that in the trading phase any division, no matter how unfair, would be considered just fine. The discrepancy was already noted by Legut et al. (1994) who kept the name of "fair division game" for the exchange model they analyzed, but noted in a footnote that

> "this name does not seem to be very appropriate in the present situation but in Legut (1990) this term has been introduced for games of this kind where the initial endowment [...] was a result of a fair division process. Since we are studying the same games it is not sensible to change the name."

Here we propose models in which the two activities of dividing the cake and cooperating among children are merged together. At the beginning of the procedure and *before* the cake is actually cut, children may form coalitions. Each coalition acts as a single player in the division by reclaiming a share proportional to the cardinality of the coalition, and distributing the share of the cake in the most efficient way within the coalition, i.e., giving each crumble of the cake to the player in the coalition who appreciates it most. A fair evaluation of the conflicting interests between competing coalitions is taken care of by means of a *maximin* social welfare function, which is widely used in the fair division literature.

We point out that, as a result of this procedure, while coalitions as a whole will be treated fairly, the allocation within a single coalition may turn out to be extremely unfair to single members of the coalition. Fairness at this level will be restored by means of side payments according to the principles of TU-models: the overall payoff of each player (inclusive of the side payments) should be high enough to discourage single players or subgroups to leave the coalition. In particular, we will be interested in payments that make the grand coalition formed by all players stable. Such payments belong to the core of the cooperative game.

The use of transferable utility is inherited from Legut (1990). Such assumption is often criticized in the classical context of fair division where children attending the division are not supposed to handle money. We have already noted, however, that the domain of fair division theory is very large and extends to applications such as the division of land, where money compensations are more common. Also, Legut et al. (1995) connect situations where side payments are allowed (TU-models) to situations where such transfers are not allowed (NTU-models) by means of a result (Theorem 5) that guarantees the existence of equilibrium payoffs in one setting whenever an equilibrium payoff exists in the other setting.

We will make two proposals for the definition of cooperative games associated to the combined model of exchange and division, emphasizing the importance of picking the right weights for the individuals, as well as for the coalitions that they form.

Section 2 recalls the mathematical framework of fair division and introduces a couple of essential assumptions. Section 3 gives a formal description of the two opposite attitudes which the players may put on: competition and cooperation, and describes how to mix the two attitudes in a single model. Furthermore, we define a cooperative game arising from cooperation in fair division before the division of the cake takes place, and show that this game is superadditive, balanced and possesses population monotonic allocation schemes. This game is compared in Sect. 4 with the cooperative

game introduced by Legut (1990), in which players have initial endowments arising from a process of fair division. To our surprise, it may be convenient for a group of players to form a coalition after the division has taken place, rather than participating to the division as a single coalition from the start. We also propose a modification of the game defined in Sect. 3, which outperforms the corresponding game of cooperation after division. We conclude with some remarks in Sect. 5.

2 Basic definitions and assumptions

A cake $X \subset \Re^n$ is to be divided among n players (children). Let $N = \{1, 2, \ldots, n\}$ denote the set of players. Each μ_i $(i = 1, 2, \ldots, n)$ is a probability measure on $(X, \mathcal{B}(X))$, $\mathcal{B}(X)$ being the Borel sets in X. For each $A \in \mathcal{B}(X)$, $\mu_i(A)$ measured on the unit scale tells us how much player i likes slice A. Throughout this work we will require some assumptions to hold. The first one guarantees the complete divisibility of the cake.

(A) *Atomless preferences* Each μ_i does not contain atoms: If $\mu_i(A) > 0$, then there exists a measurable $B \subset A$ such that $\mu_i(A \cap B) > 0$ and $\mu_i(A \cap B^c) > 0$.

Each μ_i is absolutely continuous w.r.t. $\nu = \sum_i \mu_i / n$. Consequently, by the Radon–Nikodym theorem, each μ_i admits a density function f_i w.r.t. ν such that

$$\mu_i(A) = \int_{A_i} f_i \, d\nu \quad \text{for every } A \in \mathcal{B}(X).$$

As a special case, the preferences may be absolutely continuous w.r.t. the Lebesgue measure λ, which therefore replaces ν in the above formula. In such a case, (A) holds and each density function gives a pointwise description of the corresponding player.

Another useful assumption requires the players to share a common interest to the same parts of the cake (though the liking may vary from player to player).

(B) *Common support* If $\mu_i(A) > 0$ for some i and $A \in \mathcal{B}(X)$, then $\mu_j(A) > 0$ for every other $j \neq i$.

The cake X will be partitioned into n measurable sets $(A_1, A_2, \ldots, A_n)$. The set of all measurable n-partitions of X is denoted as Π_n.

The main purpose of fair division is to find a partition $(A_1, A_2, \ldots, A_n) \in \Pi_n$ and assign "slice" A_i to player i, who will evaluate it $\mu_i(A_i)$. The goal is then to find a "good" partition that keeps the values $\mu_i(A_i)$ as high and even as possible.

