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Strong Kuhn–Tucker conditions and constraint qualifications in locally Lipschitz multiobjective optimization problems

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Abstract We consider a Pareto multiobjective optimization problem with a feasible set defined by inequality and equality constraints and a set constraint, where the objective and inequality constraints are locally Lipschitz, and the equality constraints are Fréchet differentiable. We study several constraint qualifications in the line of Maeda (J. Optim. Theory Appl. 80: 483–500, 1994) and, under the weakest ones, we establish strong Kuhn–Tucker necessary optimality conditions in terms of Clarke subdifferentials so that the multipliers of the objective functions are all positive.

Keywords Multiobjective optimization problems · Constraint qualification · Necessary conditions for Pareto minimum · Lagrange multipliers · Clarke subdifferential

Mathematics Subject Classification (2000) 90C29 · 90C46

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1 Introduction

Constraint qualifications play an important role in optimization, since they allow us to guarantee an effective intervention of the objective function in the Fritz John-type necessary conditions for a point to be an optimum. Since the first decade of the 50's, the study of these qualifications has been the aim of several researchers with different approaches, proposing various regularity conditions.

In order to avoid the case where some of the Lagrange multipliers associated with the objective function vanish for a multiobjective optimization problem, several works have been developed in recent years, and strong Kuhn–Tucker (K–T) necessary optimality conditions have been obtained. We say that strong K–T conditions hold when the Lagrange multipliers are positive for all objectives.

Maeda (1994) studies differentiable multiobjective optimization problems (in finite-dimensional spaces) and gives strong K–T necessary conditions for a Pareto minimum of a function over a feasible set defined by inequality constraints under a regularity condition, called generalized Guignard constraint qualification (CQ). He also studies other regularity conditions as the generalized Abadie CQ, showing that the generalized Guignard CQ is the weakest one.

Two of the present authors (Jiménez and Novo 1999) develop similar results by considering equality constraints not considered by Maeda. They also introduced new qualifications that are sufficient conditions for the generalized Guignard CQ.

Preda and Chitescu (1999) extend the results obtained by Maeda, considering Dini-quasiconvex and directionally differentiable functions. However, due to the requirement on the objective functions to be Dini-quasiconvex and Dini-quasiconcave with convex and concave Dini derivatives, their necessary optimality conditions (Theorems 3.1 and 3.2) are very restrictive. Their results are given by means of inequalities concerning the derivatives.

In our previous work (Giorgi et al. 2004), we generalize the above results by considering objective and inequality constraint functions that are Dini or Hadamard directionally differentiable and Fréchet differentiable equality constraints and also introduce new qualifications.

Li (2000) considers a problem with only inequality constraints and supposes that the involved functions are locally Lipschitz. In one of the most important results, he proves strong Kuhn–Tucker optimality conditions under a generalized Abadie-type CQ and assuming that the Clarke subdifferentials of the objective and constraint functions are polytopes.

Yuan et al. (2007) consider only inequality constraints and locally Lipschitz functions which are regular in the main result (Theorem 7).

Li and Zhang (2005) consider only inequality constraints and locally Lipschitz functions and provide strong K–T necessary optimality conditions that are expressed in terms of upper convexificators.

Bigi and Pappalardo (1999) point out the importance of strong K–T conditions, since if, for example, in (12) there exists an index i such that $\lambda_i = 0$, then the optimality condition loses a part of its importance because it no longer involves the corresponding objective function. These authors consider four classes of problems. Two of them are those which admit at least one positive multiplier, i.e., $\lambda > 0$ (regular problems), and those for which all the multipliers λ are positive (totally regular problems). They relate these classes to some constraint qualifications and say that this can be employed to achieve additional properties of the K–T multipliers. In this paper, we study constraint qualifications to obtain regular problems.

On the other hand, it is known that a proper efficient solution has better properties than an efficient solution. A necessary condition for a point to be a proper efficient solution is that strong K–T conditions are satisfied (Sawaragi et al. 1985, Theorem 3.5.1). These conditions are also sufficient under additional assumptions of convexity (Sawaragi et al. 1985, Theorem 3.5.2). These results show the importance of getting strong K–T conditions.

In this paper, we extend the results of Li (2000) by considering locally Lipschitz functions, Fréchet differentiable equality constraints, and an abstract set constraint. We consider general locally Lipschitz functions, that is, we do not assume that the Clarke subdifferentials are polytopes. We also establish strong K–T optimality conditions under an extended generalized Guignard CQ, but in this case the objective functions must be Fréchet differentiable.

This paper is structured as follows: Sect. 2 contains the definitions and notation we use and some previous results. In Sect. 3, several constraint qualifications are proposed and the relationships between them are studied. Finally, in Sect. 4, several necessary optimality conditions of strong Kuhn–Tucker-type are obtained, i.e., such that they assure the positivity of the multipliers under the weaker qualifications proposed.

2 Notation and preliminaries

Let x and y be two points of $\mathbb{R}^n$. Throughout this paper, we use the following notation.

$x \leq y$ if $x_i \leq y_i$, $i = 1, \ldots, n$; $x < y$ if $x_i < y_i$, $i = 1, \ldots, n$.

Let M be a subset of $\mathbb{R}^n$. As usual, $\mathrm{cl}\, M$, $\mathrm{co}\, M$, $\mathrm{cone}\, M$, and $\mathrm{lin}\, M$ denote the closure, convex hull, cone generated, and subspace generated by M, respectively. $B(x_0, \delta)$ is the open ball of center x_0 and radius $\delta > 0$. If A is a convex subset of $\mathbb{R}^n$, $\mathrm{ri}\, A$ denotes the relative interior of A.

Given a function $f : \mathbb{R}^n \to \mathbb{R}^p$, the following multiobjective optimization problem is considered:

$$\text{(MP)} \quad \mathrm{Min}\big\{f(x) : x \in M\big\}.$$

One says that the point $x_0 \in M$ is a local Pareto minimum, denoted $x_0 \in \mathrm{LMin}(f, M)$, if there exists a neighborhood U of x_0 such that

$$\big[f(M \cap U) - f(x_0)\big] \cap \big(-\mathbb{R}^p_+ \setminus \{0\}\big) = \emptyset,$$

We use the following tangent cones:

Definition 2.1 Let $M \subset \mathbb{R}^n$ and $x_0 \in \mathrm{cl}\, M$.

(a) The tangent cone to M at the point x_0 is

$$T(M, x_0) = \{v \in \mathbb{R}^n : \exists t_k \to 0^+,\ \exists x_k \to x_0 \text{ with } x_k \in M \text{ such that } (x_k - x_0)/t_k \to v\}.$$

(b) The Clarke tangent cone to M at the point x_0 is

$$T_C(M, x_0) = \{v \in \mathbb{R}^n :\ \forall t_k \to 0^+, \forall x_k \to x_0 \text{ with } x_k \in M, \exists v_k \to v \text{ such that } y_k := x_k + t_k v_k \in M\ \forall k \in \mathbb{N}\}.$$

It is well known that

$$T_C(M, x_0) \subset T(M, x_0), \tag{1}$$

and $T_C(M, x_0)$ is a closed convex cone. If M is a convex set, then $T_C(M, x_0) = T(M, x_0) = \operatorname{cl}\operatorname{cone}(M - x_0)$.

