Top (2008) 16: 1–41
DOI 10.1007/s11750-008-0044-5

INVITED PAPER

Transversality of the Shapley value

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Published online: 19 April 2008

Abstract A few applications of the Shapley value are described. The main choice criterion is to look at quite diversified fields, to appreciate how wide is the terrain that has been explored and colonized using this and related tools.

Keywords Coalitional game · Shapley value · Applied game theory · Axiomatizations · Game practice

Mathematics Subject Classification (2000) 91-02 · 91A12 · 91A80

1 Introduction

The Shapley value was introduced[1] in 1953. Seen in retrospect, a great year for cooperative games, since in that year also the core appeared (Gillies 1953): the two solution concepts most widely studied and used.

[1] Created or discovered? A long lasting debate on mathematical research.

This invited paper is discussed in the comments available at:
http://dx.doi.org/10.1007/s11750-008-0045-4, http://dx.doi.org/10.1007/s11750-008-0046-3, http://dx.doi.org/10.1007/s11750-008-0047-2, http://dx.doi.org/10.1007/s11750-008-0048-1, http://dx.doi.org/10.1007/s11750-008-0049-0, http://dx.doi.org/10.1007/s11750-008-0050-7.

The title is inspired by a tutorial that one of the authors planned to deliver at the 7th meeting on Game Theory and Practice (Montreal, 2007), but was unable to do it for personal reasons. Thanks to Georges Zaccour whose invitation sparked the present survey.

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The Shapley value addresses a problem:

How to convert information about the worth that subsets of the player set can achieve, into a personal attribution (of payoff) to each of the players?

Shapley proposes an answer to this question, which is based on the idea of defining a "value" for each player involved in the game, so that players can evaluate ex-ante the convenience to participate. It is clear that there is an immediate connection of this idea with the most well-known concept of a solution at that time: the value for a zero-sum game, whose existence was proved in 1928 by von Neumann (1928), in the so-called "minimax theorem".

The approach followed by Shapley is a clever one: to provide a set of properties that a "conversion" as described above should satisfy. In other words, he uses the so-called "axiomatic approach", that proved to be so powerful just a few years before, employed by Arrow (1951 the "dictator" theorem) and by Nash (1950 the Nash bargaining solution).

Shapley succeeded in providing three conditions on the transformation from a TU-game into an allocation that can be fairly said to be natural. Actually, two of them can be considered quite compelling, from the point of view of the standard interpretation of cooperative games. The last one, which requires additivity for the transformation, is more debatable, for sure. Not incidentally, quite similar remarks can be made about the role that the "Independence of Irrelevant Alternatives" plays (in different settings) in the approach by Arrow and Nash. It is not by chance that these axioms have a key role in allowing to extend the "solution" from a small set of situations to a much broader and more interesting one.

In the fifty years elapsed since it appeared, Shapley value has shown an amazing vitality, staying in the foreground, and prompting:

- Applications to quite diverse fields (this will be the main focus of this survey);
- Introduction of new theoretical approaches to the Shapley value;
- A lot of extensions and generalizations, but also of "particularizations", that is, restrictions to smaller classes of games, of special interest for specific applications.

It is worth mentioning also that the Shapley value is used both as a *normative* tool and as a *descriptive* tool, quite similarly to what happens for the Nash bargaining solution.

So, the literature about the Shapley value is quite large, and we refer the reader, who would like to have an idea of the kind of the results that are available, to some dedicated sources or surveys, like: Roth (1988a) and six chapters in the 3rd volume of the Handbook of Game Theory (Aumann and Hart 2002): Chap. 53 by Winter (2002), 54 by Monderer and Samet (2002), 55 by McLean (2002), 56 by Neyman (2002), 57 by Aumann and Hart (2002), and 58 by Mertens (2002). See also the on-line bibliography Hart (2006).

From a broader point of view, we mention some books that offer a general introduction to game theory and, especially, to cooperative games: Owen (1995), Myerson (1991), and Osborne and Rubinstein (1994).

The aim of this contribution is not to provide a survey of the Shapley value, its theoretical developments, and the bulk of its applications. We shall look at some *specific* *cases of application*, trying to emphasize the *diversity* of the fields and disciplines to which the Shapley value has been applied. Whenever appropriate, we shall also emphasize the variants that have been introduced to better fit the application at hand. We shall discuss applications to the following fields: cost allocation (especially, the costs of infrastructures), social networks, water-focused issues, biology, reliability theory, belief formation.

We feel the need to stress that the topics chosen are not meant to be "the most important", or the like. We used some kind of "bounded rationality" criterion, i.e., chose some examples, characterized by the fact that they are significantly distant each other, with the hidden goal to encourage the search for applications that could extend the "scope" of game-theoretical methods.

The structure of this contribution is as follows: after a section devoted to establish notations, terminology, and definitions, the Shapley value is introduced in Sect. 3. Then, Sect. 4 deals with some reformulations of the original approach to the Shapley value, while Sect. 5 offers some of the extensions, generalizations, and particularizations. The remaining sections contain the discussion of the special topics outlined above. A very short concluding section precedes the list of references.

2 Notations

Given a set N, the set of its subsets will be denoted by $\mathcal{P}(N)$, while $\mathcal{P}_{\blacksquare}(N)$ refers to all *nonempty* subsets, and $\mathcal{P}_2(N)$ denotes the set of all subsets of N with cardinality 2; $R \subseteq S$ means that R is a subset of S, while the notation $R \subsetneq S$ means $R \subseteq S$ and $R \neq S$.

A TU-game[2] in characteristic form[3] is (N, v), where:

- N is a finite set (whose elements are usually said to be the "players")
- $v : \mathcal{P}(N) \to \mathbb{R}$ is a map, with $v(\emptyset) = 0$.

As it is customary in this setting, we shall be quite loose in the notation used to identify sets, to avoid to be too cumbersome. We shall use $v(i)$ instead of $v(\{i\})$, $v(ij)$ instead of $v(\{i, j\})$, $v(S \cap i)$ instead of $v(S \cap \{i\})$, and so on. We shall also use in a systematic way lowercase letters to indicate the number of elements in a set: in particular, given a coalition S, its cardinality will be referred to as s.

We shall say that a TU-game is:

- *Superadditive*, if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$ s.t. $S \cap T = \emptyset$;
- *Cohesive*, if $v(N) \geq \sum_{k=1}^{m} v(S_k)$ for any partition $\{S_1, \ldots, S_m\}$ of N;
- *Convex*, if $v(S \cup i) - v(S) \geq v(T \cup i) - v(T)$ for all $S, T \subseteq N$ s.t. $S \supseteq T$.

The class of all TU-games with player set N will be denoted by $\mathcal{G}(N)$, while $\mathcal{SG}(N)$ denotes the set of superadditive games.

Often, instead of looking at the worth of coalitions, one focuses on the *costs* attributed (or due) to the coalitions. We shall usually employ the notation $c(S)$ when

[2] Also said: "side-payment game", or "coalitional game".