A partition $(A_1, A_2, \ldots, A_n) \in \Pi_n$ is *equitable* if

$$\mu_1(A_1) = \mu_2(A_2) = \cdots = \mu_n(A_n), \tag{1}$$

and it is *fair* if

$$\mu_i(A_i) \geq \frac{1}{n} \quad \text{for every } i \in \{1, \ldots, n\}. \tag{2}$$

Since all players assign value 1 to the whole cake and they are all treated equally, it makes sense to assign each one a part which is worth at least one nth of the whole cake.

In fair division, players are usually treated equal, since it is assumed that they have equal rights over the cake. If players have different entitlements, this is usually managed by means of different weights $w_i \geq 0$, $i \in N$, associated to the players. The notions of fairness and equitability could be adjusted by dividing each $\mu_i(A_i)$ by w_i in (1) and by replacing $1/n$ with $w_i / \sum_{i \in N} w_i$ in (2). For what follows, we consider the case of equal entitlements, but we keep in mind that, when it comes to comparing coalitions in place of single players, we will have to resort to weights taking into account the cardinality of the coalitions.

The following set,

$$\mathcal{D} = \left\{ \big(\mu_1(A_1), \mu_2(A_2), \ldots, \mu_n(A_n) \big) : (A_1, \ldots, A_n) \in \Pi_n \right\},$$

is called the *allocation range*. It plays a central role in many well-known results of fair division theory and the present work will be no exception. The importance of the allocation range can be explained by its properties, as Proposition 2.1 states.

Proposition 2.1 (Lyapunov 1940; Dubins and Spanier 1961) $\mathcal{D}$ *is a compact subset of* $\Re^n$. *Moreover, if* (A) *holds, then* $\mathcal{D}$ *is also convex.*

3 Competition and cooperation

An important issue in fair division theory regards the existence and construction of an allocation which enjoys one or more desirable properties such as fairness or envy-freeness. Usually, uniqueness of such allocation is not guaranteed, and often a whole class of such allocations can be identified. To reduce the cardinality of such class of allocations, the use of a social welfare function may come handy. The choice of such function depends on the circumstances under which the division takes place. Dubins and Spanier (1961) define two such functions related to optimization problems corresponding to two types of players' behavior:[1]

Complete competition Each player is assigned a part of the cake and no further action is possible. Therefore, an allocation is sought with players' values as high and equitable as possible. This can be achieved by maximizing the utility of the least well-off player, via a *maximin* allocation

$$u_m = \sup \left\{ \min_{i=1,\ldots,n} \mu_i(A_i) : (A_1, \ldots, A_n) \in \Pi_n \right\}. \tag{3}$$

Dubins and Spanier (1961) show in Corollary 6.10 that the supremum is always attained. Therefore "sup" can be replaced by "max" in the above definition. In the optimization problems that follow, the same reasoning applies, and we will consider the maximizing partitions.

It is easy to verify that, when (A) holds, a maximin allocation is fair. Furthermore, if also (B) holds, Dubins and Spanier note that the allocation is equitable.

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[1] Dubins and Spanier define these two optimization problems—but do not attach any meaning to them in terms of competition or cooperation.

Complete cooperation Suppose now that after the allocation players are allowed to transfer money to other players. Players are therefore likely to agree on an allocation which maximizes their joint utility, as expressed by the sum, and compensate the less fortunate players by means of side payments. The problem is now to find a partition that maximizes the average utility, i.e., find

$$u_p = \max\left\{ \frac{\sum_{i \in N} \mu_i(A_i)}{n} : (A_1, \ldots, A_n) \in \Pi_n \right\}. \tag{4}$$

The allocation is equitable by construction, and it is fairer, on average, than the previous one, since

$$u_p \geq u_m. \tag{5}$$

The inequality holds since, for any vector $(x_1, \ldots, x_n) \in \mathcal{D}$,

$$\min_{i \in N} x_i \leq \frac{\sum_{i \in N} x_i}{n}.$$

Maximizing both sides over $\mathcal{D}$ yields the result.

Dubins and Spanier (1961) show that

$$u_p = \frac{\int_X f \, dv}{n}, \tag{6}$$

where $f = \max_i f_i$. Moreover, they exhibit a maximizing allocation

$$A_1 = \{x \in X : f_1(x) = f(x)\};$$
$$A_j = \{x \in X : f_h(x) < f(x) \text{ for } h < j, \ f_j(x) = f(x)\}, \quad j = 2, \ldots, n.$$

In what follows, we study a class of intermediate situations between the two cases listed above and we address the question: What happens when players form several competing groups and are allowed to transfer money only within the coalition they belong to? Within each coalition, players will agree on maximizing their joint utility. The maximim social welfare function will take into account the average utility of each coalition.