Let $A \subset \mathbb{R}^n$. Then the (negative) polar cone to A is $A^- = \{\xi \in \mathbb{R}^n : \langle \xi, x\rangle \le 0\ \forall x \in A\}$.

The normal cone to M at x_0 is the polar to the tangent cone, i.e., $N(M, x_0) = T(M, x_0)^-$. The Clarke normal cone to M at x_0 is $N_C(M, x_0) := T_C(M, x_0)^-$.

Note that if the sets are defined through function constraints, their approximation is realized through the cones defined by the directional derivatives of the functions.

Definition 2.2 Let $f : \mathbb{R}^n \to \mathbb{R}$, $x_0, v \in \mathbb{R}^n$.

(a) The upper Dini derivative of f at x_0 in the direction v is

$$\bar{D} f(x_0, v) = \limsup_{t \to 0^+} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

(b) The upper Hadamard derivative of f at x_0 in the direction v is

$$\bar{d} f(x_0, v) = \limsup_{(t,u)\to(0^+,v)} \frac{f(x_0 + tu) - f(x_0)}{t}.$$

(c) The Clarke derivative of f at x_0 in the direction v is

$$d^0 f(x_0, v) = \limsup_{(t,x)\to(0^+,x_0)} \frac{f(x + tv) - f(x)}{t}.$$

The Fréchet derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^p$ at x_0 is denoted by $\nabla f(x_0)$.

The subdifferential (in the sense of the Convex Analysis) of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 is denoted by $\partial f(x_0)$, and the Clarke subdifferential of a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ at x_0 is

$$\partial_C f(x_0) = \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \le d^0 f(x_0, v)\ \forall v \in \mathbb{R}^n\}.$$

It is well known that if $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, then $\bar{D} f(x_0, v) = \bar{d} f(x_0, v)$ and $d^0 f(x_0, v) \ge \bar{d} f(x_0, v)\ \forall v \in \mathbb{R}^n$. The reader is referred to the book (Clarke 1983) for more details.

The following proposition collects Proposition 3.6, Lemma 3.10, and parts of Theorem 3.5 and Lemma 3.2 in Jiménez and Novo (2002).

Proposition 2.1 *Let $f_1, \dots, f_p$ be sublinear functions from $\mathbb{R}^n$ to $\mathbb{R}$, $f = (f_1, \dots, f_p)$, $h = (h_1, \dots, h_r) : \mathbb{R}^n \to \mathbb{R}^r$ be a linear function given by $h_k(u) = \langle c_k, u \rangle$, $c_k \in \mathbb{R}^n$, $k \in K = \{1, \dots, r\}$, and let $A \subset \mathbb{R}^n$ be a convex cone. Consider the following statements*:

(a) $0 \notin \mathrm{co}(\bigcup_{i=1}^{p} \partial f_i(0)) + \mathrm{lin}\{c_k : k \in K\} + A^-$.
(b) *There exists $v \in \mathbb{R}^n$ such that $f(v) < 0$, $h(v) = 0$, $v \in A$.*

Then

(i) *If $0 \in \mathrm{ri}\, h(Q)$, then (a) $\Leftrightarrow$ (b).*
(ii) *If* (a) *holds and* $\mathrm{Ker}\, h \cap \mathrm{ri}\, A \neq \emptyset$, *then*

$$D := \mathrm{cone\,co}\left(\bigcup_{i=1}^{p} \partial f_i(0)\right) + \mathrm{lin}\{c_k : k \in K\} + A^- \text{ is a closed cone.}$$

(iii) *There exists $u \in \mathbb{R}^n$ such that $f(u) < 0$, $h(u) = 0$, $u \in \mathrm{ri}\, A$, if and only if* (b) *holds and* $\mathrm{Ker}\, h \cap \mathrm{ri}\, A \neq \emptyset$.
(iv) $\mathrm{Ker}\, h \cap \mathrm{ri}\, A \neq \emptyset$ *if and only if* $0 \in \mathrm{ri}\, h(A)$.

The next theorem gives an alternative theorem which is basic in the development of this paper. It is a generalization of Proposition 2.2 in Giorgi et al. (2004), where the case $A = \mathbb{R}^n$ is considered (see also Ishizuka 1992; Luu and Nguyen 2006).

Theorem 2.1 (Generalized Tucker alternative theorem) *Let $f_1, \dots, f_p, g_1, \dots, g_m$ be sublinear functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $h_1, \dots, h_r$ linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ given by $h_k(v) = \langle c_k, v \rangle$, $k \in K = \{1, \dots, r\}$, with $c_k \in \mathbb{R}^n$, let $f = (f_1, \dots, f_p)$, $g = (g_1, \dots, g_m)$, $h = (h_1, \dots, h_r)$, and let $A \subset \mathbb{R}^n$ be a closed convex cone. Suppose that for each $i \in \{1, \dots, p\}$, the cone*

$$D_i = \mathrm{cone\,co}\left(\bigcup_{j \neq i} \partial f_j(0)\right) + \mathrm{cone\,co}\left(\bigcup_{j=1}^{m} \partial g_j(0)\right) + \mathrm{lin}\{c_k : k \in K\} + A^-$$

is closed. Then, the following statements are equivalent:

(a) *The system*

$$f(v) \leq 0, \qquad f(v) \neq 0, \qquad g(v) \leq 0, \qquad h(v) = 0, \quad v \in A,$$

has no solution $v \in \mathbb{R}^n$.
(b) *There exist $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that $\lambda > 0$, $\mu \geq 0$, and*

$$0 \in \sum_{i=1}^{p} \lambda_i \partial f_i(0) + \sum_{j=1}^{m} \mu_j \partial g_j(0) + \sum_{k=1}^{r} \nu_k c_k + A^-.$$

(c) *There exist* $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ *such that* $\lambda > 0$, $\mu \geq 0$, *and*

$$\sum_{i=1}^{p} \lambda_i f_i(u) + \sum_{j=1}^{m} \mu_j g_j(u) + \sum_{k=1}^{r} \nu_k h_k(u) \geq 0 \quad \forall u \in A. \tag{2}$$

Proof The equivalence (b) $\Leftrightarrow$ (c) is proved similarly to the equivalence (a) $\Leftrightarrow$ (b) of Theorem 3.5 in Jiménez and Novo (2002).