[3] Or in "characteristic function form", or also "in coalitional form".

we shall refer to *cost games*. The class of cost games is identical to the class of TU-games, so that the distinction is just a matter of *interpretation*. Notice, however, that unilateral conditions (superadditivity, convexity, etc.) need the reversed inequalities to have a natural (or useful) interpretation in the context of cost games: so, subadditivity, concavity, etc., are the conditions that are usually employed for cost games.

An *allocation* for a game (N, v) is an element $x \in \mathbb{R}^N$. A *pre-imputation* is an allocation which is both feasible: $\sum_{i \in N} x_i \leq v(N)$, and collectively rational: $\sum_{i \in N} x_i \geq v(N)$. A pre-imputation that satisfies also the condition: $x_i \geq v(i)$ is said to be an *imputation*. Notice that the standard interpretation of $\sum_{i \in N} x_i \leq v(N)$ as a feasibility condition (which, together with the collective rationality assumption, identifies the pre-imputations, often called efficient allocations) is disputable if a game is not cohesive (actually a non-cohesive game can have an empty set of imputations: just consider $N = \{1, 2\}$, with $v(1) = v(2) = 1$ and $v(12) = 0$. This game has no imputation, while considering the allocation $x_1 = x_2 = 1$ as unfeasible is exposed to strong criticism). Given an allocation x, we shall use the notation $x(S)$ to denote $\sum_{i \in S} x_i$.

An imputation is in the *core* of the game (N, v) if $x(S) \geq v(S)$ for all $S \subseteq N$.

3 The basic model

Given a subset $\mathcal{C}$ of $\mathcal{G}(N)$, a (point map) *solution* on $\mathcal{C}$ is a map $\Phi : \mathcal{C} \to \mathbb{R}^N$. For a solution Φ we shall be interested in various properties.

Property 1 (Efficiency, EFF) *For all games* $v \in \mathcal{C}$, $\sum_{i \in N} \Phi_i(v) = v(N)$, *i.e.*, $\Phi(v)$ *is a pre-imputation.*

Remark 1 The name "efficiency" given to this property is acceptable from the point of view of interpretation for cohesive games, and a fortiori for superadditive games. On the contrary, it could be misleading when applied to some other class of games, or to the whole set $\mathcal{G}(N)$. A trivial example is given by the game already used in the previous section: $N = \{1, 2\}$, $v(1) = v(2) = 1$, $v(1, 2) = 0$. In the class $\mathcal{G}(N)$ it could be better to see it as a normalization condition. Due to its widespread use, we shall stick to the traditional name, but the reader should bear in mind this remark.

Property 2 (Symmetry, SYM) *If* $v(S \cup i) = v(S \cup j)$ *for all* $S \subseteq N$ *s.t.* $i, j \in N \setminus S$, *then* $\Phi_i(v) = \Phi_j(v)$.

Definition 1 (Null Player) Given a game (N, v), a player $i \in N$ s.t. $v(S \cup i) = v(S)$ for all $S \subseteq N$ will be said to be a *null player*.

Property 3 (Null Player Property, NPP) $\Phi_i(v) = 0$ *if* i *is a null player.*

Property 4 (Additivity, ADD) *Given* $v, w \in \mathcal{G}(N)$, $\Phi(v + w) = \Phi(v) + \Phi(w)$.

Theorem 1 (Shapley 1953) *There is a unique map Φ defined on $\mathcal{G}(N)$ that satisfies EFF, SYM, NPP, ADD. Moreover, for any $i \in N$:*

$$\Phi_i(v) = \sum_{S \subseteq N: i \in S} \frac{(s-1)!(n-s)!}{n!} \big(v(S) - v(S \setminus i)\big). \tag{1}$$

To describe the classical proof we need a bit of terminology.

Definition 2 (Unanimity game) Given N and $S \subseteq N$, $S \neq \emptyset$, the game u_S, defined as follows:

$$u_S(T) = \begin{cases} 1, & \text{if } S \subseteq T; \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

will be called *unanimity game*.

The reason for the name is obvious: to get "something", the unanimous consent of all of the members of S is needed (and sufficient). The set of unanimity games is a basis for the vector space $\mathcal{G}(N)$, which has dimension $2^n - 1$.

The proof of the theorem makes use of the following facts. Properties EFF, SYM, NPP determine Φ on the class of all games αv, with v a unanimity game and $\alpha \in \mathbb{R}$. Since the class of unanimity games is a basis for the vector space, ADD allows to extend Φ in a unique way to $\mathcal{G}(N)$.

Every coalitional game (N, v) can be written as a linear combination of unanimity games in a unique way, i.e., $v = \sum_{S \subseteq N, S \neq \emptyset} \lambda_S(v) u_S$. The coefficients $\lambda_S(v)$, for each $S \in \mathcal{P}_{\blacksquare}(N)$, are called *unanimity coefficients* of the game (N, v) and are given by the formula: $\lambda_S(v) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$.

An alternative representation of the Shapley value can be given in terms of the unanimity coefficients $(\lambda_S(v))_{S \in \mathcal{P}_{\blacksquare}(N)}$ of a game (N, v), that is,

$$\phi_i(v) = \sum_{S \subseteq N: i \in S} \frac{\lambda_S(v)}{s}, \tag{3}$$

for each $i \in N$. As already said, s denotes the number of elements of S, i.e., its cardinality. The number $\delta_S = \frac{\lambda_S}{s}$ is called *Harsanyi dividend*, relative to the coalition S (Harsanyi 1959).

The formula (1) can be seen as a condensed version of the following:

$$\Phi_i(v) = \frac{1}{n!} \sum_{\sigma} m_i^{\sigma}(v). \tag{4}$$

Here σ is a permutation of N, while $m_i^{\sigma}(v)$ is the *marginal contribution* of player i according to the permutation σ, which is defined as:

$$v\big(\{\sigma(1), \sigma(2), \ldots, \sigma(j)\}\big) - v\big(\{\sigma(1), \sigma(2), \ldots, \sigma(j-1)\}\big),$$

where j is the unique element of N s.t. $i = \sigma(j)$.

Formula (4) can be connected with the following story (the "room parable"): players gather one by one in a room to create the "grand coalition", and each one who enters gets his marginal contribution.[4] Assuming that all the different orders in which they enter are equiprobable, one gets Formula (4).

Let us notice that the Shapley value can be characterized by the same properties also on $\mathcal{SG}(N)$, despite of the fact that it is not a vector space. The proof simply needs to take the formula which provides a game v as a linear combination of unanimity games and decompose it into two parts: where unanimity games have positive coefficient and where coefficients are nonnegative. See, e.g., Owen (1995).

The original setting in which Shapley proved his result was the setting of superadditive games. The approach was also slightly different from those outlined here. Instead of working with a fixed set N of players, Shapley used an infinite set of potential players (but games were always with a finite number of players), and had a unique condition ("support") that amounts essentially to the joining of efficiency and null player properties.