Let $\mathcal{S} = \{S_1, \ldots, S_h\}$ be a partition of N. A partition $\mathcal{S}' = \{S'_1, \ldots, S'_{h'}\}$ is *finer* than the partition $\mathcal{S}'' = \{S''_1, \ldots, S''_{h''}\}$ (and $\mathcal{S}''$ is *coarser* than $\mathcal{S}'$) if for each $S'_i \in \mathcal{S}'$ there exists $S''_j \in \mathcal{S}''$ such that $S'_i \subset S''_j$.

Now, assume that players cluster into the coalitions specified by the partition $\mathcal{S} = \{S_1, \ldots, S_h\}$. In this situation, we recommend an allocation satisfying

$$u(S_1, \ldots, S_h) = \max\left\{ \min_{j=1,\ldots,h} \left\{ \frac{\sum_{i \in S_j} \mu_i(A_i)}{|S_j|} \right\} : (A_1, \ldots, A_n) \in \Pi_n \right\}. \tag{7}$$

The cases of complete competition and complete cooperation, respectively, are included as special cases, since

$$u_m = u(\{1\}, \ldots, \{n\}) \quad \text{and} \quad u_p = u(N).$$

For a given coalition structure $\mathcal{S} = \{S_1, \ldots, S_h\}$, $u(S_1, \ldots, S_h)$ can be interpreted as the minimal average utility that each coalition is bound to receive. If $(A_1^*, \ldots, A_n^*)$ attains the maximin value in (7), then

$$\frac{\sum_{i \in S_j} \mu_i(A_i^*)}{|S_j|} \geq u(S_1, \ldots, S_h) \quad \text{for each } j = 1, \ldots, h.$$

It must be noted that, in the same context, it is *not* guaranteed that $\mu_i(A_i^*) \geq u(S_1, \ldots, S_h)$ for every $i \in N$.

We now turn to an alternative interpretation for the value defined by (7).

Proposition 3.1 *For any coalition structure $\mathcal{S} = \{S_1, \ldots, S_h\}$,*

$$u(S_1, \ldots, S_h) = \max \left\{ \min_{j=1,\ldots,h} \left\{ \frac{\mu_{S_j}(B_j)}{|S_j|} \right\} : (B_1, \ldots, B_h) \in \Pi_h \right\}, \quad (8)$$

where for each $B \in \mathcal{B}(X)$ and $j = 1, \ldots, h$

$$\mu_{S_j}(B) = \int_B f_{S_j} \, d\nu \quad \text{with } f_{S_j}(x) = \max_{i \in S_j} f_i(x).$$

Proof The main ideas for this proof are derived from Theorem 2 in Dubins and Spanier (1961).

Denote with $\tilde{u}$ the right-hand side in (8). For any partition $(A_1, \ldots, A_n) \in \Pi_n$ and the given coalition structure $\{S_1, \ldots, S_h\}$ define a partition $(B_1^*, \ldots, B_h^*)$ in Π_h by $B_j^* = \bigcup_{i \in S_j} A_i$, $j = 1, \ldots, h$. The following inequality holds

$$\sum_{i \in S_j} \mu_i(A_i) = \sum_{i \in S_j} \int_{A_i} f_i \, d\nu \leq \sum_{i \in S_j} \int_{A_i} f_{S_j} \, d\nu = \int_{B_j^*} f_{S_j} \, d\nu = \mu_{S_j}(B_j^*).$$

Consequently,

$$\min_{j=1,\ldots,h} \left\{ \frac{\sum_{i \in S_j} \mu_i(A_i)}{|S_j|} \right\} \leq \min_{j=1,\ldots,h} \left\{ \frac{\mu_{S_j}(B_j^*)}{|S_j|} \right\}.$$

Take the supremum over the two classes of partitions Π_n and Π_h to obtain

$$u(S_1, \ldots, S_h) \leq \tilde{u}.$$

Following the same lines of Corollary 6.10 in Dubins and Spanier (1961), we can show that the optimal value $\tilde{u}$ is attained by some $(\tilde{B}_1, \ldots, \tilde{B}_h) \in \Pi_h$. We now show that, for each $j = 1, \ldots, h$, there exists a partition $(\tilde{A}_i)_{i \in S_j}$ of $\tilde{B}_j$ (and therefore a partition of the whole space) such that

$$\sum_{i \in S_j} \mu_i(\tilde{A}_i) = \mu_{S_j}(\tilde{B}_j). \quad (9)$$

In fact, write $S_j = \{i_1, \ldots, i_r\}$ and let $\tilde{A}_{i_p}$ ($p = 1, \ldots, r$) be the subset of $\tilde{B}_j$ where $f_{i_\ell}(x) < f_{S_j}(x)$ for $\ell < p$ and $f_{i_p}(x) = f_{S_j}(x)$. Then, $(\tilde{A}_{i_1}, \ldots, \tilde{A}_{i_r})$ is a measurable partition of $\tilde{B}_j$ with

$$\sum_{\ell=1,\ldots,r} \int_{\tilde{A}_{i_\ell}} f_{i_\ell}\, dv = \sum_{\ell=1,\ldots,r} \int_{\tilde{A}_{i_\ell}} f_{S_j}\, dv = \int_{\tilde{B}_j} f_{S_j}\, dv = \mu_{S_j}(\tilde{B}_j).$$