Now suppose that (c) is true and (a) is false, and let v_0 be a solution of the system in (a). Then $\sum_{i=1}^{p} \lambda_i f_i(v_0) < 0$, $\sum_{j=1}^{m} \mu_j g_j(v_0) \leq 0$ and $\sum_{k=1}^{r} \nu_k h_k(u) = 0$, and therefore

$$\sum_{i=1}^{p} \lambda_i f_i(v_0) + \sum_{j=1}^{m} \mu_j g_j(v_0) + \sum_{k=1}^{r} \nu_k h_k(v_0) < 0,$$

which contradicts (2).

Finally, suppose that (a) holds, then for each $i = 1, \dots, p$, the system

$$f_i(v) < 0, \qquad f_s(v) \leq 0, \quad s \neq i, \qquad g(v) \leq 0, \qquad h(v) = 0, \quad v \in A,$$

has no solution $v \in \mathbb{R}^n$. By Theorem 3.13 in Jiménez and Novo (2002), there exist $\lambda_{ii} > 0$, $\lambda_{is} \geq 0$, $s \neq i$, $\mu_{ij} \geq 0$, $j = 1, \dots, m$, $\nu_{ik} \in \mathbb{R}$, $k = 1, \dots, r$, such that

$$\lambda_{ii} f_i(u) + \sum_{s=1, s\neq i}^{p} \lambda_{is} f_s(u) + \sum_{j=1}^{m} \mu_{ij} g_j(u) + \sum_{k=1}^{r} \nu_{ik} h_k(u) \geq 0$$
$$\forall u \in A, \; i = 1, \dots, p.$$

Summing up in $i = 1, \dots, p$, we obtain

$$\sum_{i=1}^{p} \lambda_i f_i(u) + \sum_{j=1}^{m} \mu_j g_j(u) + \sum_{k=1}^{r} \nu_k h_k(u) \geq 0 \quad \forall u \in A,$$

where $\lambda_i = \sum_{s=1}^{p} \lambda_{si}$, $i = 1, \dots, p$; $\mu_j = \sum_{i=1}^{p} \mu_{ij}$, $j = 1, \dots, m$; $\nu_k = \sum_{i=1}^{p} \nu_{ik}$, $k = 1, \dots, r$, and obviously $\lambda > 0$, $\mu \geq 0$, which proves (b). □

Notice that (b) $\Leftrightarrow$ (c) and (c) $\Rightarrow$ (a) are true without the assumption on the closedness of the cones D_i.

If the functions f_i and g_j are linear and $A = \mathbb{R}^n$, Theorem 2.1 becomes the classic Tucker alternative theorem (see, for example, Mangasarian 1969). Notice that in this case the cones D_i are closed because are finite generated.

In order to decide if the cones D_i are closed, one has the following criterion.

Remark 2.1 If

$$0 \notin \operatorname{co}\left(\bigcup_{j \neq i} \partial f_j(0) \cup \bigcup_{j=1}^{m} \partial g_j(0) \right) + \operatorname{lin}\{c_k : k \in K\} + A^-$$

and $\mathrm{Ker}\, h \cap \mathrm{ri}\, A \neq \emptyset$, then D_i is closed. This follows from Proposition 2.1(ii).

From now on, we shall assume that the feasible set of problem (MP) is $M = S \cap Q$, where Q is an arbitrary closed subset of $\mathbb{R}^n$, and S is defined by

$$S = \left\{x \in \mathbb{R}^n : g(x) \leq 0,\ h(x) = 0\right\} \tag{3}$$

with $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^r$, whose component functions are, respectively, g_j, $j \in J := \{1, \ldots, m\}$, h_k, $k \in K := \{1, \ldots, r\}$. We shall adopt the following notation. Given $x_0 \in S$, the active index set at x_0 is $J_0 = \{j \in J : g_j(x_0) = 0\}$. The sets defined by the constraints g and h are denoted, respectively, by

$$G = \left\{x \in \mathbb{R}^n : g(x) \leq 0\right\}, \qquad H = \left\{x \in \mathbb{R}^n : h(x) = 0\right\},$$

so $S = G \cap H$. We suppose that the functions g_j are locally Lipschitz and that h is continuously Fréchet differentiable (i.e., C^1). The cones that will be used are the following (linearized cones):

$$C_0(S) = \left\{v \in \mathbb{R}^n : d^0 g_j(x_0, v) < 0\ \forall j \in J_0,\ \nabla h_k(x_0)v = 0\ \forall k \in K\right\},$$

$$C(S) = \left\{v \in \mathbb{R}^n : d^0 g_j(x_0, v) \leq 0\ \forall j \in J_0,\ \nabla h_k(x_0)v = 0\ \forall k \in K\right\}.$$

$C_0(G)$ and $C(G)$ are defined in an analogous way, and we denote

$K(H) = \mathrm{Ker}\, \nabla h(x_0)$.

Consequently, $C_0(S) = C_0(G) \cap K(H)$ and $C(S) = C(G) \cap K(H)$.

Lemma 2.1 *If A is a closed convex cone and $C_0(S) \cap A \neq \emptyset$, then* $\mathrm{cl}[C_0(S) \cap A] = C(S) \cap A$.

Proof By assumption $C_0(G) \cap W \neq \emptyset$, where $W = K(H) \cap A$. Since $U := C_0(G)$ is an open set, one has:

$$U \cap W = \emptyset \quad \Leftrightarrow \quad W \subset U^c \quad \Leftrightarrow \quad \mathrm{cl}\, W \subset U^c \quad \Leftrightarrow \quad \mathrm{ri}\, W \subset U^c$$
$$\Leftrightarrow \quad U \cap \mathrm{ri}\, W = \emptyset$$

(the third equivalence is true because W is a closed convex set, and so $\mathrm{cl}\,\mathrm{ri}\, W = \mathrm{cl}\, W = W$). Hence $C_0(G) \cap \mathrm{ri}\, W \neq \emptyset$. From Theorem 6.5 in Rockafellar (1970), taking into account that $\mathrm{cl}\, C_0(G) = C(G)$, we conclude that

$$\mathrm{cl}\left[C_0(G) \cap \mathrm{ri}\, W\right] = \mathrm{cl}\left[C_0(G) \cap W\right] = C(G) \cap W = C(S) \cap A. \qquad \square$$

Proposition 2.2 *Let $h : \mathbb{R}^n \to \mathbb{R}^r$ be a C^1 function, let $g : \mathbb{R}^n \to \mathbb{R}^m$ be locally Lipschitz, and let Q be a closed subset of $\mathbb{R}^n$. If the following basic constraint qualification holds at $x_0 \in G \cap H \cap Q$:*

$$\text{(BCQ)} \quad 0 \in \sum_{k \in K} \nu_k \nabla h_k(x_0) + N_C(Q, x_0) \quad \Rightarrow \quad \nu = 0,$$

then

$$\operatorname{Ker}\nabla h(x_0)\cap T_C(Q,x_0)\subset T_C(H\cap Q,x_0). \tag{4}$$

If, in addition, $C_0(S)\cap T_C(Q,x_0)\neq\emptyset$, *then* $C(S)\cap T_C(Q,x_0)\subset T(S\cap Q,x_0)$.