As it was said earlier, ADD is the property which is less "obvious" among those introduced by Shapley. Its acceptability was questioned, in particular, by Luce and Raiffa (1957, pp. 250–252). Notice that the criticism (which makes use of a special example) is referred to the (normative) interpretation of the Shapley value as an "arbitration scheme"; it does not apply directly to the original interpretation proposed by Shapley. Additivity is, moreover, often a quite natural condition in the context of cost allocation.

If the game is superadditive, it is immediate to prove that the Shapley value is individually rational, and hence it is an imputation.

One of the difficulties of the Shapley value is that it is not guaranteed that the value is in the core, even when it is not empty, pointing at problems about the stability of the Shapley value allocation.

Example 1 (Glove game) Let $N = \{1, 2, 3\}$, $v(\{1, 2, 3\}) = v(\{1, 2\}) = v(\{1, 3\}) = 1$, and $v(S) = 0$ for the other coalitions S. This is the simplest example of a so-called "glove game". The core contains a unique imputation, $(1, 0, 0)$, while the Shapley value is $(4/6, 1/6, 1/6)$.

The fact that the Shapley value does not belong to the core is not a serious drawback if one looks at it as an *expected* outcome, not as a specific allocation to be realized ex post. An important class of games for which the Shapley value lies always in the core is the class of convex games.

A couple of conditions that often replace NPP and SYM are DPP and ANON. Both of these conditions are stronger than their counterparts, but it is worth mentioning them, since they describe interesting properties.

[4] An alternative approach could be to share this marginal contribution between the existing group and the new entrant. This different approach, that would lead to a final allocation different, in general, from the Shapley value, is largely unexplored. Notice that this solution does not satisfy NPP.

Property 5 (Anonymity, ANON) *Let* (N, v) *be a game and* $\sigma : N \to N$ *be a permutation. Then,* $\Phi_{\sigma(i)}(\sigma v) = \Phi_i(v)$ *for all* $i \in N$*. Here* σv *is the game defined by*: $\sigma v(S) = v(\sigma(S))$, *for all* $S \subseteq N$.

Definition 3 (Dummy Player) Given a game (N, v), a player $i \in N$ s.t. $v(S \cup i) = v(S) + v(i)$ for all $S \subseteq N$ will be said to be a *dummy player*.

Property 6 (Dummy Player Property, DPP) *If* i *is a dummy player, then* $\Phi_i(v) = v(i)$.

Clearly ANON is usually stated on a class of games $\mathcal{C}$ s.t. $\sigma v \in \mathcal{C}$, provided that $v \in \mathcal{C}$, a property which is satisfied by most of the interesting classes of games. The meaning of ANON is that whatever a player gets via Φ should depend only on the *structure* of the game v, not on his "name", i.e., the way in which he is labeled.

As a last comment on the classical results about the Shapley value, we remind that Shapley and Shubik (1954) proposed to use the Shapley value as a *power index*, when restricted to the class of simple games $\mathcal{S}(N)$, which is the class of games such that $v(S) \in \{0, 1\}$ (often is added the requirement that $v(N) = 1$). From the point of view of its axiomatic characterization, it is important to notice that the ADD property does not impose any restriction on a solution map defined on $\mathcal{S}(N)$.[5] Therefore, the classical conditions are not enough to characterize the Shapley–Shubik value on $\mathcal{S}(N)$. A condition that resembles ADD and can substitute it to get a characterization of the Shapley–Shubik index on $\mathcal{S}(N)$ is the following ($v \vee w$ is defined as: $(v \vee w)(S) = (v(S) \vee w(S)) = \max\{v(S), w(S)\}$, and $v \wedge w$ is defined analogously, using min instead):

Property 7 (Transfer, TRNSF) *For any* $v, w \in \mathcal{S}(N)$, *it holds*:

$$\Phi(v \vee w) + \Phi(v \wedge w) = \Phi(v) + \Phi(w).$$

The characterization is due to Dubey (1975).

4 Reformulations

Other axiomatic approaches have been provided for the Shapley value, of which we shall briefly describe those by Young and by Hart and Mas-Colell. We shall also mention the work of Roth on the interpretation of the Shapley value as a von Neumann–Morgenstern utility.

It is quite interesting that approaches which, like those quoted above, seem to be far from the original one by Shapley, eventually provide the same "value". Not only this offers new insights, together with some additional "defense" of the assumptions used by Shapley, but it can be of interest from the point of view of applications. Some

[5] From this point of view, considering $\mathcal{S}(N)$ as a subset of $\mathcal{SG}(N)$ or of $\mathcal{G}(N)$ does not make any difference.

of the properties may be debatable in a given context, while others will have a much greater appeal: we shall see this fact discussing some of the applications.

Let us start with the axiomatization provided by Young, which is interesting for at least two reasons: it shows that one can (apparently) dispose of the additivity condition, substituting it with another one with a good appeal; it makes clear that the NPP is just a very partial expression of a property behind the Shapley value, the marginalistic principle.

Property 8 (Marginalism, MARG) *A map* $\Psi : \mathcal{G}(N) \to \mathbb{R}^N$ *satisfies MARG if, given* $v, w \in \mathcal{G}(N)$, *for any player* $i \in N$ *s.t.*

$$v(S \cup i) - v(S) = w(S \cup i) - w(S) \quad \textit{for every coalition } S \subseteq N,$$

the following is true:

$$\Psi_i(v) = \Psi_i(w).$$

Theorem 2 (Young 1988) *There is a unique map* Ψ *defined on* $\mathcal{G}(N)$ *that satisfies EFF, SYM, and MARG. Such a* Ψ *coincides with the Shapley value.*

A proof (by induction) of this result can be found in Young (1988), where it is noticed that this axiomatization is valid also on $\mathcal{SG}(N)$. What is interesting in this approach is that it makes clear that the Shapley value is based in a *fundamental* way on a "marginalistic principle": whenever symmetry considerations do not apply, the *unique* guiding principle is: look at the marginal contributions. For sure, a quite clear-cut statement.

Property MARG can be substituted by SMON (see Young 1985). Property SMON is the following:

Property 9 (Strong Monotonicity, SMON) *A map* $\Psi : \mathcal{G}(N) \to \mathbb{R}^N$ *satisfies SMON if, given* $v, w \in \mathcal{G}(N)$, *for any player* $i \in N$ *s.t.*

$$v(S \cup i) - v(S) \geq w(S \cup i) - w(S) \quad \textit{for every coalition } S \subseteq N,$$

the following holds:

$$\Psi_i(v) \geq \Psi_i(w).$$

Just to quote a couple of solutions that do not satisfy MARG, one can refer to the "proportional to marginal product rule", which is discussed in Young (1988), and is defined as:

$$F_i(v) = v(N) \frac{v(N) - v(N \setminus i)}{\sum_{j \in N} [v(N) - v(N \setminus j)]};$$

also the solution suggested in footnote 4 clearly fails to satisfy MARG, since it does not satisfy NPP.