From (9) we derive

$$u(S_1, \ldots, S_h) \geq \min_{j=1,\ldots,h} \left\{ \frac{\sum_{i \in S_j} \mu_i(\tilde{A}_i)}{|S_j|} \right\} = \min_{j=1,\ldots,h} \left\{ \frac{\mu_{S_j}(\tilde{B}_j)}{|S_j|} \right\} = \tilde{u},$$

which completes the proof. □

Proposition 3.1 suggests an alternative interpretation: players who coalesce into S_j state their joint preferences as μ_{S_j} and participate in a maximin division with the competing coalitions. Each coalition is given a weight which is proportional to the cardinality of the group.

Inequality (5) shows that moving from complete competition to complete cooperation is beneficial on average to the players. This improvement carries on to the intermediate situations as well: merging subcoalitions into larger ones improves the average value of the division.

Proposition 3.2 *If* $\mathcal{S}' = \{S_1', \ldots, S_{h'}'\}$ *is finer than* $\mathcal{S}'' = \{S_1'', \ldots, S_{h''}''\}$, *then*

$$u(S_1', \ldots, S_{h'}') \leq u(S_1'', \ldots, S_{h''}''). \tag{10}$$

Proof Since $\mathcal{S}'$ is finer than $\mathcal{S}''$, each $S_j'' \in \mathcal{S}''$ is partitioned into elements of $\mathcal{S}'$, say $S_j'' = \{S_1', \ldots, S_q'\}$.

For any $(x_1, \ldots, x_n) \in \mathcal{D}$, the following holds:

$$\min_{\ell=1,\ldots,q} \left\{ \frac{\sum_{i \in S_\ell'} x_i}{|S_\ell'|} \right\} \leq \frac{|S_1'|\left(\frac{\sum_{i \in S_1'} x_i}{|S_1'|}\right) + \cdots + |S_q'|\left(\frac{\sum_{i \in S_q'} x_i}{|S_q'|}\right)}{|S_1'| + \cdots + |S_q'|} = \frac{\sum_{i \in S_j''} x_i}{|S_j''|}.$$

The inequality is preserved once we minimize over all $S_j'' \in \mathcal{S}''$ and then again maximize this result over all $(x_1, \ldots, x_n) \in \mathcal{D}$. □

As a straightforward consequence, for any coalition structure $\mathcal{S} = \{S_1, \ldots, S_h\}$, a division attaining (7) is fair on average, since

$$u(S_1, \ldots, S_h) \geq u(\{1\}, \ldots, \{n\}) \geq \frac{1}{n}.$$

The first inequality derives from Proposition 3.2, while the last one is a consequence of the fairness of the maximin allocation reaching (3) in the complete competitive setting.

Our maximin objective function controls the average value of the least well-off coalition. The other coalitions will get on average at least as much as that coalition, but little is known, in general, about their value. If the preferences have common support, however, all players are treated equal on average.

Proposition 3.3 *If $(A_1^*, \ldots, A_n^*)$ is a maximin partition with respect to* (7) *for a given coalition structure $\mathcal{S} = (S_1, \ldots, S_h)$, then*

$$\frac{\sum_{i \in S_1} \mu_i(A_i^*)}{|S_1|} = \cdots = \frac{\sum_{i \in S_h} \mu_i(A_i^*)}{|S_h|}.$$

Therefore, all players get the same average value, no matter what coalitions they belong to.

Proof First of all, it is easy to verify that since $\mathcal{D}$ is a convex subset of $\Re^n$ and by virtue of the Lyapunov theorem, the set

$$\mathcal{H} = \left\{ \left(\frac{\sum_{i \in S_1} x_i}{|S_1|}, \ldots, \frac{\sum_{i \in S_h} x_i}{|S_h|} \right) : (x_1, \ldots, x_n) \in \mathcal{D} \right\}$$

is also a convex subset of $\Re^h$.

Now, assume that for some coalition structure $\mathcal{S} = \{S_1, \ldots, S_h\}$ the partition $(A_1^*, \ldots, A_n^*)$ attaining the maximin value in (7) is not equitable and, say,

$$\frac{\sum_{i \in S_1} \mu_i(A_i^*)}{|S_1|} > \frac{\sum_{i \in S_j} \mu_i(A_i^*)}{|S_j|} \geq \frac{1}{n} \quad \text{for } j = 2, \ldots, h.$$

Denote $\tilde{A} = \bigcup_{i \in S_1} A_i^*$. Since $\sum_{i \in S_1} \mu_i(A_i^*) > 0$, then $\mu_{p(1)}(\tilde{A}) \geq \mu_{p(1)}(A_{p(1)}^*) > 0$ for some $p(1) \in S_1$.