Proof By Proposition 3.2 in Jourani (1994), h is metrically regular at x_0 with respect to Q, i.e., there exist $\alpha>0$ and $\delta>0$ such that

$$d\big(x,h^{-1}(z)\cap Q\big)\leq\alpha\|h(x)-z\|\quad\forall x\in Q\cap B(x_0,\delta),\ z\in B\big(h(x_0),\delta\big). \tag{5}$$

Let $v\in\operatorname{Ker}\nabla h(x_0)\cap T_C(Q,x_0)$, and take $t_n\to 0^+$ and $x_n\to x_0$ with $x_n\in H\cap Q$. Since $v\in T_C(Q,x_0)$, there exists a sequence $v_n\to v$ such that $y_n:=x_n+t_nv_n\in Q$. Since h is C^1, by the mean-value theorem we have

$$\big\|h(y_n)-h(x_n)-\nabla h(x_0)(y_n-x_n)\big\|\leq\|y_n-x_n\|\sup_{c\in[x_n,y_n]}\big\|\nabla h(c)-\nabla h(x_0)\big\|.$$

Consequently, since $\|\nabla h(\cdot)-\nabla h(x_0)\|$ is continuous and the set $[x_n,y_n]$ is compact, we have

$$\big\|t_n^{-1}\big(h(y_n)-h(x_n)\big)-\nabla h(x_0)(v_n)\big\|\leq\|v_n\|\big\|\nabla h(c_n)-\nabla h(x_0)\big\|$$

for some $c_n\in[x_n,y_n]$. Since $\nabla h(\cdot)$ is continuous at x_0, $\lim_{n\to\infty}\|\nabla h(c_n)-\nabla h(x_0)\|=0$ for the sequence $c_n\to x_0$. Therefore $\lim_{n\to\infty}t_n^{-1}\|h(y_n)-h(x_n)\|=0$, since $\nabla h(x_0)(v_n)\to\nabla h(x_0)v=0$. From (5) and using that $x_n\in H=h^{-1}(0)$, it follows that

$$d\big(y_n,h^{-1}(0)\cap Q\big)\leq\alpha\big\|h(y_n)-0\big\|=\alpha\big\|h(y_n)-h(x_n)\big\|=o(t_n).$$

Hence, there exists a sequence $z_n\in h^{-1}(0)\cap Q$ such that $\lim_{n\to\infty}(y_n-z_n)/t_n=0$. Now,

$$\lim_{n\to\infty}(x_n-z_n)/t_n=\lim_{n\to\infty}\big[(x_n-y_n)/t_n+(y_n-z_n)/t_n\big]=v,$$

and therefore $v\in T_C(H\cap Q,x_0)$.

In order to prove the second part, first let us see that

$$C_0(S)\cap T_C(Q,x_0)\subset T(S\cap Q,x_0). \tag{6}$$

Take $v\in C_0(G)\cap\operatorname{Ker}\nabla h(x_0)\cap T_C(Q,x_0)$. From the first part it follows that $v\in T_C(H\cap Q,x_0)\subset T(H\cap Q,x_0)$, and therefore there exist sequences $t_n\to 0^+$ and $v_n\to v$ such that $x_0+t_nv_n\in H\cap Q$. Since $v\in C_0(G)$, we have that $d^0g_j(x_0,v)<0$ for all $j\in J_0$. However,

$$\limsup_{n\to\infty}\frac{g_j(x_0+t_nv_n)-g_j(x_0)}{t_n}\leq\bar{d}g_j(x_0,v)\leq d^0g_j(x_0,v)<0.$$

Since $g_j(x_0)=0$ for each $j\in J_0$, there is $n_0(j)\in\mathbb{N}$ such that $g_j(x_0+t_nv_n)<0$ $\forall n\geq n_0(j)$. For each $j\in J\setminus J_0$, since $g_j(x_0)<0$ and g_j is continuous at x_0, there is

$n_0(j) \in \mathbb{N}$ such that $g_j(x_0 + t_n v_n) < 0\ \forall n \geq n_0(j)$. So, taking $n_0 = \max\{n_0(j) :\ j \in J\}$, we have $g_j(x_0 + t_n v_n) < 0\ \forall n \geq n_0$, which proves that $x_0 + t_n v_n \in G$, and therefore $x_0 + t_n v_n \in S \cap Q$, which implies that $v \in T(S \cap Q, x_0)$, and (6) is proved.

Finally, the conclusion follows from Lemma 2.1 taking into account (6) and the closeness of $T(S \cap Q, x_0)$. □

Remark 2.2 If in Proposition 2.2 some g_j is Fréchet differentiable at x_0 instead of locally Lipschitz, then the conclusion is also true, considering, in the definitions of $C_0(S)$ and $C(S)$, $\nabla g_j(x_0)v$ instead of $d^0 g_j(x_0, v)$, i.e., denoting $J_{0F} = \{j \in J_0 : g_j$ is Fréchet differentiable at $x_0\}$ and defining

$$C_0(G) = \big\{v \in \mathbb{R}^n : \nabla g_j(x_0)v < 0\ \forall j \in J_{0F}, d^0 g_j(x_0, v) < 0\ \forall j \in J_0 \setminus J_{0F}\big\},$$

$$C(G) = \big\{v \in \mathbb{R}^n : \nabla g_j(x_0)v \leq 0\ \forall j \in J_{0F}, d^0 g_j(x_0, v) \leq 0\ \forall j \in J_0 \setminus J_{0F}\big\},$$

$C_0(S) = C_0(G) \cap K(H)$ and $C(S) = C(G) \cap K(H)$. To prove this assertion, it suffices to observe that, for the sequence $x_0 + t_n v_n \in H \cap Q$ in the proof of Proposition 2.2 and $v \in C_0(G)$, one has $\nabla g_j(x_0)v < 0$, $\forall j \in J_{0F}$. However, $\lim_{n\to\infty} t_n^{-1}(g_j(x_0 + t_n v_n) - g_j(x_0)) = \nabla g_j(x_0)v < 0$, and so $g_j(x_0 + t_n v_n) < 0$ for all n large enough, and the proof follows as in the proof of Proposition 2.2.