A quite different approach was pursued by Hart and Mas-Colell (1987). In a few words, the basic idea is to see the Shapley value as a (discrete) gradient of a conveniently defined *potential*. That is, to each game (N, v) one can associate a real number $P(N, v)$ (or, simply, $P(v)$), its potential. P has the key property that its "partial derivative", that is: $D^i(P)(N, v) = P(N, v) - P(N \setminus i, v)$, coincides with $\Phi_i(N, v)$. Here, $(N \setminus i, v)$ is defined simply as the restriction of v to the subsets of $N \setminus i$.

The nice point is that it can be proved that such a function exists and is unique, up to a constant (that can be fixed, defining $P(\emptyset, v_0) = 0$, where v_0 is the unique game defined on the empty set, being $v_0(\emptyset) = 0$, of course).

The result that delivers these properties is the following:

Theorem 3 (Hart and Mas-Colell 1987) *There is a unique map* P, *defined on the set of all finite games*, *that satisfies*:

$$\begin{cases} P(\emptyset, v_0) = 0, \\ \sum_{i \in N} D^i P(N, v) = v(N). \end{cases}$$

Moreover, $D^i(P)(N, v) = \Phi_i(N, v)$.

That such a P is uniquely defined is a consequence of its recursive definition, while the identity $D(P) = (D^i(P))_{i \in N} = \Phi$ may be proved showing that $D(P)$ satisfies Shapley's axioms.

An interesting point is that the potential P of a game (N, v) is determined only by v and its subgames. This is in contrast with the previous axiomatizations which use a "big" class of games, so that this parsimonious characterization may be of interest in applications.

Due to the connection with the Shapley value it is not surprising that there are formulas for the calculation of the potential that are similar to those that we have seen in the previous section. For example, $P(N, v) = \sum_{S \subseteq N} \frac{1}{s} \lambda_S$, where λ_S are the coefficients that appear when v is expressed as a linear combination of unanimity games. Or in terms of the Harsanyi's dividends: $P(N, v) = \sum_{i \in S} \delta_S$.

One additional contribution from Hart and Mas-Colell (1987) to the understanding of the structural properties of Shapley value is the emphasis that is given to the *consistency* requirement. In words, the idea is quite simple: given a game v, assume that some players leave the scene, taking with them what they should receive according to the Shapley value. Is there any connection between what the Shapley value assigned, in the original game v, to the players who don't leave the game, and what the Shapley value would give them in the new (smaller) game?

The key point is the appropriate definition of the reduced game, which will depend on the point map solution concept that is being considered. Assume that ϕ is the solution used and that players in T remain, while T^c is the group of players who leave the game. The reduced game (T, v_T^ϕ) is defined as follows, for all $S \subseteq T$,

$$v_T^\phi(S) = v\big(S \cup T^c\big) - \sum_{i \in T^c} \phi_i\big(S \cup T^c, v\big).$$

Notice that $v(S \cup T^c)$ represents what coalition S would have earned if they had been together with all of the players in T^c. The other term, to be subtracted, is what the members of T^c would have received if they had played the game restricted to $S \cup T^c$. As it can be easily seen, some calculations are needed to get the reduced game. Anyway, the consistency condition is easy to state:

$$\phi_i(T, v_T^{\phi}) = \phi_i(N, v), \quad \forall (N, v),\ \forall T \subseteq N,\ \forall i \in T.$$

The last reformulation that we shall see is the interpretation of the Shapley value as a von Neumann–Morgenstern utility function, due to Roth (1977), who has investigated the possibility of giving a definite meaning to Shapley's idea that his value could be seen as an "a priori" evaluation of the prospect of playing a cooperative game.

As customary in modern utility theory, the starting point is a preference relation (of some rational decision maker), described by means of a total preorder on $N \times \mathcal{SG}(N)$. As for interpretation, an element $(i, v) \in N \times \mathcal{SG}(N)$ represents a *position* (the position i) in the game v, that the decision maker is evaluating (and comparing with other positions in other games).

That is, $(i, v) \succeq^* (j, w)$ should be interpreted as: "the decision maker weakly prefers to play the position i in game v than position j in game w".

It is assumed that these preferences can be represented by means of a von Neumann–Morgenstern utility function $\theta : N \times \mathcal{SG}(N) \to \mathbb{R}$.

Roth imposes additional restrictions to the preferences on $N \times \mathcal{SG}(N)$. A key point is to allow that θ can be seen as an "extension", on the same scale, of the utility values used to describe the game. In particular, it happens that $\theta(i, cv_i) = c$, where v_i is the game defined by $v_i(S) = 1$ iff $i \in S$, and zero otherwise.

We refer to Roth (1988b) for the details (see his conditions R1, R2, and R3). The restrictions imposed by these conditions are not enough to prove the coincidence of θ with the Shapley value, but two additional conditions are needed, which can be connected with the ADD and EFF properties (ordinary risk neutrality gives ADD, while strategic risk neutrality gives EFF).

Let us see their detailed formulation.

Property 10 (Neutrality to ordinary risk over games) *Given* $(i, v),\ (i, w) \in N \times \mathcal{SG}(N)$ *and* $q \in [0, 1]$,

$$\big(i, qv + (1-q)w\big) \sim^* \big[q, (i, v); (1-q)(i, w)\big].$$

The indifference is between the alternative of being in position i in the "expected game" $qv + (1-q)w$, and the alternative of having to face the lottery according to which one should be with probability q in position i in the game v, and with probability $1-q$ in the same position, but in game w.

Roth proves that in his setting this property is equivalent to the additivity of θ.

The second condition refers to what Roth calls "strategic risk", and is connected with the expectation of the decision maker about what he could get from a pure bargaining game.

Given a unanimity game u_R (seen as a pure bargaining game), its "certainty equivalent" is defined as the real number $f(r)$ s.t. $(i, u_R) \sim^* (i, f(r)v_i)$, for a position $i \in R$. Notice that, in the game $f(r)v_i$ (v_i is the game defined above), player i should receive $f(r)$, for sure. So, this number provides one estimate of what the decision maker would expect to get in the pure bargaining game u_R.

Roth introduces the following:

Property 11 (Neutrality to strategic risk) *For all* $r \in \{1, \ldots, n\}$, $f(r) = 1/r$.

and proves that, for a decision maker who satisfies both kinds of neutrality, the utility θ coincides with the Shapley value.

It is worth mentioning that a different f, that is $f(r) = \frac{1}{2^{r-1}}$, yields the Banzhaf value (Banzhaf 1965), provided that the decision maker is neutral to ordinary risk.

5 Extensions, generalizations and particularizations

The Shapley value has been extended and "adapted" to different settings. Typically, the picture is as follows: additional characteristics are added to the standard model of a TU-game, and one tries to understand whether and how these add-ons influence the "classical" Shapley value.

