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An approach for solving a modification of the extended rapid transit network design problem

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Abstract In this paper we deal with a slight modification of the extended rapid transit network design problem to allow circular lines. A two-stage approach is proposed for solving this problem. In the first stage, an integer model is solved for selecting the stations to be constructed and the links between them. It drastically reduces the dimension of a modification of a 0–1 model given in the literature to adapt it to our problem. In the second stage, the line design problem is solved by means of a procedure that assigns each selected link to exactly one line under certain constraints. We report some computational experiments that show that our approach also produces a drastic reduction on the computational effort required for solving the modification of the 0–1 model given in the literature.

Keywords Station and link location · Circular line · Line designing · Degree of a node

Mathematics Subject Classification (2000) 90B06 · 90B80 · 90C10 · 90C35

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1 Introduction

The extended rapid transit network design problem was stated in Marín (2007). Given a set of potential station locations and a set of potential links between them, this problem basically consists in selecting which stations and links to construct without exceeding the available budget, and determining noncircular lines from them, to maximise the total expected number of users.

Some related works are the following: Gendreau et al. (1995) describe the main criteria used to design rapid transit alignments and show how Operational Research tools can assist the design process. Laporte et al. (1997) analyse under certain assumptions a number of basic rapid transit network configurations with respect to the passenger–network effectiveness and the passenger–plane effectiveness. Bruno et al. (1998) introduce a bicriterion model which evaluates the attractiveness of a rapid transit line by taking into consideration the mobility demand, described in terms of an origin–destination matrix, and a bimodal transportation system. Hamacher et al. (2001) study the effects of introducing new train stops in an existing railway network. Bruno et al. (2002) present a heuristic for the location of a rapid transit line for increasing as much as possible the number of people covered by the alignment. Laporte et al. (2002) deal with the problem of locating a prefixed number of stations in a given line so that the weighted coverage will be maximised. Laporte et al. (2005) propose several heuristics for the construction of a rapid transit alignment, with the goal of maximising the total origin–destination demand covered by the alignment. Laporte et al. (2007) provide a pure 0–1 model for an extension of the problem considered in the above paper, where some budget constraints are introduced as well as the possibility of defining an upper bounded number of noncircular lines, each of them with a fixed origin and destination. Escudero and Muñoz (2008) contains a preliminary version of the present work, where a pure 0–1 model is considered in the first stage of the proposed approach, and in the computational experiments all of the variables have priority zero in the branching process. Marín and García-Ródenas (2008) present a nonlinear programming model for locating the infrastructure of a rapid transit network.

In this work, we consider a modification of the extended rapid transit network design problem to allow the definition of circular lines, and we present a two-stage approach for solving this new problem.

The remainder of the paper is structured as follows: Section 2 introduces some modifications in the pure 0–1 model given in Marín (2007) to allow circular lines. Section 3 presents a procedure for solving our problem. Specifically, Sect. 3.1 provides an integer model for selecting the stations and links to be constructed without exceeding the available budget, so that the total expected number of users will be maximised; it drastically reduces the dimension of the model considered in Sect. 2. Section 3.2 proposes an algorithm that assigns each selected link to exactly one line, in such a way that the number of lines that go through each selected station is as small as possible. Section 4 reports some computational experience on several instances considered in Marín (2007) that shows that the procedure presented in Sect. 3 produces a drastic reduction on the computational effort required for solving the model given in Sect. 2; moreover, for one of the instances it also obtains a more efficient network design. Finally, Sect. 5 draws some conclusions and future research from this work.

2 Modification of the extended rapid transit network design problem model

Let $V = \{1, \dots, n\}$ be the set of potential locations for the stations, let E be the set of (nonordered) pairs of locations that can be linked, i.e. $E = \{\{i, j\} \in V \times V \mid i \neq j$ and it is possible to link i and $j\}$, and let $m = |E|$. Without loss of generality, whenever we refer to an edge $\{i, j\} \in E$ it will be assumed that $i < j$.

For each $i \in V$, let $\Gamma(i)$ be the set of locations that can be linked to i (notice that $\Gamma(i)$ is the set of nodes adjacent to i in the simple graph $G = (V, E)$, and $|\Gamma(i)|$ is the degree of i).

Given that Marín (2007) distinguishes among nonordered and ordered pairs of locations, let us also define U as the set of ordered pairs of locations that can be linked, i.e. $U = \{(i, j) \mid \{i, j\} \in E\} \cup \{(j, i) \mid \{i, j\} \in E\}$.

Let W be the set of origin–destination pairs of locations in demand, and let us denote $w = (o_w, d_w)\ \forall w \in W$, where o_w and d_w are the origin and the destination of pair w, respectively. Throughout the paper, we shall consider $W = \{(i, j) \in V \times V \mid i \neq j\}$.

Let $L = \{1, \dots, q\}$ be the set of potential lines to be constructed. Theoretically, q should be large enough to allow the achievement of the maximum possible total expected number of users. Below it will be shown that the model presented in this section allows nonconnected lines consisting of one noncircular "sub-line" and various circular "sub-lines"; therefore, q is an upper bound for the number of noncircular lines to be constructed.

Let a_i denote the cost of constructing a station at location i, c_{ij} the cost of linking locations i and j, b the available budget for constructing the rapid transit network, d_{ij} the distance between locations i and j, and g_w the number of potential users for pair w, i.e. the demand for pair w. (We are assuming that $c_{ji} = c_{ij}$ and $d_{ji} = d_{ij}\ \forall\{i, j\} \in E$.)