Remark 2.3 (a) By Lemma 3.2 in Jiménez and Novo (2002), one has

$$\text{(BCQ)} \quad \Leftrightarrow \quad 0 \in \operatorname{int} \nabla h(x_0)\big(T_C(Q, x_0)\big) \quad \Rightarrow \quad \operatorname{Ker} \nabla h(x_0) \cap \operatorname{ri} T_C(Q, x_0) \neq \emptyset.$$

(b) Note that, by Proposition 2.1(iii) and (Jiménez and Novo 2002, Theorem 3.9 and Lemma 3.2), we have that (BCQ) and $C_0(S) \cap T_C(Q, x_0) \neq \emptyset$ are fulfilled if and only if the following implication is true:

$$0 \in \sum_{j \in J_0} \mu_j \partial_C g_j(x_0) + \sum_{k \in K} \nu_k \nabla h_k(x_0) + N_C(Q, x_0), \quad \mu \geq 0 \quad \Rightarrow \quad \mu = 0, \quad \nu = 0.$$

We finish this section with the following well-known result.

Lemma 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}$. If x_0 is a local minimizer of f on $S \subset \mathbb{R}^n$, then $\bar{d} f(x_0, v) \geq 0\ \forall v \in T(S, x_0)$.*

3 Constraint qualifications in multiobjective optimization

Let us consider the multiobjective optimization problem

$$\text{(MP)} \quad \operatorname{Min}\big\{f(x) : x \in S \cap Q\big\},$$

where the set S is given by (3), $Q \subset \mathbb{R}^n$ is closed, h is C^1, g is locally Lipschitz, and $f : \mathbb{R}^n \to \mathbb{R}^p$ is locally Lipschitz or Fréchet differentiable and has component functions f_i, $i \in I := \{1, \dots, p\}$. We denote $d^0 f(x_0, v) = (d^0 f_1(x_0, v), \dots, d^0 f_p(x_0, v))$.

Keeping the notation of Sect. 2, given $x_0 \in S \cap Q$, we consider the following sets: $F = \{x \in \mathbb{R}^n : f(x) \leq f(x_0)\}$, $S^0 = F \cap S$, and for each $i \in I$,

$$F^i = \left\{x \in \mathbb{R}^n : f_j(x) \leq f_j(x_0)\ \forall j \in I \setminus \{i\}\right\}$$

and $S^i = F^i \cap S$. Obviously $F = \bigcap_{i=1}^p F^i$ and $S^0 = \bigcap_{i=1}^p S^i$. Since the sets given above are defined by inequality and equality constraints, the corresponding linearized cones can be defined. Let us remark that for the set F, all the functions f_i, $i \in I$, are active at x_0 and for the set F^i, the same is true for the functions f_j, $j \in I \setminus \{i\}$. We have $C_0(S^i) = C_0(F^i) \cap C_0(G) \cap K(H)$, $C(S^i) = C(F^i) \cap C(G) \cap K(H)$, and similar expressions for $C_0(S^0)$ and $C(S^0)$. When f is Fréchet differentiable, in the definitions of the linearized cones $C_0(F^i)$, $C(F^i)$, $C_0(F)$, and $C(F)$, we use $\nabla f_j(x_0)$ instead of $d^0 f_j(x_0, \cdot)$, as in Remark 2.2.

It is a well-known result that x_0 is a local Pareto minimum to problem (MP) if and only if for each $i = 1, \ldots, p$, x_0 is a local minimum of the scalar problem

$$(\mathrm{P}_i) \quad \mathrm{Min}\left\{f_i(x) : x \in S^i \cap Q\right\}.$$

We present now different qualifications for problem (MP) as in the approaches of Maeda (1994), Preda and Chitescu (1999), Jiménez and Novo (1999), Giorgi et al. (2004), and Li (2000). The implications between them are also analyzed.

Definition 3.1 The following constraint qualifications are considered:

1. *Generalized Guignard* (*GGCQ*):

$$C\left(S^0\right) \cap T_C(Q, x_0) \subset \bigcap_{i=1}^{p} \mathrm{cl}\,\mathrm{co}\, T\left(S^i \cap Q, x_0\right).$$

2. *Abadie* (*ACQ*):

$$C\left(S^0\right) \cap T_C(Q, x_0) \subset T\left(S^0 \cap Q, x_0\right).$$

3. *Generalized Abadie* (*GACQ*):

$$C\left(S^0\right) \cap T_C(Q, x_0) \subset \bigcap_{i=1}^{p} T\left(S^i \cap Q, x_0\right).$$

4. *Cottle* (*CCQ*): (BCQ) and for each $i = 1, \ldots, p$, $C_0(S^i) \cap T_C(Q, x_0) \neq \emptyset$.

In the next proposition we give alternative formulations of the above constraint qualifications. We denote $P_F = \bigcup_{i \in I} \partial_C f_i(x_0)$, $P_G = \bigcup_{j \in J_0} \partial_C g_j(x_0)$, $P_{F_i} = \bigcup_{j \in I \setminus \{i\}} \partial_C f_j(x_0)$; if f is Fréchet differentiable, we replace $\partial_C f_i(x_0)$ by $\{\nabla f_i(x_0)\}$. For a set $A \subset \mathbb{R}^n$, $A^s = \{v \in \mathbb{R}^n : \langle a, v\rangle < 0\ \forall a \in A\}$.

Proposition 3.1

(i) (*GGCQ*) *holds* $\Leftrightarrow$ $P_F^- \cap P_G^- \cap K(H) \cap T_C(Q, x_0) \subset \bigcap_{i=1}^p \mathrm{cl}\,\mathrm{co}\, T(S^i \cap Q, x_0)$.

(ii) (ACQ) *holds* $\Leftrightarrow P_F^- \cap P_G^- \cap K(H) \cap T_C(Q, x_0) \subset T(S^0 \cap Q, x_0)$.
(iii) $(GACQ)$ *holds* $\Leftrightarrow P_F^- \cap P_G^- \cap K(H) \cap T_C(Q, x_0) \subset \bigcap_{i=1}^p T(S^i \cap Q, x_0)$.
(iv) (CCQ) *holds* $\Leftrightarrow$ (BCQ) *holds, and for each* $i \in I$, $P_{F_i}^s \cap P_G^s \cap K(H) \cap T_C(Q, x_0) \neq \emptyset$.

Proof If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, it is well known that $\partial_C\varphi(x_0)$ is a convex compact set and $d^0\varphi(x_0, v) = \max_{\xi \in \partial_C\varphi(x_0)}\langle \xi, v\rangle$. Therefore, the following statements are obvious:

(a) $d^0\varphi(x_0, v) \leq 0 \Leftrightarrow v \in \partial\varphi(x_0)^-$,
(b) $d^0\varphi(x_0, v) < 0 \Leftrightarrow v \in \partial\varphi(x_0)^s$.