We shall discuss here just some of the main cases of this kind. About this, we would like to stress that we shall not discuss a lot of these "variations on the theme". Among the main "omissions" we quote: fuzzy games and multi-level games (see Aubin 1981 and Branzei et al. 2005); games with infinitely many players (Aumann and Shapley 1974, is a classical reference); stochastic cooperative games (Suijs and Borm 1999; Suijs et al. 1999; Timmer et al. 2004), who propose three different approaches for the Shapley value); NTU-games, for which there are a couple of outstanding extensions of Shapley value: the λ-transfer value by Shapley (1969) and the Harsanyi (1963) NTU value.

We shall describe briefly some relevant cases:

- Simple games;
- Restricted set of coalitions;
- Coalition structures;
- Communication links;
- Weighted value.

Simple games have been already considered in Sect. 3. We just notice that the standard interpretation for these games is to consider coalitions as "winning" ($v(S) = 1$) or "losing" ($v(S) = 0$). We denote by $\mathcal{S}(N)$ the class of simple games, whose standard field of application is the analysis of power, especially political power, but also the power of shareholders.

On the class $\mathcal{S}(N)$ there is a "competing" value that is often used in their political applications: the Banzhaf value (Banzhaf 1965). The Banzhaf value for a game can be evaluated using a formula similar to the one for the Shapley value:

$$\beta_i(v) = \sum_{S \subseteq N : i \in S} \frac{1}{2^{n-1}} \big(v(S) - v(S \setminus i)\big). \tag{5}$$

The restriction that defines the class of simple games is a restriction on the values that the characteristic function can take. Somehow dually one can consider the case in which the characteristic function for a TU-game is undefined for some coalitions. It is easy to give examples for which this restriction arises in a natural way: connection problems (some players, i.e., nodes in a network, for some reason cannot be directly connected), various kinds of incompatibility between players (political reasons, linguistic barriers, etc.), flow problems (connecting two nodes some intermediate nodes cannot be left out), etc.

Different approaches can be used, whose acceptability depends on the context. A simple idea is to define $v(S)$ just as the sum of the values of its subcoalitions. In general, this operation could lead to different values, in case of different possible ways[6] of aggregating the values of subcoalitions. One way to get out of this problem is to define a new game v' as follows (assuming that v is always defined for singletons, a mild restriction):

$$v'(S) = \max\left\{\sum_{k=1}^{m} v(S_k) \ : \ \{S_1, \ldots, S_m\} \text{ a partition of } S\right\}.$$

The formula above, where only coalitions for which $v(S)$ exists are considered, can be seen as a reasonable approach in a context in which superadditivity is considered as a condition that "should" hold (notice that, having in mind the interpretation as a "cost game", the use of *min* instead of *max* would be appropriate).

Another approach can be adapting the "room parable": we shall come back to this issue in Sect. 8.

The first case that we consider in which there is some additional structure, is when players can be partitioned into a set of mutually disjoint coalitions, that represent the existence of special ties among the players. A typical example of this kind is the division of citizens, or of parliament members, into political parties. For a less obvious example, players could be aircraft landings, and the special ties could be provided by the fact that planes belong to the same flight company. It is, of course, of interest to take into account in some way this additional structure, and see which kind of adaptation can/must be done to the Shapley value to catch it in some way.

Different approaches have been proposed to this issue. We shall mention here just a couple of them.

Assume that the set of players is partitioned into a coalition structure, so that each player belongs to exactly one of these coalitions.

The question is: what is the payoff that a player i, belonging to a coalition S in this coalition structure, would get?

A straightforward answer, provided by Aumann and Dréze (1974), is that to player i its Shapley value (that we shall denote by $\Phi_i(v, S)$) of the game v, restricted to the subsets of S (we shall continue to denote it by v), should be imputed. Of course, $\Phi_i(v, N)$ is just the Shapley value for player i in the game (N, v).

[6] For example, for $N = \{1, 2, 3\}$, assume that only one and two-person coalitions are directly meaningful. There is no reason, in principle, why it could not happen that, e.g.: $v(1) + v(23) \neq v(12) + v(3)$.

It has been proved by Myerson (1980) that the coalitional value can be characterized by means of the following conditions:

$$\begin{cases} \sum_{i \in S} \Phi_i(v, S) = v(S), & \forall S \subseteq N; \\ \Phi_i(v, S) - \Phi_i(v, S \setminus j) = \Phi_j(v, S) - \Phi_j(v, S \setminus i), & \forall S \subseteq N \; \forall i, j \in S. \end{cases} \tag{6}$$

Since, as noticed, $\Phi(v, N)$ is just the Shapley value for (N, v), clearly (6) provides, as a byproduct, another characterization for the Shapley value. The force of the conditions (6) lies, clearly, in the fact that they work on all of the subsets of N.

It is interesting to notice how the second condition in (6), which can be seen as a "balanced contributions", can be used (conveniently adapted) to provide a characterization of an extension of the Shapley value to the case in which a *graph-based conference structure* is added to the game, which we shall see later.

Another approach to this value, in presence of a coalition structure, has been provided by Owen (1977). As before, the additional structure is represented by a partition of the set N. One can find in Owen (1995), to which we refer, both an axiomatic characterization of the coalitional value, as a formula to evaluate it, given v and the coalition structure $\mathcal{T} = \{T_1, \ldots, T_k\}$, where $\mathcal{T}$ is a partition of N.

We shall describe here another approach (also in Owen 1995), because it emphasizes in a direct way the difference between the solution provided by Owen and that discussed above, exploiting in an explicit way the fact that the "a priori" coalitions are not worlds apart.

Given (N, v) and a coalition structure, i.e., a partition $\mathcal{T}$, one can build the *quotient game*, which is the game played between the "a priori unions": players are the coalitions T_j and the characteristic form is defined in an obvious way. The value $v(N)$ is divided among the coalitions T_j according to the Shapley value for the quotient game. The interesting part comes when one looks at the way in which the value is divided among the players of T_j: this is done taking into account the possibility that a player $i \in T_j$ can cooperate with players *outside* T_j.

This is done through a *reduced game* w_j. A coalition $S \subseteq T_j$ bargains with his partners in $T_j \setminus S$, taking into account what can get with the help of players in $N \setminus T_j$ and without the help of players in $T_j \setminus S$. So, the game w_j (again, its characteristic function is defined in an obvious way) involves, as players, S and the remaining unions T_m for $m \neq j$, and the coalition S will get, as its worth, the Shapley value of this game. Having built the game v_j, its Shapley value in the game w_j will be assigned to a player $i \in T_j$.

Along with this line of thought, it is worth to mention that it is possible to adapt formula (4) to get the Owen value: the value for a player i is obtained as his average marginal contributions, taking into account only the orderings of the players which respect the way in which players are grouped into unions.

In the case considered by Myerson (1977), the additional structure is a non-oriented graph G on N, that is $G \subseteq \mathcal{P}_2(N)$, which is used to formalize the conference structure mentioned above, with the intended meaning that G identifies the direct communication possibilities existing among the players.