For each $j = 2, \ldots, h$, take a player $p(j) \in S_j$, fix ε, $0 < \varepsilon < 1$, and consider the following convex combination of elements in $\mathcal{H}$

$$(1 - \varepsilon) \left(\frac{\sum_{i \in S_1} \mu_i(A_i^*)}{|S_1|}, \frac{\sum_{i \in S_2} \mu_i(A_i^*)}{|S_2|}, \frac{\sum_{i \in S_3} \mu_i(A_i^*)}{|S_3|}, \ldots, \frac{\sum_{i \in S_h} \mu_i(A_i^*)}{|S_h|} \right)$$

$$+ \frac{\varepsilon}{h-1} \left(0, \frac{\sum_{i \in S_2} \mu_i(A_i^*) + \mu_{p(2)}(\tilde{A})}{|S_2|}, \frac{\sum_{i \in S_3} \mu_i(A_i^*)}{|S_3|}, \ldots, \frac{\sum_{i \in S_h} \mu_i(A_i^*)}{|S_h|} \right)$$

$$+ \frac{\varepsilon}{h-1} \left(0, \frac{\sum_{i \in S_2} \mu_i(A_i^*)}{|S_2|}, \frac{\sum_{i \in S_3} \mu_i(A_i^*) + \mu_{p(3)}(\tilde{A})}{|S_3|}, \ldots, \frac{\sum_{i \in S_h} \mu_i(A_i^*)}{|S_h|} \right)$$

$$\cdots$$

$$+ \frac{\varepsilon}{h-1} \left(0, \frac{\sum_{i \in S_2} \mu_i(A_i^*)}{|S_2|}, \frac{\sum_{i \in S_3} \mu_i(A_i^*)}{|S_3|}, \ldots, \frac{\sum_{i \in S_h} \mu_i(A_i^*) + \mu_{p(h)}(\tilde{A})}{|S_h|} \right)$$

$$= \left((1 - \varepsilon) \frac{\sum_{i \in S_1} \mu_i(A_i^*)}{|S_1|}, \frac{\sum_{i \in S_2} \mu_i(A_i^*)}{|S_2|} + \frac{\varepsilon \mu_{p(2)}(\tilde{A})}{(h-1)|S_2|}, \right.$$

$$\left. \frac{\sum_{i \in S_3} \mu_i(A_i^*)}{|S_3|} + \frac{\varepsilon \mu_{p(3)}(\tilde{A})}{(h-1)|S_3|}, \ldots, \frac{\sum_{i \in S_h} \mu_i(A_i^*)}{|S_h|} + \frac{\varepsilon \mu_{p(h)}(\tilde{A})}{(h-1)|S_h|} \right).$$

Since $\mathcal{H}$ is convex, there exists a partition $(\bar{A}_1, \ldots, \bar{A}_n)$ with the same values for the players as the right-hand side term in the above equality. Since $\mu_{p(1)}(\tilde{A}) > 0$, by assumption (B), also $\mu_{p(j)}(\tilde{A}) > 0$ for every $j = 2, \ldots, h$ and, for ε positive close to 0, we have exhibited a partition with a better maximin value than $(A_1^*, \ldots, A_n^*)$ for the coalition structure $\mathcal{S}$. This is in contradiction with the previous assumptions. □

The partition maximizing (7) depends on the whole coalition structure. We change the perspective and look at the division from the point of view of a single coalition $S \subset N$.

Players may want to explore the advantage of joining a particular coalition S, independently of the behavior of the players outside that coalition. By Propositions 3.2 and 3.3 we know that players in S will get at least the value of the coalition structure defined when all the players outside S decide not to cooperate. Consequently, we propose the following value for coalition S:

$$v(S) = |S|u\big(S, \{j\}_{j \notin S}\big) \quad \text{for } S \subset N, \tag{11}$$

i.e., the minimal value that the coalition S as a whole is bound to receive when the coalition is formed, irrespective of the behavior of the other players. The function is suitable for analysis in a cooperative game theoretical setting.

Proposition 3.4 *The function v defines a superadditive game.*

Proof First of all, we note that the empty coalition has value zero

$$v(\emptyset) = |\emptyset|u\big(\emptyset, \{j\}_{j \in N}\big) = 0.$$

Next, we consider S_1, S_2, disjoint subsets of N. Then,

$$\begin{aligned}
v(S_1 \cup S_2) &= |S_1 \cup S_2|u\big(S_1 \cup S_2, \{j\}_{j \notin S_1 \cup S_2}\big) \\
&= |S_1|u\big(S_1 \cup S_2, \{j\}_{j \notin S_1 \cup S_2}\big) + |S_2|u\big(S_1 \cup S_2, \{j\}_{j \notin S_1 \cup S_2}\big) \\
&\geq |S_1|u\big(S_1, \{j\}_{j \notin S_1}\big) + |S_2|u\big(S_2, \{j\}_{j \notin S_2}\big) = v(S_1) + v(S_2).
\end{aligned}$$

The inequality is motivated by Proposition 3.2, since $\{S_1 \cup S_2, \{j\}_{j \notin S_1 \cup S_2}\}$ is coarser than both $\{S_1, \{j\}_{j \notin S_1}\}$ and $\{S_2, \{j\}_{j \notin S_2}\}$. □

In the cooperative game just defined, players are encouraged to form the grand coalition N since the equal-share vector belongs to the core of v. The same equal share principle can be applied to the smaller coalitions to provide a Population Monotonic Allocation Scheme (PMAS, see Sprumont 1990).