Now, taking into account Lemma 2.1 in Li (2000), the proposition is clear because $C_0(G) = P_G^s$, $C(G) = P_G^-$, and similarly for the sets F and F_i. If f is Fréchet differentiable, it is clear that $P_F^- = C(F)$ and $P_{F_i}^s = C_0(F^i)$. □

Theorem 3.1 *The following implications are satisfied*:

(a) *Abadie* $\Rightarrow$ *Generalized Abadie.*
(b) *Cottle* $\Rightarrow$ *Generalized Abadie* $\Rightarrow$ *Generalized Guignard.*

Proof (a) It is clear because $S^0 = \bigcap_{i=1}^p S^i$ and so $T(S^0 \cap Q, x_0) \subset \bigcap_{i=1}^p T(S^i \cap Q, x_0)$.

(b) The second implication is obvious and for the first one, it is enough to apply Proposition 2.2 (see also Remark 2.2) to each set S^i, obtaining $C(S^i) \cap T_C(Q, x_0) \subset T(S^i \cap Q, x_0)$. From here we deduce that $C(S^0) \cap T_C(Q, x_0) = \bigcap_{i=1}^p C(S^i) \cap T_C(Q, x_0) \subset \bigcap_{i=1}^p T(S^i \cap Q, x_0)$ □

4 Optimality conditions under generalized qualifications

In this section strong Kuhn–Tucker-type necessary optimality conditions are given for a point to be local Pareto minimum. These conditions are obtained both in primal form and in dual form, with a feasible set defined by inequality and equality constraints and set constraint, and the objective functions and the inequality constraints are locally Lipschitz. In order to obtain the positivity of the multipliers associated with the objective function, a generalized constraint qualification will be assumed. In this way we generalize Maeda's results (Maeda 1994), which are valid for differentiable programs with only inequality constraints, Jiménez and Novo's results (Jiménez and Novo 1999), valid for differentiable problems with inequality and equality constraints, and Li's results (Li 2000) with locally Lipschitz data (only inequality constraints) and Clarke subdifferentials that are polytopes.

Theorem 4.1 *Let* f_i, $i \in I$, *and* g_j, $j \in J_0$, *be locally Lipschitz, let* h *be* C^1 *at* x_0, *and suppose that the generalized Abadie constraint qualification is satisfied at* x_0. *If*

$x_0 \in \mathrm{LMin}(f, S \cap Q)$, *then there exists no solution* $v \in \mathbb{R}^n$ *of the system*

$$\begin{cases} d^0 f(x_0, v) \leq 0, \qquad d^0 f(x_0, v) \neq 0, \\ d^0 g_j(x_0, v) \leq 0 \quad \forall j \in J_0, \\ \nabla h(x_0) v = 0, \\ v \in T_C(Q, x_0). \end{cases} \tag{7}$$

Proof Assume that the conclusion is not true. Then there exist $v \in \mathbb{R}^n$ and $i \in \{1, \ldots, p\}$ such that

$$\begin{cases} d^0 f_i(x_0, v) < 0, \\ d^0 f_j(x_0, v) \leq 0 \quad \forall j \neq i, \\ v \in C(S) \cap T_C(Q, x_0). \end{cases} \tag{8}$$

These conditions imply that $v \in C(S^0) \cap T_C(Q, x_0)$ and, by the generalized Abadie constraint qualification, $v \in \bigcap_{j=1}^p T(S^j \cap Q, x_0)$, and in particular $v \in T(S^i \cap Q, x_0)$. Since x_0 is a local Pareto minimum, it also is a local minimum of each scalar problem (P_j), in particular, $x_0 \in \mathrm{LMin}(f_i, S^i \cap Q)$. By Lemma 2.2 we have $\bar{d} f_i(x_0, u) \geq 0$ $\forall u \in T(S^i \cap Q, x_0)$. Choosing $u = v$, we have $\bar{d} f_i(x_0, v) \geq 0$. Since $d^0 f_i(x_0, v) \geq \bar{d} f_i(x_0, v)$, it follows that $d^0 f_i(x_0, v) \geq 0$, in contradiction to (8). □

Since $d^0 f_i(x_0, \cdot)$ is a sublinear function, if $d^0 f_i(x_0, \cdot)$ is also concave, then $d^0 f_i(x_0, \cdot)$ is linear, i.e., $d^0 f_i(x_0, \cdot) = \langle a_i, \cdot \rangle$ for some $a_i \in \mathbb{R}^n$, and so $\partial_C f_i(x_0) = \{a_i\}$ is a singleton; consequently, f is strictly differentiable at x_0 (Clarke 1983, Proposition 2.2.4). Hence, as in the following theorem we need the concavity of the derivative, we directly require f to be Fréchet differentiable (which is a slight weaker condition, and moreover we do not need f to be locally Lipschitz).

Theorem 4.2 *Let* f *be Fréchet differentiable at* x_0, *let* g_j, $j \in J_0$, *be locally Lipschitz,* h *be* C^1 *at* x_0, *and suppose that the generalized Guignard constraint qualification is satisfied at* x_0. *If* $x_0 \in \mathrm{LMin}(f, S \cap Q)$, *then there is no solution* $v \in \mathbb{R}^n$ *of system* (7) *with* $\nabla f(x_0)v$ *instead of* $d^0 f(x_0, v)$.

Proof Assume that the conclusion is false. Then for some $i \in \{1, \ldots, p\}$, the system

$$\begin{cases} \nabla f_i(x_0) v < 0, \\ \nabla f_j(x_0) v \leq 0 \quad \forall j \neq i, \\ v \in C(S) \cap T_C(Q, x_0), \end{cases} \tag{9}$$

has a solution $v \in \mathbb{R}^n$. These conditions imply that $v \in C(S^0) \cap T_C(Q, x_0)$. From the generalized Guignard constraint qualification it follows that $v \in \bigcap_{j=1}^p \mathrm{cl}\,\mathrm{co}\, T(S^j \cap Q, x_0)$, and in particular, $v \in \mathrm{cl}\,\mathrm{co}\, T(S^i \cap Q, x_0)$. Since $x_0 \in \mathrm{LMin}(f_i, S^i \cap Q)$ and f_i is Fréchet differentiable, from Lemma 2.2 we obtain $\nabla f_i(x_0) u \geq 0$ $\forall u \in T(S^i \cap Q, x_0)$ (in this case, $\bar{d} f_i(x_0, u) = \nabla f_i(x_0) u$). By the linearity and continuity of $\nabla f_i(x_0)(\cdot)$, it follows that $\nabla f_i(x_0) u \geq 0$ $\forall u \in \mathrm{cl}\,\mathrm{co}\, T(S^i \cap Q, x_0)$. Choosing $u = v$, we have a contradiction to (9). □

It is possible to obtain the dual form of these two last theorems by applying the generalized Tucker alternative theorem (Theorem 2.1).