If there are λ lines going through a location i or linking two locations i and j, then the associated construction costs will be λa_i and λc_{ij}, respectively, since it is assumed that we construct as many stations at i and as many links between i and j as the number of lines involved.

The reasons for modifying the model considered in Marín (2007) to allow circular lines are the following: Laporte et al. (1997) proved that circumferential configurations can increase the effectiveness of a rapid transit network. Furthermore, many cities have incorporated circular lines into their rapid transit networks (see the website http://www.urbanrail.net). On the other hand, the Marín model allows cycles consisting of more than one line (for example, two lines whose endpoints coincide); in this case, it would be preferable to define a unique circular line as the union of the initial lines, since this would reduce the construction costs at their endpoints, as well as the number of transfers that should be done by the users to arrive at their destinations.

We define the following variables:

$$y_i^l = \begin{cases} 1 & \text{if line } l \text{ goes through } i; \\ 0 & \text{otherwise} \end{cases} \qquad \forall i \in V, \forall l \in L;$$

$$x_{ij}^l = \begin{cases} 1 & \text{if arc } (i, j) \text{ belongs to line } l; \\ 0 & \text{otherwise} \end{cases} \qquad \forall (i, j) \in U, \forall l \in L;$$

$$f_{ij}^w = \begin{cases} 1 & \text{if the users of pair } w \\ & \text{are recommended to utilise arc } (i,j); \\ 0 & \text{otherwise} \end{cases} \quad \forall w \in W, \forall (i,j) \in U;$$

$$p_w = \begin{cases} 1 & \text{if the users of pair } w \\ & \text{will utilise the rapid transit network;} \\ 0 & \text{otherwise} \end{cases} \quad \forall w \in W.$$

It is assumed that the users of pair w will utilise the rapid transit network if and only if $\sum_{(i,j)\in U} d_{ij} f_{ij}^w \leq \mu\, u_w^{\text{pri}}$, where u_w^{pri} is the *generalised cost* of satisfying the demand of pair w through an existing private network and μ is a *congestion factor* (see Marín 2007 for more details). Notice that $\sum_{(i,j)\in U} d_{ij} f_{ij}^w$ is the total distance covered by the users of pair w whenever they follow the recommended path.

The model below contains the modifications that we propose for the extended rapid transit network design problem model to allow circular lines.

Model 1

$$\text{Maximise } z = \sum_{w\in W} g_w p_w$$

subject to:

$$\sum_{l\in L}\left(\sum_{i\in V} a_i y_i^l + \sum_{\{i,j\}\in E} c_{ij} x_{ij}^l\right) \leq b; \tag{1}$$

$$x_{ij}^l \leq y_i^l \quad \forall \{i,j\} \in E, \forall l \in L; \tag{2}$$

$$x_{ij}^l \leq y_j^l \quad \forall \{i,j\} \in E, \forall l \in L; \tag{3}$$

$$x_{ij}^l = x_{ji}^l \quad \forall \{i,j\} \in E, \forall l \in L; \tag{4}$$

$$\sum_{j\in \Gamma(i)} x_{ij}^l \leq 2 \quad \forall i \in V, \forall l \in L; \tag{5}$$

$$1 + \sum_{\{i,j\}\in E} x_{ij}^l \geq \sum_{i\in V} y_i^l \quad \forall l \in L; \tag{6}$$

$$\sum_{j\in \Gamma(i)} f_{ji}^w - \sum_{j\in \Gamma(i)} f_{ij}^w = \begin{cases} -1 & \text{if } i = o_w; \\ 1 & \text{if } i = d_w; \\ 0 & \text{otherwise} \end{cases} \quad \forall w \in W, \forall i \in V; \tag{7}$$

$$\sum_{(i,j)\in U} d_{ij} f_{ij}^w - \mu\, u_w^{\text{pri}} - M(1 - p_w) \leq 0 \quad \forall w \in W; \tag{8}$$

$$f_{ij}^w + p_w - 1 \leq \sum_{l\in L} x_{ij}^l \quad \forall w \in W, \forall (i,j) \in U; \tag{9}$$

$$y_i^l \in \{0,1\} \quad \forall i \in V, \forall l \in L;$$

$$x_{ij}^l \in \{0,1\} \quad \forall (i,j) \in U, \forall l \in L;$$

$$f_{ij}^{w} \in \{0, 1\} \quad \forall w \in W, \forall (i, j) \in U;$$

$$p_w \in \{0, 1\} \quad \forall w \in W,$$

where M is a big enough number (we have taken $M = \sum_{(i,j)\in U} d_{ij} - \mu \min\{u_w^{\text{pri}} \mid w \in W\}$ in the computational experience reported in Sect. 4).

The objective function is the same as the objective function from Marín (2007) where we have fixed $\eta = 1$, since this assumption has been made in its computational experiments. We have removed the constraints (1) from Marín model, since it seems that they have not been considered in its computational experiments. Constraint (1) is the constraint (2) from Marín model where we have fixed $c_{\min} = 0$, since it seems that this assumption has been made in its computational experiments. Constraints (2)–(5) are the constraints (12)–(15) from Marín model. Constraints (6) are the constraints (16) from Marín model where we have forced the inequality to allow circular lines. We have removed the constraints (17), (19) and (20) from Marín model, since they do not allow circular lines. Constraints (7)–(9) are the constraints (9)–(11) from Marín model.

Notice that Model 1 has $(n + 2m)q + (n^2 - n)(2m + 1)$ variables and $(n + 3m + 1)q + (n^2 - n)(n + 2m + 1) + 1$ constraints, since $|W| = n^2 - n$.