Proposition 3.5 *The game v has non-empty core and admits a PMAS.*

Proof We show that the equal share vector

$$\left(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n}\right)$$

belongs to the core of v. To prove that a reward vector $(x_1, \ldots, x_n)$ is in the core of a game, we need to show that

$$
\text{(i)} \quad \sum_{i \in S} x_i \geq v(S) \quad \text{for each } S \in 2^N \setminus \{\emptyset\};
$$

$$
\text{(ii)} \quad \sum_{i \in N} x_i = v(N).
$$

To prove (i) for the vector $(\frac{v(N)}{n}, \ldots, \frac{v(N)}{n})$ consider

$$
\sum_{i \in S} x_i = \sum_{i \in S} \frac{v(N)}{n} = \sum_{i \in S} u(N) = |S| u(N) \geq |S| u\big(S, \{j\}_{j \notin S}\big) = v(S),
$$

where the inequality holds by Proposition 3.2. Statement (ii) is trivial.

In a similar fashion we show that, for each non-empty $S \subset N$, the payoff vector $(x_{S,i})_{i \in S}$, with $x_{S,i} = v(S)/|S|$, $i \in S$, generates a PMAS for v, since it is easy to verify that

$$
\sum_{i \in S} x_{S,i} = v(S) \quad \text{for every non-empty } S \subset N;
$$

$$
x_{S,i} = \frac{v(S)}{|S|} \leq \frac{v(T)}{|T|} = x_{T,i} \quad \text{whenever } i \in S \subset T. \qquad \square
$$

4 Cooperation after the division versus cooperation before the division

We now consider a two-stage model. At first, players take part into a fully competitive scheme, and receive their share of the cake. Afterwards, they may trade parts of their slices with other players for mutual benefit. A formal model for the cooperative behavior of players who already own slices of the cake and exchange their endowments was first examined in Legut (1990) and Legut et al. (1994) in the context of economies with land. In that setting, the players' endowments are arbitrary as long as they form a partition of X. Here, we specify that players just took part in a competitive maximin division (3). Denote as $(A_1^m, \ldots, A_n^m)$ the resulting maximin partition. A coalition $S \subset N$ of players will redistribute the total wealth of the players in the coalition, $A^m(S) = \bigcup_{i \in S} A_i^m$, to maximize the joint utility. The following *post-division game* (posterior to the actual division) can be therefore defined by

$$
v_{\text{post}}(S) = \sup \left\{ \sum_{i \in S} \mu_i(C_i) \big| \{C_i\}_{i \in S} \text{ is a partition of } A^m(S) \right\}
$$

for each $S \subset N$.

Legut et al. (1994) show that

$$
v_{\text{post}}(S) = \int_{A^m(S)} f_S \, dv = \mu_S\big(A^m(S)\big) \quad \text{for each } S \subset N. \tag{12}
$$

More importantly, the same authors give a two-fold characterization of this game: (i) as a sum of n different games, each one defined on the initial endowment of the single players, and (ii) as a linear combination of simple games characterizing information market games with one informed player.

The multiple characterization allows a fairly detailed description of the core of the game. In particular, it is non-empty and each element belonging to a specific non-empty subset of the core can be extended to a PMAS. For more details we refer to Legut et al. (1994).

Here, we are interested in the relationship between the game v defined in the previous section, in which cooperation occurs before the division of the cake, and the game, which we denote here by v_{post}, where cooperation takes place after the division of the cake. The two games coincide in the extreme cases of complete competition, where by definition

$$v(\{i\}) = v_{\text{post}}(\{i\}), \quad i \in N,$$

and that of total cooperation, i.e.,

$$v(N) = v_{\text{post}}(N),$$

equality which holds by virtue of the definitions of the games v and v_{post} and the results (6) and (12).

In the game v, players cooperate at an earlier stage than in the game v_{post}, and before the partition is actually performed. Thus, when earlier agreements are allowed, the optimal division of the cake takes into account the coalitions that have already formed. In the v_{post} game, conversely, cooperation comes into play only after the cake has already been divided. Therefore, one would expect that the earlier the cooperation occurs, the better a coalition will perform, and the game v yields values at least as high as those of v_{post}. Quite surprisingly, this is not always the case, as the following counterexample shows.