Given a coalition S, the players of S will be able to negotiate effectively only if they are all connected via the given graph, *restricted* to S; in such a case, we say that S is internally connected via G. To express formally this, a piece of notation is useful. $S|_G$ will denote the partition of S into equivalence classes defined as the sets of players connected by the graph G *within* S. Of course, $S|_G$ will reduce to a singleton, $\{S\}$, if and only if S is internally connected via G.

The following two conditions appear as reasonable restrictions for an allocation rule Ψ that takes into account the structure provided by the communication possibilities among players, provided by G.

We assume that Ψ is defined for each game (N, v) and any graph G on N, and we shall use the notation $\Psi(v, G)$:

$$\begin{cases} \sum_{i \in S} \Psi_i(v, G) = v(S), & \forall G, \forall S \in N|_G; \\ \Psi_i(v, G) - \Psi_i(v, G \setminus \{i, j\}) = \Psi_j(v, G) - \Psi_j(v, G \setminus \{i, j\}), & \forall G\ \forall \{i, j\} \in G. \end{cases} \tag{7}$$

The first one is an "efficiency" condition that is assumed to hold only for those coalitions whose players are able to communicate effectively among them and *are not connected to other players*.

The second one says that two players should gain or lose in exactly the same way, when a direct link between them is established (or deleted).

Myerson (1977) has proved that conditions (7) characterize a unique allocation rule Ψ. Moreover, $\Psi_i(v, \mathcal{P}_2(S)) = \Phi_i(v, S)$. That is, whenever all players in S are directly connected, this rule coincides with the coalitional value. In particular, $\Psi_i(v, \mathcal{P}_2(N))$ is just the Shapley value for (N, v).

An interesting and immediate consequence of the second condition in (7) is that, in a *superadditive* game, players will never lose in forming a link between them.

To introduce a weighted Shapley value, one needs a family of positive weights $\lambda = (\lambda_i)_{i \in N}$. An immediate way to define the weighted Shapley value is to use the weights to split the gains in the unanimity games (and their multiples), and then simply use additivity to extend it to all of $\mathcal{G}(N)$. Given a coalition $S \subseteq N$ and its corresponding unanimity game u_S, the weighted Shapley value for u_S is defined as follows:

$$(\Phi_\lambda)_i(u_S) = \begin{cases} \frac{\lambda_i}{\sum_{k \in S} \lambda_k}, & \text{for } i \in S; \\ 0, & \text{for } i \notin S. \end{cases}$$

We refer to Kalai and Samet (1988) for a survey on weighted Shapley value (included the use of "weight systems" in which zero weights are also allowed).

6 Shapley value and infrastructure cost games

The topics chosen in this section have in common not only the subject, but also a technical aspect: the calculation of the Shapley value in practice. In applications, the calculation of the Shapley value often does not create computational problems: for a "generic" game (i.e., a TU-game without a special structure) the main difficulty[7] is

[7] We had investigated this issue in Moretti and Patrone (2004), where we propose an extended model of cooperative games (TUIC games) to take into account the costs incurred in getting the data and the potential arbitrary choices done choosing the level of detail.

the effort required for collecting the 2^n data needed to have a TU-game on a set of n players.

It can happen, however, that the data have a potentially simple structure, so that it is possible to treat (in applications) games with a huge number of players. As it is easy to imagine, in such cases one can exploit the specific structure of the data to get a much more manageable formula than, e.g., (1).

We shall see here a couple of examples of this kind. Both of them refer to the use of an infrastructure: a landing strip or the railway infrastructure. Another example can be found in Moretti et al. (2007), where the evaluation of the Shapley value for a game with thousands of players is possible (and easy) using an approach that bypasses the explicit construction of the microarray game (see Sect. 9.2).

The first example is one of the most classical examples in game theory: we are referring to the so-called *airport game*.

Airport games were analyzed via the Shapley value by Littlechild and Thompson (1977), and Littlechild and Owen (1973). Notice, however, that a cost allocation rule which coincides with the Shapley value for the airport game was earlier proposed in Baker and Associates (1965) and by Thompson (1971). There is also an axiomatization of the Shapley value for the class of airport games, due to Dubey (1982).

The issue is: how to divide the costs due to the landing strip of an airport among the planes that use it? One idea has to do with the identification of the players that will give rise to a cooperative (cost) TU-game: a reasonable modeling approach brings to the idea that the players are the *landings* that occur during the lifetime[8] of the landing strip.

Clearly, such a modeling choice gives a game with a lot of players. So, it becomes crucial to have an "easy" way to calculate the Shapley value, if one wants seriously to take into account this approach.

The first step is to define $c(S)$ for a set S of landings. Since not all players will need a landing strip of the same length, one can reasonably assume that the cost associated with a landing strip long enough to accommodate all of the landings in S can be imputed to S.

Formally, we partition the set of all landings, N, into groups of landings that require a strip of the same length: $N_1, N_2, \ldots, N_k$, ordered in an increasing way w.r.t. costs. For each group N_i, let C_i be its cost.

So, we can define

$$c(S) = \max\{C_t \ : \ S \cap N_t \neq \emptyset\}.$$

If we introduce the quantities $c_i = C_i - C_{i-1}$ ($C_0 = 0$), we see that we are facing a cost allocation problem in which we can identify the cost elements c_i (see Young 1994). Since, for the way in which the problem has been approached, there is no worry about the intensity of use of the various components by the players (landings), a sensible accounting principle suggests to divide the cost due to an element evenly among those who use it.

In such a way, it is easy to get a sensible cost allocation, for whose straightforward computation we need very few data:

[8] Or one can limit the analysis to a "typical" year, referring to the amortization costs.

– the cardinality of each of the homogeneous groups N_i,
– $C_1, C_2, \dots, C_k$, the costs induced by each of the groups.

The resulting allocation, for a player m belonging to N_i, is given by the formula

$$\Phi_m(N, c) = \sum_{j=1}^{i} \frac{c_j}{|\bigcup_{r=j}^{k} N_r|}.$$

The allocation has been derived applying an accounting principle, without making any explicit reference to the (cost) game that was defined above. So, an interesting aside is that this approach provides exactly the Shapley value for the given game (the decomposition property is well suited for the exploitation of the ADD property).

A slight generalization of this line of thought has been provided by Fragnelli et al. (1999) (see also González and Herrero 2004). The issue was, again, a fair imputation of the costs arising from the use of an infrastructure (the railway network, in this case) among its users. Here, again, one faces a problem whose modeling leads "naturally" to a game with a lot of players, that in this case are *trains* running on the infrastructure during, e.g., one year.

To tackle such a problem, the authors used a couple of simplifications, one of which is a decomposition of the game into additive components.

One decomposition that has been used is the decomposition of the railway infrastructure into different "facilities" (tracks, catenary, bridges, etc.), that are assumed to enter in an additive fashion into the overall costs. Of course, the acceptability of the decomposition is justified on modeling grounds, and has no special game-theoretical content. On the other hand, if one believes that the Shapley value offers a sensible way to allocate costs, this additive decomposition proves to be quite useful.