It is worth noting that, since the relation between the number of stations and links of each line is given by constraints (6), Model 1 allows nonconnected lines consisting of one noncircular "sub-line" and various circular "sub-lines" (see the instance defined by $b = 48$ and $\mu = 0.75$ in Table 5, Sect. 4). To avoid this type of lines, the original nonconnected line should be redefined as the noncircular "sub-line", and a new line should be defined for each circular "sub-line". This would increase the number of lines for the rapid transit network, but it would not affect its construction cost.

An undesirable property of the feasible solutions for Model 1 is that, given two opposite pairs $w = (o_w, d_w)$ and $\overline{w} = (d_w, o_w)$, it can occur that $\sum_{(i,j)\in U} d_{ij} f_{ij}^{w} \neq \sum_{(i,j)\in U} d_{ij} f_{ij}^{\overline{w}}$, i.e. the paths recommended to the users of these pairs have distinct total lengths. In this case, it would be desirable that the users that have been recommended to follow the longest path, were recommended to follow the opposite of the shortest path, since we are assuming that $d_{ji} = d_{ij}\ \forall \{i, j\} \in E$.

Another undesirable property of the feasible solutions for Model 1 is that the users of an origin–destination pair of locations can be recommended to follow a path that goes through the same location more than once (i.e. that contains a circuit), since constraints (7) allow such possibility. In this case, it would be desirable that those users were recommended to follow the path obtained by eliminating all of those circuits from the initial path.

3 Alternative approach for solving the modification of the extended rapid transit network design problem

Model 1 allows the possibility of more than one line linking two locations. However, if there were λ lines linking two locations, where $\lambda \geq 2$, it can easily be shown that it would be possible to eliminate all but one of those links and redefine the lines for the rapid transit network in such a way that its number would be increased by $\lambda - 1$ units at most (for some situations, this number would even be decreased). Obviously, this would reduce the total construction cost without modifying the value of the objective function for the considered feasible solution of Model 1 (notice that, by constraints (8) and (9), the value of the objective function of Model 1 depends on whether each pair of locations are linked or not, but, in case that they are linked, it does not depend on the number of lines linking them). Thus, there will always exist an optimal solution to Model 1 such that whichever two locations are linked by one line at most.

On the other hand, the variables $\{f_{ij}^w\}_{w\in W,(i,j)\in U}$ in Model 1 do not depend on the lines, and, consequently, if there is more than one line linking two locations, the users are not advised about which of them to choose.

Therefore, without loss of generality, from now on it will be assumed that whichever two locations are linked at most by one line. (Notice that, for imposing this condition in Model 1, it would suffice to append the constraints $\sum_{l\in L} x_{ij}^l \leq 1$ $\forall \{i,j\} \in E$ to its formulation; nevertheless, we have not performed this modification, since none of the obtained optimal solutions to the instances considered in Sect. 4 violate these constraints, see Tables 4 and 5.) Although this assumption is quite often verified in real-life rapid transit networks, it can also be violated (see the website http://www.urbanrail.net).

In this section we propose a two-stage approach for solving the problem into consideration. In the first stage an integer model is solved for selecting the stations and links to be constructed without exceeding the available budget, so that the total expected number of users will be maximised (see Sect. 3.1). In the second stage the line design problem is solved by assigning each selected link to exactly one line, in such a way that the number of lines that go through each selected station is as small as possible (see Sect. 3.2).

3.1 Improvement of the modified extended rapid transit network design problem model

Given a generic set W of origin–destination pairs of locations in demand, let $\overline{w} = (d_w, o_w)$ $\forall w \in W$ and $W' = \{w \mid w \in W, o_w < d_w\} \cup \{\overline{w} \mid w \in W, o_w > d_w\}$. (Notice that, for our considered set $W = \{(i,j) \in V \times V \mid i \neq j\}$, it will be $W' = \{(i,j) \in V \times V \mid i < j\}$.)

We define the following structural variables:

$$x_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are linked;} \\ 0 & \text{otherwise} \end{cases} \qquad \forall \{i,j\} \in E;$$

$$f_{ij}^w = \begin{cases} 1 & \text{if the users of pair } w \\ & \text{are recommended to utilise edge } \{i,j\}; \\ 0 & \text{otherwise} \end{cases} \qquad \forall w \in W', \forall \{i,j\} \in E;$$

$$p_w = \begin{cases} 1 & \text{if the users of pair } w \\ & \text{will utilise the rapid transit network;} \\ 0 & \text{otherwise} \end{cases} \qquad \forall w \in W.$$

This new definition for the variables $\{f_{ij}^w\}_{w\in W',\{i,j\}\in E}$ avoids the first undesirable situation that was pointed out at the end of Sect. 2. The second undesirable situation that was also pointed out there, will be avoided by defining the following auxiliary variables (jointly with constraints (12) in Model 2 below):

$$\varepsilon_i^w = \begin{cases} 1 & \text{if the users of pair } w \\ & \text{are recommended to go through } i; \\ 0 & \text{otherwise} \end{cases} \quad \forall w \in W', \forall i \in V \setminus \{o_w, d_w\}.$$

Notice that the definition of the variables $\{p_w\}_{w \in W}$ is the same as in Sect. 2.

The variables defined so far do not suffice to impose the budget constraint. For this purpose, it is required to determine the minimum possible number of lines going through each location (Algorithm 1 in Sect. 3.2 proves that this minimum is always reachable).