Example 4.1 Consider $X = [0, 3]$ and the preferences of three players defined by the following density functions:

$$f_1(x) = 0.3I_{[0,1)}(x) + 0.4I_{[1,2)}(x) + 0.3I_{[2,3]}(x),$$
$$f_2(x) = 0.2I_{[0,1)}(x) + 0.3I_{[1,2)}(x) + 0.5I_{[2,3]}(x),$$
$$f_3(x) = 0.4I_{[0,1)}(x) + 0.3I_{[1,2)}(x) + 0.3I_{[2,3]}(x),$$

where $I_{[a,b)}$ is the indicator function of the interval $[a, b)$. The values of v and v_{post} can be computed by means of simple linear programs (Table 1).

The trouble with the given definition of v lies in one of its apparent strength: average equitability, assessed by Proposition 3.3. For a given coalition structure $\{S, \{i\}_{i \notin S}\}$ every player gets on average the same value, $u(S, \{i\}_{i \notin S})$, whether or not he belongs to the coalition S. Suppose now that no coalition is formed before the division, and after the division, exchanges are allowed only among players in the

Table 1 Values of v and v_{post} in Example 4.1

S	v	v_{post}
$\{i\}, i = 1, 2, 3$	0.4231	0.4231
$\{1, 2\}$	0.86	0.8615
$\{1, 3\}$	0.8462	0.8462
$\{2, 3\}$	0.86	0.8615
$\{1, 2, 3\}$	1.3	1.3

coalition S. Clearly, the coalition S will receive $v_{\text{post}}(S)$, while each player i outside S will receive $v_{\text{post}}(\{i\})$. Now, it is easy to verify that

$$\frac{v_{\text{post}}(S)}{|S|} \geq v_{\text{post}}(\{i\}), \tag{13}$$

with strict inequality whenever the slice received by some player in S contains a part which is strictly more valuable for some other player in the same coalition. In other terms, the resulting allocation will not be equitable, with players in the coalition S taking advantage of exchanges internal to the coalition.

Therefore, any coalition $S \subset N$ deciding to cooperate before the division will expect an advantage over players outside S proportional to the bonus they would get if they cooperated after the division. Accordingly, we can modify the weights of the coalitions in v by replacing the cardinality of each coalition with the corresponding value of the v_{post} game. For each $S \subset N$, define the *pre-division game* (precedent to the actual division) as

$$v_{\text{pre}}(S) = v_{\text{post}}(S) \max \left\{ \min \left\{ \frac{\sum_{i \in S} \mu_i(A_i)}{v_{\text{post}}(S)}, \frac{\mu_j(A_j)}{v_{\text{post}}(\{j\})}_{j \notin S} \right\} : (A_1, \ldots, A_n) \in \Pi_n \right\} \tag{14}$$

with the convention

$$v_{\text{pre}}(\emptyset) = v_{\text{post}}(\emptyset) = 0. \tag{15}$$

The game v_{pre} makes sure that when the players in S coalesce before the cake is cut, they maintain the same advantage over the players outside S, illustrated by (13), that they would get if they coalesced after the division.

The games v and v_{pre} differ in the system of weights contrasting the players in S to those outside S. Some features of v_{pre}, however, are inherited from those of v with little effort.

Proposition 4.2 *If $(A_1^*, \ldots, A_n^*) \in \Pi_n$ belongs to argmax with respect to (14), then*

$$\frac{\sum_{i \in S} \mu_i(A_i^*)}{v_{\text{post}}(S)} = \frac{\mu_j(A_j^*)}{v_{\text{post}}(\{j\})} \quad \textit{for all } j \notin S.$$

Moreover, we can write

$$v_{\text{pre}}(S) = v_{\text{post}}(S) \max \left\{ \min \left\{ \frac{\mu_S(B_S)}{v_{\text{post}}(S)}, \frac{\mu_j(B_j)}{v_{\text{post}}(\{j\})}_{j \notin S} \right\} : \right.$$
$$\left. \left(B_S, B_j (j \notin S) \right) \in \Pi_{|S^c|+1} \right\}.$$

Proof Repeat the proofs of Propositions 3.3 and 3.1 with the modified system of weights. □

We now turn to the properties of v_{pre}.