Theorem 4.3 *Assume the hypotheses of Theorems 4.1 or 4.2. If the cones*

$$D_i = \operatorname{cone}\operatorname{co}\left(\bigcup_{j\neq i} \partial_C f_j(x_0)\right) + \operatorname{cone}\operatorname{co}\left(\bigcup_{j\in J_0} \partial_C g_j(x_0)\right) + \operatorname{lin}\left\{\nabla h_k(x_0) : k \in K\right\} + N_C(Q, x_0), \tag{10}$$

$i = 1, \dots, p$, are closed, then there exists $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that

$$\text{(a)} \quad \lambda > 0, \qquad \mu \geq 0, \qquad \mu_j g_j(x_0) = 0, \quad j = 1, \dots, m, \tag{11}$$

$$\text{(b)} \quad 0 \in \sum_{i=1}^{p} \lambda_i \partial_C f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial_C g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N_C(Q, x_0). \tag{12}$$

As usual, we take $\mu_j = 0$ if $g_j(x_0) < 0$.

Note that, by Theorem 2.1, condition (12) is equivalent to

$$\sum_{i=1}^{p} \lambda_i d^0 f_i(x_0, v) + \sum_{j=1}^{m} \mu_j d^0 g_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) v \geq 0 \quad \forall v \in T_C(Q, x_0).$$

Theorem 4.3 generalizes Theorem 5.1 in Li (2000), where it is assumed that $\partial_C f_i(x_0)$ and $\partial_C g_j(x_0)$ are convex polytopes and h and Q are not considered.

Now we investigate conditions that allow us to ensure that the cones in (10) are closed.

Proposition 4.1 (i) *If for each $i = 1, \dots, p$,*

$$C_0(S^i) \cap T_C(Q, x_0) \neq \emptyset \quad \text{and} \quad \operatorname{Ker} \nabla h(x_0) \cap \operatorname{ri} T_C(Q, x_0) \neq \emptyset, \tag{13}$$

then the cones D_i, $i = 1, \dots, p$, given by (10), are closed.

(ii) *If $\partial_C f_i(x_0)$, $i \in I$, $\partial_C g_j(x_0)$, $j \in J_0$ are polytopes and if $T_C(Q, x_0)$ is a polyhedral convex cone, then the cones D_i, $i = 1, \dots, p$, given by (10), are closed.*

Part (i) follows from Proposition 2.1(ii). Part (ii) follows from Lemma 4.3.3 in Hiriart-Urruty and Lemaréchal (1996, Chap. III), since a polytope is the convex hull of a finite number of points, a polyhedral convex cone is the intersection of finitely many half-spaces, and its normal cone is finite generated.

Note that, by Proposition 2.1(iii), condition (13) is equivalent to saying that there exists $v \in \mathbb{R}^n$ such that

$$d^0 f_j(x_0, v) < 0 \quad \forall j \in I \setminus \{i\}, \qquad d^0 g_j(x_0, v) < 0 \quad \forall j \in J_0, \qquad \nabla h(x_0) v = 0, \quad v \in \operatorname{ri} T_C(Q, x_0), \tag{14}$$

i.e., $C_0(S^i) \cap \operatorname{ri} T_C(Q, x_0) \neq \emptyset$. We point out that if the Cottle constraint qualification holds, then it is unnecessary to use the generalized Tucker alternative theorem to obtain positive multipliers, since this result can be obtained directly as follows.

Proposition 4.2 *Let f be locally Lipschitz, and suppose that the Cottle constraint qualification (CCQ) is satisfied. If $x_0 \in \operatorname{LMin}(f, S \cap Q)$, then conditions (11)–(12) hold.*

Proof For each $i = 1, \dots, p$, we have $x_0 \in \operatorname{LMin}(f_i, S^i \cap Q)$, and by Lemma 2.2, $d^0 f_i(x_0, v) \geq \bar{d} f_i(x_0, v) \geq 0\ \forall v \in T(S^i \cap Q, x_0)$. By Proposition 2.2, for each $i = 1, \dots, p$, $C(S^i) \cap T_C(Q, x_0) \subset T(S^i \cap Q, x_0)$. Therefore, none of the p systems ($i = 1, \dots, p$):

$$\begin{cases} d^0 f_i(x_0, v) < 0, & \\ d^0 f_j(x_0, v) \leq 0 & \forall j \neq i, \\ d^0 g_j(x_0, v) \leq 0 & \forall j \in J_0, \\ \nabla h_k(x_0) v = 0 & \forall k \in K, \\ v \in T_C(Q, x_0), & \end{cases} \tag{15}$$

has a solution $v \in \mathbb{R}^n$. Let us consider the convex problem in the variable v

$$\begin{aligned} (\mathrm{CP}_i) \quad \alpha_i = \operatorname{Min}\big\{ & d^0 f_i(x_0, v) : d^0 f_j(x_0, v) \leq 0\ \forall j \neq i, \\ & d^0 g_j(x_0, v) \leq 0\ \forall j \in J_0,\ \nabla h_k(x_0) v = 0\ \forall k \in K, \\ & v \in T_C(Q, x_0) \big\}. \end{aligned}$$

Because of the incompatibility of the system (15) above, we have $\alpha_i \geq 0$. Since $v = 0$ is a feasible solution and $d^0 f_i(x_0, 0) = 0$, it is $\alpha_i = 0$. From Theorem 28.2 in Rockafellar (1970) (we can use it because (14) holds for some $v \in \mathbb{R}^n$) it follows that there exist $\lambda_{ij} \geq 0$, $j \neq i$; $\mu_{ij} \geq 0$, $j \in J_0$; $\nu_{ik} \in \mathbb{R}$, $k \in K$, such that

$$d^0 f_i(x_0, v) + \sum_{j=1, j \neq i}^{p} \lambda_{ij} d^0 f_j(x_0, v) + \sum_{j \in J_0} \mu_{ij} d^0 g_j(x_0, v) + \sum_{k=1}^{r} \nu_{ik} \nabla h_k(x_0) v \geq 0$$

for all $v \in T_C(Q, x_0)$ and for $i = 1, \dots, p$. Summing over $i = 1, \dots, p$, we have

$$\sum_{i=1}^{p} \lambda_i d^0 f_i(x_0, v) + \sum_{j \in J_0} \mu_j d^0 g_j(x_0, v) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) v \geq 0 \quad \forall v \in T_C(Q, x_0),$$

where, in order to simplify, we have denoted $\lambda_i = 1 + \sum_{j=1, j \neq i}^{p} \lambda_{ji}$, $i = 1, \dots, p$; $\mu_j = \sum_{i=1}^{p} \mu_{ij}$, $j \in J_0$; $\nu_k = \sum_{i=1}^{p} \nu_{ik}$, $k = 1, \dots, r$, and obviously we have $\lambda > 0$ and $\mu \geq 0$. The conclusion is obtained using the equivalence between parts (b) and (c) in Theorem 2.1. □