For each $i \in V$, let

$$r(i) = \begin{cases} \frac{|\Gamma(i)|}{2} & \text{if } |\Gamma(i)| \text{ is even;} \\ \frac{|\Gamma(i)|-1}{2} & \text{if } |\Gamma(i)| \text{ is odd} \end{cases}$$

and let $\Delta_i \in \{0, \dots, r(i)\}$ and $\gamma_i \in \{0, 1\}$ be such that $\sum_{j \in \Gamma(i), j>i} x_{ij} + \sum_{j \in \Gamma(i), j<i} x_{ji} = 2\Delta_i + \gamma_i$ (notice that $\sum_{j \in \Gamma(i), j>i} x_{ij} + \sum_{j \in \Gamma(i), j<i} x_{ji}$ is an integer number ranging from 0 to $|\Gamma(i)|$ that indicates the number of selected links with an endpoint at i). Then, it can easily be verified that the minimum possible number of lines going through i is $\Delta_i + \gamma_i$, hence the budget constraint will be imposed by the constraints $\sum_{j \in \Gamma(i), j>i} x_{ij} + \sum_{j \in \Gamma(i), j<i} x_{ji} = 2\Delta_i + \gamma_i$ $\forall i \in V$ and $\sum_{i \in V} a_i(\Delta_i + \gamma_i) + \sum_{\{i,j\} \in E} c_{ij} x_{ij} \leq b$.

Thus, we define the following additional auxiliary variables:

$$\gamma_i = \begin{cases} 1 & \text{if } \sum_{j \in \Gamma(i), j>i} x_{ij} + \sum_{j \in \Gamma(i), j<i} x_{ji} \text{ is odd;} \\ 0 & \text{otherwise} \end{cases} \quad \forall i \in V;$$

$$\Delta_i \in \{0, \dots, r(i)\} \quad \forall i \in V,$$

where

$$\Delta_i = \frac{\sum_{j \in \Gamma(i), j>i} x_{ij} + \sum_{j \in \Gamma(i), j<i} x_{ji} - \gamma_i}{2} \quad \forall i \in V.$$

Let

$$s(w) = \begin{cases} w & \text{if } o_w < d_w; \\ \overline{w} & \text{if } o_w > d_w \end{cases} \quad \forall w \in W.$$

The model below contains the improvements that we propose for Model 1.

Model 2

$$\text{Maximise } z = \sum_{w \in W} g_w p_w$$

subject to:

$$\sum_{j \in \Gamma(i), j>i} x_{ij} + \sum_{j \in \Gamma(i), j<i} x_{ji} = 2\Delta_i + \gamma_i \quad \forall i \in V; \tag{10}$$

$$\sum_{i\in V} a_i\,(\Delta_i+\gamma_i)+\sum_{\{i,j\}\in E} c_{ij}x_{ij}\le b; \tag{11}$$

$$\sum_{j\in\Gamma(i),j>i} f_{ij}^w+\sum_{j\in\Gamma(i),j<i} f_{ji}^w=\begin{cases}1 & \text{if } i\in\{o_w,d_w\};\\ 2\varepsilon_i^w & \text{otherwise}\end{cases}\quad \forall w\in W',\forall i\in V; \tag{12}$$

$$\sum_{\{i,j\}\in E} d_{ij}\,f_{ij}^{s(w)}-\mu\,u_w^{\text{pri}}-M_w(1-p_w)\le 0\quad \forall w\in W; \tag{13}$$

$$f_{ij}^{s(w)}+p_w-1\le x_{ij}\quad \forall w\in W,\forall\{i,j\}\in E; \tag{14}$$

$$x_{ij}\in\{0,1\}\quad \forall\{i,j\}\in E;$$

$$f_{ij}^w\in\{0,1\}\quad \forall w\in W',\forall\{i,j\}\in E;$$

$$p_w\in\{0,1\}\quad \forall w\in W;$$

$$\varepsilon_i^w\in\{0,1\}\quad \forall w\in W',\forall i\in V\setminus\{o_w,d_w\};$$

$$\gamma_i\in\{0,1\}\quad \forall i\in V;$$

$$\Delta_i\in\{0,\dots,r(i)\}\quad \forall i\in V,$$

where $M_w=\sum_{\{i,j\}\in E} d_{ij}-\mu\,u_w^{\text{pri}}\ \forall w\in W$.

The budget constraint (1) of Model 1 is now imposed by constraints (10) and (11). Constraints (2)–(6) of Model 1 are not required here. Constraints (7), (8) and (9) of Model 1 reduce to constraints (12), (13) and (14), respectively.

Notice that Model 2 has $(n^2-n)\frac{n+m}{2}+2n+m$ variables and $(n^2-n)(\frac{n}{2}+m+1)+n+1$ constraints, since $|W|=n^2-n$ and $|W'|=\frac{n^2-n}{2}$.

It is worth noting that Laporte et al. (2005) and Marín and García-Ródenas (2008) make use of the Logit distribution to define the objective functions for their proposed models. Models 1 and 2 could be modified following their guidelines, and this would probably result in network designs more adjusted to the requirements of the potential users.

3.2 Line designing from a given set of links to be constructed

Let $V'\subseteq V$ and $E'\subseteq E$ be the set of locations where a station is to be constructed and the set of links to be constructed, respectively, that have been selected by solving Model 2, and, for each $i\in V'$, let $\Gamma'(i)$ be the set of nodes adjacent to i in the partial subgraph $G'=(V',E')$.

In this section we propose a method for solving the line design problem for G'. On the contrary to Model 1, it has the advantage of requiring no upper bound for the number of noncircular lines for the rapid transit network.