Proposition 4.3 *The characteristic function v_{pre} defines a monotonic game.*

Proof In the definition (14), the factor $v_{\text{post}}(S)$ is a constant, and can be distributed among the terms composing the maximin objective. Thus, we can write

$$v_{\text{pre}}(S) = \max_{(A_1,\dots,A_n)\in\Pi_n} \min \left\{ \sum_{i\in S} \mu_i(A_i), v_{\text{post}}(S) \frac{\mu_j(A_j)}{v_{\text{post}}(\{j\})}_{j\notin S} \right\}. \tag{16}$$

Further, by (15), v_{pre} is a characteristic function. Now, take $S, T \subset N$ with $S \subset T$ and consider $(A_1^*, \dots, A_n^*) \in \Pi_n$, a partition that attains the maximin value defining $v_{\text{pre}}(S)$ in (16). We can write

$$v_{\text{pre}}(T) \geq \min \left\{ \sum_{i\in T} \mu_i(A_i^*), v_{\text{post}}(T) \frac{\mu_j(A_j^*)}{v_{\text{post}}(\{j\})}_{j\notin T} \right\}$$
$$\geq \min \left\{ \sum_{i\in S} \mu_i(A_i^*), v_{\text{post}}(S) \frac{\mu_j(A_j^*)}{v_{\text{post}}(\{j\})}_{j\notin S} \right\} = v_{\text{pre}}(S).$$

The second inequality is justified by the fact that $\sum_{i\in T\setminus S} \mu_i(A_i^*) \geq 0$ and $v_{\text{post}}(T) \geq v_{\text{post}}(S)$. □

Next, we show that v_{pre} overcomes the difficulties associated with v.

Proposition 4.4 *The game v_{pre} dominates v_{post} in the sense that for each $S \subset N$,*

$$v_{\text{pre}}(S) \geq v_{\text{post}}(S) \tag{17}$$

holds, with a strict equality sign holding for the extreme cases $S = N$ and $S = \{i\}$, $i \in N$.

Proof Take $i \in N$. When $S = \{i\}$, $v_{\text{post}}(\{i\}) = v_{\text{post}}(\{j\})$, $j \neq i$. Let $(A_1^m, \dots, A_n^m)$ be a maximin partition achieving (3). Then, the same partition attains the maximin value defining $v_{\text{pre}}(\{i\})$. In fact,

$$v_{\text{pre}}(\{i\}) = v_{\text{post}}(\{i\}) \max_{(A_1,\dots,A_n)\in\Pi_n} \min_{j\in N} \frac{\mu_j(A_j)}{v_{\text{post}}(\{i\})} = \mu_i(A_i^m) = v_{\text{post}}(\{i\}).$$

Table 2 Values of v_{pre} and v_{post} in Example 4.1

S	v_{pre}	v_{post}
$\{i\}, i = 1, 2, 3$	0.4231	0.4231
$\{1, 2\}$	0.8662	0.8615
$\{1, 3\}$	0.8462	0.8462
$\{2, 3\}$	0.8662	0.8615
$\{1, 2, 3\}$	1.3	1.3

For a generic $S \subset N$, it holds

$$v_{\text{pre}}(S) = v_{\text{post}}(S) \max_{(B_S, B_j (j \notin S)) \in \Pi_{|S^c|+1}} \min \left\{ \frac{\mu_S(B_S)}{v_{\text{post}}(S)}, \frac{\mu_j(B_j)}{v_{\text{post}}(\{j\})}_{j \notin S} \right\}$$

$$\geq v_{\text{post}}(S) \min \left\{ \frac{\mu_S(A^m(S))}{v_{\text{post}}(S)}, \frac{\mu_j(A_j^m)}{v_{\text{post}}(\{j\})}_{j \notin S} \right\} = v_{\text{post}}(S).$$

Finally, both $v_{\text{pre}}(N)$ and $v_{\text{post}}(N)$ coincide, up to the scale factor $1/n$, with the fully cooperative approach (4). $\qquad\square$

Apart from the extreme cases, the values of the two games usually differ. Consider again the situation presented in Example 4.1. The values of the games v_{pre} and v_{post}, shown in Table 2, make clear that v_{pre} dominates v_{post} with a strict inequality for the coalitions $\{1, 2\}$ and $\{2, 3\}$.

We note that the game v_{pre} overcomes a major difficulty of the first proposal v and, therefore, it seems highly preferable to it. The game v_{pre}, however, requires the use of non-additive weights for the single coalitions, and poses new technical challenges which make it more difficult to examine than its predecessor.

5 Conclusions

The proposed models are an attempt to overcome the limitations of the existing models of cooperation in the allocation of a divisible good. Here, players can cooperate as soon as they are involved in the division process. The results show that, if side payments are allowed, it is beneficial for the players to join the grand coalition. Moreover, in the modified game where coalitions are given incentives, it is better to form coalitions as soon as possible.

More investigation of the topic is needed. It would be useful to provide a description of the core of v and v_{pre} following the lines of what has been done in Legut et al. (1994) for the game of cooperation after the division v_{post}. The most evident difficulty lies in the fact that, while v_{post} can be seen as the sum of n games, each defined on the endowments of the single players, the new context we are analyzing dispenses altogether with the notion of endowments, since the players will receive a share of the cake only at a second stage.

One of the main concerns in fair division is the design of procedures to achieve a partition with the required properties. So it would be advisable to devise a procedure

that achieves the optimal partitions in (11) and (14) or at least a good approximation of them. On a less ambitious scale, it would be advisable to find an easy way to compute the values of the two games v and v_{pre}.

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