A second simplification was done assuming that the costs induced by a group of trains could be approximately described in a "linear" way, where linearity refers to the number of trains running. Concentrating the attention on one facility (e.g., tracks) the assumption made was that the costs imputed to a homogeneous group of trains could be approximately given as the sum of a fixed part, plus another part linear in the number of trains.

Dropping for the moment the necessary technicalities, clearly this approach represents a generalization of the airport game; moreover, the fact that the costs for a coalition can be expressed by means of a restricted number of parameters, allows for an easy computation of the Shapley value, even in presence of thousands of players (trains) involved.

Formally, the class of the maintenance games is introduced. The set N of players divided into groups[9] $N_1, \dots, N_k$. We are given $\frac{k(k+1)}{2}$ non-negative numbers $\{\alpha_{ij}\}_{i,j \in \{1,\dots,k\}, j \geq i}$.

[9]The same set of groups introduced for the "airport game" is used. Even if the maintenance game will be added to an airport game with the same groups of players, it is immediate to see that such an assumption can be done without loss of generality.

The maintenance cost game corresponding to $N_1, \ldots, N_k$ and to the α_{ij}'s is the cooperative cost game (N, c) defined as follows:

$$c(S) = \sum_{i=1}^{j(S)} |S \cap N_i| A_{ij(S)},$$

for $S \subseteq N$, and where $A_{ij} = \alpha_{ii} + \cdots + \alpha_{ij}$ for all $i, j \in \{1, \ldots, k\}$, with $j \geq i$, and $j(S) = \max\{r : S \cap N_r \neq \emptyset\}$.

The meaning of the numbers α_{ij} is the following. Assume that one player in N_i has used the corresponding facility i. To restore the facility up to a level j (with $j \geq i$), the maintenance costs are $A_{ij} = \alpha_{ii} + \cdots + \alpha_{ij}$.

So, $c(S)$ expresses the maintenance cost corresponding to the facility up to the level $j(S)$ (i.e., *the most sophisticated level* required by *players in* S).

In Fragnelli et al. (1999) a formula for the calculation of the Shapley value for a maintenance cost game is provided. Using the data already introduced, the Shapley value for a player i, belonging to the group $N_{j(i)}$ is:

$$\begin{aligned} \Phi_i(N, c) = \alpha_{j(i)j(i)} + \sum_{m=j(i)+1}^{k} \alpha_{j(i)m} \frac{n_m + \cdots + n_k}{n_m + \cdots + n_k + 1} \\ + \sum_{m=2}^{j(i)} \sum_{l=1}^{m-1} \alpha_{lm} \frac{n_l}{(n_m + \cdots + n_k)(n_m + \cdots + n_k + 1)} \end{aligned} \tag{8}$$

(n_r denotes, as usual, the cardinality of the set N_r).

As said earlier, in Fragnelli et al. (1999) it was assumed that the costs for the use of the infrastructure can be described as an "infrastructure cost game", which is the sum of an airport game and a maintenance cost game. Clearly, the additivity of the Shapley value allows for an easy utilization of the two formulas described above.

To show the effectiveness of the approach, in Fragnelli et al. (1999) a calculation is provided in a realistic case involving 20.000 trains. The main point is to show that estimates of the costs provided in Baumgartner (1997) can be converted into an infrastructure cost game. The remaining calculations are straightforward, even with a huge number of "players".

The problem of determining landing fees has been approached also in Vázquez-Brage et al. (1997). The authors suggest that the Owen value is more appropriate in this context, to take into account the fact that the airplanes are organized into airlines, which represent a priori unions. The set of players and the cost structure are identical to the airport game. In addition, a system $P = \{P^1, \ldots, P^A\}$ of a priori unions is given.

One result is the establishment of a formula for the Owen value in this special context. Not surprisingly, the formula allows for an easy calculation of the value for an airline a and an airplane of type t:

$$\Phi_{a,t}(N, c, P) = \sum_{\tau=1}^{t} \frac{c_\tau}{|\mathcal{A}_{\geq\tau}| \cdot |N^a_{\geq\tau}|},$$

where $N^a_{\geq\tau} = \bigcap_{k=\tau}^{T} N_k \cap P^a$ is the set of planes of airline a that are of type τ or larger, and where $\mathcal{A}_{\geq\tau} = \{a \in \{1, 2, \dots, A\} \,:\, N^a_{\geq\tau} \neq \emptyset\}$ is the set of the airlines that have airplanes of type τ or bigger.

A consequence of this formula is that the fee which will be paid by an airline depends on the types of its airplanes, but *not on the number of movements made*. This means that the cost for movement will be lower for the airlines which use the airport most intensively. Since this result could raise doubts about its fairness, the authors notice that the *total fees* paid by the companies will also include other parts, as in the classical airport game.

Connected to this remark is the behavior of Owen value w.r.t. mergers: it is proved in Vázquez-Brage et al. (1997) that mergers are always profitable, in the sense that the total costs to be paid will not increase for a group of airlines that would merge.

As a last point, which is interesting from the point of view of applying a specific value, we notice that Vázquez-Brage et al. (1997) provide a characterization of the Owen value that can be considered especially meaningful for the class of airport games. Contrary to the classical axiomatization by Owen (1977), the authors provide a characterization in which *the characteristic form is kept fixed*, while changes are allowed for the a priori unions. The search is restricted to coalitional values which would restitute the Shapley value for degenerate sets of a priori unions (i.e., in the cases in which the unions are all singletons). Under this assumption, it is shown that the Owen value is *characterized* by a couple of additional properties:

Property 12 (Balanced Contributions, BC) *Φ satisfies the property of balanced contributions if, for any system of a priori unions P, for all $P^a \in P$ and $i, j \in P^a$,*

$$\Phi_i(N, c, P) - \Phi_i(N, c, P_{-j}) = \Phi_j(N, c, P) - \Phi_j(N, c, P_{-i}),$$

where P_{-r} is the system of unions obtained when player r separates from the union P^a to which belongs, i.e.,

$$P_{-r} = \{P^1, \dots, P^{a-1}, P^a \setminus \{r\}, P^{a+1}, \dots, P^A, \{r\}\}.$$

Property 13 (Quotient game, QG) *Φ satisfies the condition*

$$\sum_{i \in P^a} \Phi_i(N, c, P) = \Phi_{P^a}(P, c^P, \mathcal{P}),$$

where P is the set of players for the game on the right, $c^P(S) = c(\bigcup_{r \in S} P^r)$ and $\mathcal{P}$ is the trivial system of unions for the set of players P, that is,

$$\mathcal{P} = \{\{P^1\}, \dots, \{P^A\}\}.$$

7 Social networks

An application of the Shapley value, which uses both the classical one and the one by Myerson (1977), has been proposed by Gómez et al. (2003), to provide a definition of *centrality* in social networks. We refer to the paper for an interesting discussion, focusing on different notions of centrality and its relation with the notion of power, and for a comparison between their proposal and other centrality measures available in the literature. Here we confine ourselves to mention that they look at a network of *social binary* relations, formally represented by a (finite) undirected graph G.