The algorithm below can be outlined as follows: A node i with odd degree is chosen as starting node (if such a node does not exist, then a node i with positive even degree is chosen). Next, another node j adjacent to i is chosen and the edge $\{i,j\}$ is eliminated from E'; we set $i=j$ and repeat this procedure until we reach a node j which either has already been visited or it has no adjacent nodes apart from i. In the first case, a circular line is defined, and the above procedure is carried on from the last reached node which is an endpoint of an edge that has been eliminated from E' but has not yet been assigned to a line, if any. In the second case, a noncircular line is defined. This approach is repeated until we get $E' = \emptyset$. (Notice that, proceeding in this way, the number of lines going through each location $i \in V'$ will be $\frac{|\Gamma'(i)|}{2}$ if $|\Gamma'(i)|$ is even, or $\frac{|\Gamma'(i)|+1}{2}$ if $|\Gamma'(i)|$ is odd; in both cases, this number coincides with the minimum possible number of lines going through i, since $\sum_{j\in\Gamma(i),j>i} x_{ij} + \sum_{j\in\Gamma(i),j<i} x_{ji} = |\Gamma'(i)|$; see Sect. 3.1.)

In order to store the sequence of nodes chosen at each iteration, which will allow the definition of the lines to be constructed, we associate a value $p(i)$ to each node $i \in V'$ in such a way that, initially, we set $p(i) = 0\ \forall i \in V'$; next, for the starting node i we set $p(i) = i$, and for each node j chosen subsequently we set $p(j) = i$, where i is the node from which j has been reached; thus, for each $i \in V'$ we have that $p(i) > 0$ if and only if i has been visited at the current iteration. We also consider a counter l for the number of lines that are being defined.

Algorithm 1

Step 1. Set $p(i) = 0\ \forall i \in V'$ and $l = 0$.
Step 2. If $|\Gamma'(i)| = 0\ \forall i \in V'$, STOP.
Step 3. If $|\Gamma'(i)|$ is even $\forall i \in V'$, choose $i_0 \in V'$ such that $|\Gamma'(i_0)| > 0$; otherwise, choose $i_0 \in V'$ such that $|\Gamma'(i_0)|$ is odd. Set $l = l + 1$, $L(l) = \emptyset$, $p(i_0) = i_0$ and $i = i_0$.
Step 4. Choose $j \in \Gamma'(i)$ and set $\Gamma'(i) = \Gamma'(i) \setminus \{j\}$ and $\Gamma'(j) = \Gamma'(j) \setminus \{i\}$. If $p(j) > 0$, set $j_0 = j$ and go to Step 6.
Step 5. If $|\Gamma'(j)| > 0$, set $p(j) = i$, $i = j$ and go to Step 4; otherwise, set $j_0 = i_0$.
Step 6. Set $L(l) = L(l) \cup \{\{j, i\}\}$. If $i \neq j_0$, set $j = i$, $i = p(i)$, $p(j) = 0$ and repeat Step 6.
Step 7. If $j_0 = i_0$, set $p(i_0) = 0$ and go to Step 2.
Step 8. Set $l = l + 1$ and $L(l) = \emptyset$. If $|\Gamma'(i)| > 0$, go to Step 4; otherwise, set $j = i$, $i = p(i)$, $p(j) = 0$, $j_0 = i_0$ and go to Step 6.

Remark 1 If $|\Gamma'(i_0)|$ is even, then it always will be $|\Gamma'(j)| > 0$ in Step 5 and $|\Gamma'(i)| > 0$ in Step 8.

Example 1 Consider the graph $G' = (V', E')$, where $V' = \{1, 2, 3, 4, 5\}$ and $E' = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{4, 5\}\}$ (see Fig. 1). Then $\Gamma'(1) = \{2, 3, 4, 5\}$, $\Gamma'(2) = \{1, 4, 5\}$, $\Gamma'(3) = \{1, 4\}$, $\Gamma'(4) = \{1, 2, 3, 5\}$ and $\Gamma'(5) = \{1, 2, 4\}$.

Algorithm 1 proceeds as follows:

Step 1. $p(1) = p(2) = p(3) = p(4) = p(5) = 0,\ l = 0$
Step 3. $i_0 = 2,\ l = 1,\ L(1) = \emptyset,\ p(2) = 2,\ i = 2$
Step 4. $j = 1,\ \Gamma'(2) = \{4, 5\},\ \Gamma'(1) = \{3, 4, 5\}$
Step 5. $p(1) = 2,\ i = 1$
Step 4. $j = 3,\ \Gamma'(1) = \{4, 5\},\ \Gamma'(3) = \{4\}$