Remark 4.1 Luu (2007) deals with problem (MP) with h locally Lipschitz instead of C^1. He considers the next constraint qualification (MCQ). We say that (MCQ) holds at x_0 if for each $i = 1, \dots, p$, there is no scalars $\lambda_s \geq 0$, $s \neq i$, $\mu_j \geq 0$, $j \in J_0$, $\nu_k \in \mathbb{R}$, $k \in K$, not all zero, satisfying

$$0 \in \sum_{s \neq i} \lambda_s \partial_C f_s(x_0) + \sum_{j \in J_0} \mu_j \partial_C g_j(x_0) + \sum_{k \in K} \nu_k \nabla h_k(x_0) + N_C(Q, x_0).$$

Let us observe that $\partial_C h_k(x_0) = \{\nabla h_k(x_0)\}$ $\forall k \in K$, since h is C^1. According to Remark 2.3(b), (MCQ) holds if and only if (BCQ) holds and for each $i = 1, \dots, p$, $C_0(S^i) \cap T_C(Q, x_0) \neq \emptyset$, i.e., if and only if Cottle CQ holds. In view of Remark 2.3(a), (BCQ) implies $\operatorname{Ker} \nabla h(x_0) \cap \operatorname{ri} T_C(Q, x_0) \neq \emptyset$, and by applying Proposition 4.1(i) and Theorem 3.1 it follows that (MCQ) implies that (GACQ) holds and that the cones D_i, $i = 1, \dots, p$, given by (10), are closed. In consequence, if (MCQ) holds, the assumptions of Theorem 4.3 are satisfied.

Therefore, when h is C^1, (MCQ) collapses into (CQI) in Luu (2007), and Theorem 4.3 is more general than Luu (2007, Theorem 3.1). In fact, Example 4.1(a) shows that Theorem 4.3 is applicable and (Luu 2007, Theorem 3.1) is not applicable, because (MCQ) is not satisfied.

As consequences of Theorem 4.3 and Proposition 4.1(ii), we obtain the following corollaries.

Corollary 4.1 *Assume the hypotheses of Theorem 4.1 or 4.2. If, in addition, $\partial_C f_i(x_0)$, $i \in I$, $\partial_C g_j(x_0)$, $j \in J_0$, are polytopes and $T_C(Q, x_0)$ is a polyhedral convex cone, then there exists $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that*

(a) $\lambda > 0, \qquad \mu \geq 0, \qquad \mu_j g_j(x_0) = 0, \quad j = 1, \dots, m,$

(b) $$0 \in \sum_{i=1}^{p} \lambda_i \partial_C f_i(x_0) + \sum_{j=1}^{m} \mu_j \partial_C g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N_C(Q, x_0).$$

Of course, under the hypotheses of Theorem 4.2, we must substitute $\partial_C f_i(x_0)$ by $\nabla f_i(x_0)$.

In particular, if h is not considered and $Q = \mathbb{R}^n$, Corollary 4.1 (under the assumptions of Theorem 4.1) collapses into Theorem 5.1 in Li (2000), taking Proposition 3.1(iii) into account.

Corollary 4.2 *Let f, g, and h be continuously Fréchet differentiable at x_0. Suppose that $T_C(Q, x_0)$ is a polyhedral convex cone and that the generalized Guignard qualification is satisfied. If $x_0 \in \operatorname{LMin}(f, S \cap Q)$, then there exists $(\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r$ such that*

(a) $\lambda > 0, \qquad \mu \geq 0, \qquad \mu_j g_j(x_0) = 0, \quad j = 1, \dots, m,$

(b) $$0 \in \sum_{i=1}^{p} \lambda_i \nabla f_i(x_0) + \sum_{j=1}^{m} \mu_j \nabla g_j(x_0) + \sum_{k=1}^{r} \nu_k \nabla h_k(x_0) + N_C(Q, x_0).$$

In this case, $\partial_C f_i(x_0) = \{\nabla f_i(x_0)\}$, $\partial_C g_j(x_0) = \{\nabla g_j(x_0)\}$ and $N_C(Q, x_0)$ is finite-generated, and so the cones D_i are closed.

This corollary extends Theorem 3.2 in Maeda (1994), where only inequality constraints are considered, and Corollary 8 in Jiménez and Novo (1999), where Q is not considered.

The following example illustrates our results. Notice that in part (a) Cottle qualification is not satisfied, but Theorem 4.3 can be applied. Part (b) shows that the closeness condition on the cones D_i cannot be removed of Theorem 4.3.

Example 4.1 (a) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x_1, x_2) = (2x_1 + x_2, x_2)$, let $B = \text{co}\{(-t, -t^2) : 0 \le t \le 1\}$, $x_0 = (0, 0)$, and let g be the support function of the compact convex set B, i.e., $g(x) = \sup_{b \in B} \langle b, x\rangle$, where $x \in \mathbb{R}^2$. It is easy to check the following facts:

- The feasible set is $S = \{(x_1, x_2) : x_1 \ge 0, x_1 + x_2 \ge 0\}$, and x_0 is a Pareto minimum.
- $d^0 g(x_0, v) = g(v)$ and $\partial_C g(x_0) = B$, since g is convex and positively homogeneous.
- Abadie CQ is satisfied (and by Theorem 3.1, (GACQ) and (GGCQ) are also satisfied).
- f is linear (so is C^1) with $\nabla f_1(x_0) = (2, 1)$ and $\nabla f_2(x_0) = (0, 1)$.
- The cones $D_1 = \text{cone}\{\nabla f_2(x_0)\} + \text{cone}\, B$ and $D_2 = \text{cone}\{\nabla f_1(x_0)\} + \text{cone}\, B$ are closed.
- The equation

$$0 \in \lambda_1 \nabla f_1(x_0) + \lambda_2 \nabla f_2(x_0) + \mu \partial_C g(x_0) \quad \text{with } \lambda_1, \lambda_2 > 0 \text{ and } \mu \ge 0 \tag{16}$$

  has a solution, for example, $\lambda_1 = \lambda_2 = 1$, $\mu = 2$.

Let us observe that $\partial_C g(x_0)$ is not a polytope (hence Theorem 5.1 in Li (2000) is not applicable) and that Cottle CQ is not satisfied since $C_0(S^1) = C_0(S^2) = \emptyset$.

(b) With the same data as in part (a), but now $f(x_1, x_2) = (x_1, x_1)$.

The point x_0 is a Pareto minimum, Abadie CQ is satisfied, but the cones

$$D_1 = D_2 = \text{cone}\, \text{co}\big(\big\{(1,0)\big\} \cup \partial_C g(x_0)\big) = \mathbb{R} \times (-\mathbb{R}_+) \setminus \big\{(x_1, 0) : x_1 < 0\big\}$$

are not closed. Now (16) has no solution.

Taking into account Example 4.1(b), we notice that the following results should not be correct: Theorem 3.2 in Preda and Chitescu (1999), Theorems 3.1 and 3.2 and Corollaries 3.1 and 3.2 in Li and Zhang (2005) and Theorems 6 and 7 in Yuan et al. (2007).

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