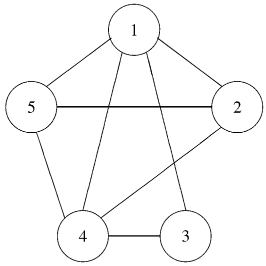

Fig. 1 Graphic representation of $G' = (V', E')$

Step 5. $p(3) = 1,\ i = 3$
Step 4. $j = 4,\ \Gamma'(3) = \emptyset,\ \Gamma'(4) = \{1, 2, 5\}$
Step 5. $p(4) = 3,\ i = 4$
Step 4. $j = 1,\ \Gamma'(4) = \{2, 5\},\ \Gamma'(1) = \{5\},\ j_0 = 1$
Step 6. $L(1) = \{\{1, 4\}\},\ j = 4,\ i = 3,\ p(4) = 0$
Step 6. $L(1) = \{\{1, 4\}, \{4, 3\}\},\ j = 3,\ i = 1,\ p(3) = 0$
Step 6. $L(1) = \{\{1, 4\}, \{4, 3\}, \{3, 1\}\}$
Step 8. $l = 2,\ L(2) = \emptyset$
Step 4. $j = 5,\ \Gamma'(1) = \emptyset,\ \Gamma'(5) = \{2, 4\}$
Step 5. $p(5) = 1,\ i = 5$
Step 4. $j = 2,\ \Gamma'(5) = \{4\},\ \Gamma'(2) = \{4\},\ j_0 = 2$
Step 6. $L(2) = \{\{2, 5\}\},\ j = 5,\ i = 1,\ p(5) = 0$
Step 6. $L(2) = \{\{2, 5\}, \{5, 1\}\},\ j = 1,\ i = 2,\ p(1) = 0$
Step 6. $L(2) = \{\{2, 5\}, \{5, 1\}, \{1, 2\}\}$
Step 7. $p(2) = 0$
Step 3. $i_0 = 2,\ l = 3,\ L(3) = \emptyset,\ p(2) = 2,\ i = 2$
Step 4. $j = 4,\ \Gamma'(2) = \emptyset,\ \Gamma'(4) = \{5\}$
Step 5. $p(4) = 2,\ i = 4$
Step 4. $j = 5,\ \Gamma'(4) = \emptyset,\ \Gamma'(5) = \emptyset$
Step 5. $j_0 = 2$
Step 6. $L(3) = \{\{5, 4\}\},\ j = 4,\ i = 2,\ p(4) = 0$
Step 6. $L(3) = \{\{5, 4\}, \{4, 2\}\}$
Step 7. $p(2) = 0.$

Consequently, two circular lines $L(1) = \{\{1, 4\}, \{4, 3\}, \{3, 1\}\}$ and $L(2) = \{\{2, 5\}, \{5, 1\}, \{1, 2\}\}$, and one noncircular line $L(3) = \{\{5, 4\}, \{4, 2\}\}$ have been defined.

Remark 2 Since the objective functions of Models 1 and 2 are the same, and the unique additional constraints that have been imposed to the feasible solutions for the above two-stage approach compared to those for Model 1 are that whichever two locations are linked by one line at most, and the number of lines that go through each selected station is as small as possible, we can conclude that the optimal values of the objective functions of Models 1 and 2 always coincide, and any set of optimal lines obtained by applying the two-stage approach is also optimal for Model 1 (see the beginning of Sect. 3).

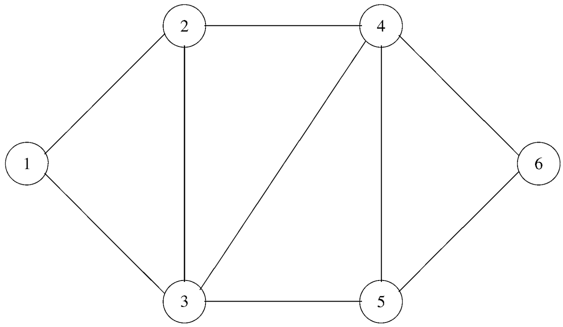

Fig. 2 Graphic representation of $G_1 = (V_1, E_1)$

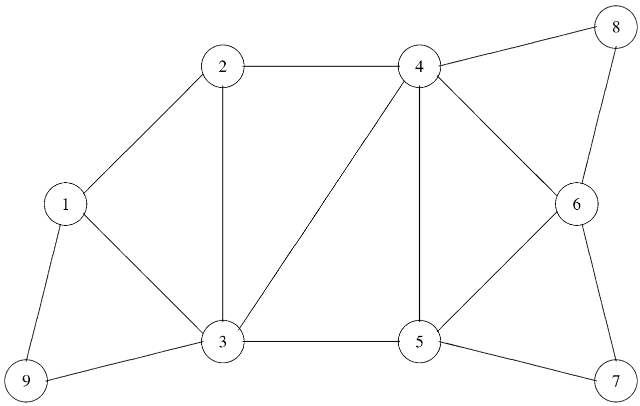

Fig. 3 Graphic representation of $G_2 = (V_2, E_2)$

Table 1 Station construction costs $\{a_i\}_{i \in V_2}$

i	1	2	3	4	5	6	7	8	9
a_i	2	3	2.2	3	2.5	1.3	2.8	2.2	3.1

4 Computational experience

We consider the same two networks as in the computational experiments given in Marín (2007), namely $R1$ and $R2$ ($R1$ was also previously considered in Laporte et al. 2007). For the sake of completeness, we reproduce the underlying graphs and data for these networks.

Network $R1$ consists of six nodes and nine edges, and network $R2$ is an extension of network $R1$ with nine nodes and fifteen edges. Figures 2 and 3 show their respective underlying graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$.

Table 1 shows the cost a_i of constructing a station at location i, for each $i \in V_2$.

Table 2 shows the cost c_{ij} of linking locations i and j, and the distance d_{ij} between locations i and j, for each $\{i, j\} \in E_2$.

The demand $g_{(i,j)}$ for pair (i, j) and the generalised cost $u^{\text{pri}}_{(i,j)}$ of satisfying the demand of pair (i, j), for each pair of locations $i, j \in V_2$ with $i \neq j$, are given in the

Table 2 Linking construction costs $\{c_{ij}\}_{\{i,j\}\in E_2}$ and distances $\{d_{ij}\}_{\{i,j\}\in E_2}$

$\{i,j\}$	{1, 2}	{1, 3}	{1, 9}	{2, 3}	{2, 4}	{3, 4}	{3, 5}	{3, 9}	{4, 5}	{4, 6}	{4, 8}	{5, 6}	{5, 7}	{6, 7}	{6, 8}
c_{ij}	1.7	2.7	2.9	2.1	3.0	2.6	1.7	2.5	2.8	2.4	3.2	1.9	3.0	2.7	2.8
d_{ij}	0.5	0.7	0.9	0.6	1.1	1.1	0.5	0.7	0.8	0.7	0.8	0.5	0.7	0.5	0.4

matrices **G** and $\mathbf{U}^{\text{pri}}$, respectively:

$$\mathbf{G}=\begin{pmatrix} - & 9 & 26 & 19 & 13 & 12 & 13 & 8 & 11\\ 11 & - & 14 & 26 & 7 & 18 & 3 & 6 & 12\\ 30 & 19 & - & 30 & 24 & 8 & 15 & 12 & 5\\ 21 & 9 & 11 & - & 22 & 16 & 25 & 21 & 23\\ 14 & 14 & 8 & 9 & - & 20 & 16 & 22 & 21\\ 26 & 1 & 22 & 24 & 13 & - & 16 & 14 & 12\\ 8 & 6 & 9 & 23 & 6 & 13 & - & 11 & 11\\ 9 & 2 & 14 & 20 & 18 & 16 & 11 & - & 4\\ 8 & 7 & 11 & 22 & 27 & 17 & 8 & 12 & - \end{pmatrix};$$

$$\mathbf{U}^{\text{pri}}=\begin{pmatrix} - & 1.6 & 0.8 & 2 & 2.6 & 2.5 & 3 & 2.5 & 0.8\\ 2 & - & 0.9 & 1.2 & 1.5 & 2.5 & 2.7 & 2.4 & 1.8\\ 1.5 & 1.4 & - & 1.3 & 0.9 & 2 & 1.6 & 2.3 & 0.9\\ 1.9 & 2 & 1.9 & - & 1.8 & 2 & 1.9 & 1.2 & 2\\ 3 & 1.5 & 2 & 2 & - & 1.5 & 1.1 & 1.8 & 1.7\\ 2.1 & 2.7 & 2.2 & 1 & 1.5 & - & 0.9 & 0.9 & 2.9\\ 2.8 & 2.3 & 1.5 & 1.8 & 0.9 & 0.8 & - & 1.3 & 2.1\\ 2.8 & 2.2 & 2 & 1.1 & 1.5 & 0.8 & 1.9 & - & 0.3\\ 1 & 1.5 & 1.1 & 2.7 & 1.9 & 1.8 & 2.4 & 3 & - \end{pmatrix}.$$

Remark 3 It is reasonable to expect that the generalised costs corresponding to any two opposite origin–destination pairs of locations will not be too much different. However, we have $u^{\text{pri}}_{(9,8)}=3=10u^{\text{pri}}_{(8,9)}$. We believe that this is a misprint, and it should be $u^{\text{pri}}_{(8,9)}=3$ instead of $u^{\text{pri}}_{(8,9)}=0.3$ (notice that locations 8 and 9 are far away from each other); nevertheless, we have taken $u^{\text{pri}}_{(8,9)}=0.3$.

We have taken $q=3$ in Model 1, since this assumption has been made in the computational experiments reported in Marín (2007).

For networks $R1$ and $R2$, the number of nodes and edges, as well as the number of variables and constraints for Models 1 and 2, are given in Table 3.

The implementation platform has been Microsoft Visual C++ 2005, CPLEX v11.0, and Pentium 4, 3.00 GHz, 1.00 Gb RAM.

We have run the CPLEX mixed integer optimiser by using the default rules, except that the relative and absolute optimality tolerances have been set to zero, and, in the branching process, the priorities for the variables $\{p_w\}_{w\in W}$ have been set to one in Models 1 and 2, and the priorities for the variables $\{x^l_{ij}\}_{(i,j)\in U,l\in L}$ and $\{x_{ij}\}_{\{i,j\}\in E}$ have been set to two in Models 1 and 2, respectively (we have considered many other

Table 3 Network and model dimensions

Network	n	m	Model 1		Model 2	
			Variables	Constraints	Variables	Constraints
$R1$	6	9	642	853	246	397
$R2$	9	15	2349	3046	897	1486

Table 4 Computational comparison for $R1$

b	μ	z^*	Model 1			Model 2 + Algorithm 1		
			Nodes	Time	Optimal lines	Nodes	Time	Optimal lines
24	0.75	316	29	1.33	L_1: 1-2-3-5-6-4	0	0.13	L_1: 1-2-3-5-6-4
24	1	365	2069	17.80	L_1: 1-2-3-5-6-4	101	1.23	L_1: 1-2-3-5-6-4
24	1.5	470	294	2.48	L_1: 1-2-3-5-6-4	75	0.30	L_1: 1-2-3-5-6-4
32	0.75	339	0	0.22	L_2: 3-4-6-5-3	0	0.05	L_1: 1-2-3-1
					L_3: 1-2-3-1			L_2: 3-4-6-5-3
32	1	470	8	0.41	L_1: 1-3	0	0.20	L_1: 1-3-2
					L_2: 3-4-6-5-3			L_2: 3-4-6-5-3
					L_3: 2-3			
32	1.5	496	0	0.22	L_3: 1-2-4-6-5-3-1	0	0.06	L_1: 3-2-4-6-5-3
								L_2: 1-3

settings for the priority values, but the best general computational results have been obtained with these ones).

In Tables 4 and 5 a number of instances from networks $R1$ and $R2$ have been considered, respectively; they have been defined by assigning several values to the available budget b and to the congestion factor μ. For each one of these instances, the optimal value z^* of the objective function of Models 1 and 2 is provided (see Remark 2), as well as the number of branch-and-cut nodes evaluated, the CPU time expressed in seconds and the optimal lines obtained by solving Model 1 and by applying the approach proposed in Sect. 3 (i.e. solving Model 2 and applying Algorithm 1).

The CPU time of Algorithm 1 has been inappreciable for all the instances. In its Steps 3 and 4 we have chosen, respectively,

$$i_0 = \begin{cases} \min\{i' \in V' \mid |\Gamma'(i')| > 0\} & \text{if } |\Gamma'(i)| \text{ is even } \forall i \in V'; \\ \min\{i' \in V' \mid |\Gamma'(i')| \text{ is odd}\} & \text{otherwise} \end{cases} \quad \text{and}$$

$$j = \min\{j' \mid j' \in \Gamma'(i)\}.$$

Each lth optimal line is denoted as L_l and defined by the sequence of locations through which it goes. The empty optimal lines for Model 1, if any, are not displayed in Tables 4 and 5. (Notice that none of the constraints of Model 1 impose that if an optimal line L_l is empty, then $L_{l'}$ is empty $\forall l' \in \{l+1, \ldots, q\}$; therefore, for this model there can exist two optimal lines L_l and $L_{l'}$ such that $l < l'$, L_l is empty and $L_{l'}$ is nonempty.)

Table 5 Computational comparison for $R2$

b	μ	z^*	Model 1			Model 2 + Algorithm 1		
			Nodes	Time	Optimal lines	Nodes	Time	Optimal lines
28	0.75	361	23142	861.17	L_1: 7-6-8 L_2: 3-5-6-4	758	21.25	L_1: 3-5-6-4 L_2: 7-6-8
28	1	466	79017	4915.88	L_1: 6-5-3-9 L_3: 1-3-4	5072	110.84	L_1: 1-3-4 L_2: 6-5-3-9
28	1.5	522	183597	9129.56	L_2: 1-3-5-6-4 L_3: 6-8	14483	290.42	L_1: 1-3-5-6-4 L_2: 6-8
48	0.75	672	4388	120.11	L_1: 2-3-5-6-7 L_2: 1-3-9; 4-6-8-4	133	3.63	L_1: 1-3-2 L_2: 4-6-8-4 L_3: 7-6-5-3-9
48	1	912	82631	10419.60	L_2: 1-2-3-4-6-8 L_3: 7-6-5-3-9	1266	30.31	L_1: 3-4-6-5-3 L_2: 1-2-3-9 L_3: 7-6-8
48	1.5	1035	530	19.48	L_1: 7-6-4-2-1-9 L_3: 1-3-5-6-8	270	3.55	L_1: 1-2-4-6-5-3-1 L_2: 1-9 L_3: 7-6-8

The unique instances of Table 4 where the optimal lines obtained by solving Model 1 and by applying the approach proposed in Sect. 3 are not the same are the two last ones. Moreover, for the instance defined by $b = 32$ and $\mu = 1$ it is clear that the line design obtained by solving Model 1 is less efficient than the line design obtained by applying the approach proposed in Sect. 3, since the line L_1 of the second one is the union of the lines L_1 and L_3 of the first one. For this instance, the optimal sets of links to be constructed obtained by solving Models 1 and 2 are the same, whereas for the instance defined by $b = 32$ and $\mu = 1.5$ they are distinct.

The instances of Table 5 where the optimal lines obtained by solving Model 1 and by applying the approach proposed in Sect. 3 are not the same are the three last ones, although for all of them the optimal sets of links to be constructed obtained by solving Models 1 and 2 are the same.

We can observe from Tables 4 and 5 that the approach proposed in Sect. 3 outperforms the solving of Model 1.

The instances and the implementation platform considered in the computational experiments from Escudero and Muñoz (2008) are the same as in here, but the branching priorities for the variables were set to zero, and, in Model 2, neither the set W' nor the values $\{s(w)\}_{w\in W}$ were defined, and the substitutions $\Delta_i = \sum_{k=1}^{r(i)} \delta_{ik}\ \forall i \in V$ were made, where $\delta_{ik} \in \{0, 1\}\ \forall k \in \{1, \dots, r(i)\}$, since slightly smaller computational times were obtained by making such substitutions. However, for the priority values considered in this work, slightly smaller general computational times have been obtained without making these substitutions. By comparing Tables 4 and 5 in Escudero and Muñoz (2008) with Tables 4 and 5 above, one can observe that, for most of the instances, the CPU times from Escudero and Muñoz (2008) are greater; this fact shows the crucial importance of the priority settings.

5 Conclusions and future research

In this paper we have presented a two-stage approach for solving a modification of the extended rapid transit network design problem to allow the definition of circular lines. The integer model considered in the first stage makes possible to select the stations and links to be constructed, and it gives much better computational results than a modification of a 0–1 model reported in the literature to solve our problem; furthermore, the proposed model manages to avoid certain undesirable properties of the feasible solutions for the modification of the model taken from the literature. Once the stations and links to be constructed have been selected, in the second stage each one of these links is assigned to a unique line, so that the number of lines going through each selected station is minimised; the computational effort required for performing these assignations has been inappreciable for all of the instances under consideration. Consequently, our approach is expected to solve larger instances than the ones solved so far in the literature.

As for an ongoing research, we are working on the exploitation of the real-life uncertainty in the demand for the distinct origin–destination pairs of locations and in the costs of constructing the stations and links.

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