The proposal is to look at the difference between:

– $\Psi(v_G)$: the Myerson value, that takes into account the communication structure;
– $\Phi(v)$: the Shapley value, that disregards completely the information provided by the graph G.

So, the point of view adopted by the authors is clear and clearly stated in their paper: the centrality of a node refers to the *variation* in power due to the social situation (represented by the graph), whilst the power is approached via a game theoretical approach. More precisely, it is the Shapley value of a game that is used as a power index.

A couple of simple examples are used to illustrate the approach. In both cases a weighted majority game with the winning quote fixed to $2/3$ is considered. In the first case, the votes of players 1, 2, and 3 are, respectively, 40, 20, and 40%; in the second case the votes are 30, 40, and 30%. In both cases, the graph contains two links: $\{1,2\}$ and $\{2,3\}$. In the first case, the Shapley value for the players is: $1/2$ for 1 and 3, 0 for 2, while the Myerson value is $1/3$ for each of the players. So, the centrality value is $1/3$ for player 2 and $-1/6$ for 1 and 3. In the second example there is no difference between the Shapley and Myerson value, so that the centrality is 0 for all of the players. As the authors stress, the relevant role that the chosen game has in determining the centrality of a player is apparent.

Clearly, this approach needs to have a game v defined on $\mathcal{P}(N)$ (N is, of course, the set of nodes of the graph G). The general approach proposed by Gómez et al. is to restrict the choice only to *symmetric* games, aiming at emphasizing the role of the graph structure. A game is symmetric if there is some $f : \mathbb{N} \to \mathbb{R}$ s.t. $v(S) = f(s)$ for all coalitions (s is, as usual, the number of elements in S): notice that the Shapley value for a symmetric game is equal to $\frac{v(N)}{n}$ for each player. Due to the symmetry condition imposed on the game, the centrality measure inherits a couple of properties from the Myerson value. The "balanced contributions" is one, the other is the fact that, removing an edge, the centrality of the incident nodes does not increase (assuming that the game is superadditive).

The authors describe general properties of their centrality measure, and in particular, how the abstract structure of the graph influences it. Some specific symmetric games ("message", "overhead", and "conference" games) are proposed as games of special interest for the analysis of centrality, and how it is affected by the specific choice of the game v. The authors provide also an interesting decomposition of centrality into *communication* centrality and *betweenness* centrality; we refer to the paper for the details.

8 Water related issues

The modeling tool of TU-games has been often applied to the context of issues related, in various ways, with water: allocation of water, allocation of costs related with

Table 1 Table for the generalized Shapley value

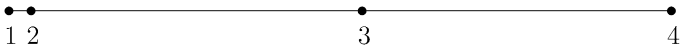
1234	2134	3124	~~4123~~
1234	2143	3142	~~4132~~
1324	2314	3214	~~4213~~
1342	2341	3241	~~4231~~
~~1423~~	~~2413~~	3412	4312
~~1432~~	~~2431~~	3421	4321

various kinds of projects (water reservoirs, irrigation systems, wastewater treatments and reuse, etc.).

The use of TU-games can be seen as a useful approach, since other methods (*in primis*: the working of a competitive market) have difficulties, due to reasons like the small number of agents involved, the absence of anonymity in the relations, and others.

So, it is far from being surprising that the Shapley value has appeared often in this context and that some interesting comments about it can be found in the literature.

One of the most relevant contributions, from this point of view, can be traced back to the work of Loehman and Whinston (1976) and related papers (e.g., Loehman et al. 1979). They take into account the fact that, for some structural reason, it is not plausible that all coalitions could be potentially considered. For example, to build a piping system



it is quite possible that coalitions like $\{1, 4\}$ or $\{2, 4\}$ will not form, possibly due to high costs due to the distance from a "source" located close to 1 and 2, while the presence of player 3 could allow some savings for coalitions like $\{1, 3, 4\}$ and $\{2, 3, 4\}$.

Such kind of situation makes disputable the applicability of the symmetry axiom. The issue is not so much, however, of assigning different weights to the players *per se*, but taking into account how these asymmetries influence the coalition building process. The proposal made by Loehman and Whinston (1976) uses the idea of a randomly generated process of building the "grand coalition" N via incremental addition of single players (the "room parable"). Analogously to the idea that we have already seen here for the Owen coalitional value, the proposal of Loehman and Whinston is to consider that some of these permutations should not be taken into account, since in the process they would require to build up an 'impossible" coalition. In the example, if we assume that coalitions $\{1, 4\}$ and $\{2, 4\}$ cannot form, then one should delete (in Table 1, these are striked out) the permutations that involve, in the process, the creation of an "impossible coalition". Notice that it is assumed that this is the *unique* reason to break the symmetry. The solution thus obtained is named the "generalized Shapley value" by Loehman and Whinston (1976).

It is clearly impossible to compare the results found above with the classical Shapley value, since the values of $\{1, 4\}$ and $\{2, 4\}$ are missing and we cannot find the Shapley value for this "partial" game. One could follow, however, a different route, simply defining $v(14) = v(1) + v(4)$ and similarly $v(24) = v(2) + v(4)$. This idea is far from being exotic, since it is rooted in a quite standard approach to define $v(S)$, as *the highest possible value that can be achieved through cooperation between the players of S* (see Sect. 5).

It is interesting to notice that this last approach, which is quite close in spirit to considerations that were expressed in a paper by Loehman et al. (1979) and that are often used to justify the superadditivity condition, provides results that are *different* from those of Loehman and Whinston (1976). So, it appears to be a delicate issue to choose between the two approaches.

Other kinds of interesting considerations can be found in other papers devoted to the analysis of water related issues. For example, Young et al. (1982) bring the attention to the fact that often in a project one has to choose the allocation rule before the actual costs are known (at best, good estimates are available). True costs will be known later. This means that a property of *monotonicity* w.r.t. variations in the total costs (i.e., the costs for the grand coalition N) is a desirable one for the rule which is being used to find the allocation of costs (or benefits). That the Shapley value satisfies this kind of condition is straightforward.

Other issues, even if they do not apply in a specific way to the Shapley value, are worth being mentioned, since they point at a cautious use of TU-games to approach allocation problems in the context of multi-faceted issues, like water.

It is interesting, for example, to look at the restrictions that are imposed on the calculations by law: Loehman et al. (1979) offer evidence of an approach, dictated by law, that imposes a *separation* between piping costs and the costs for building the wastewater treatment facility. Correctly, Loehman et al. (1979) point at the risks of *inefficiencies* which could be generated by such a separation (discouraging users located too far from the treatment facility).

Another interesting issue comes from the ways that are allowed to collect fees to pay the costs for the wastewater treatment. The use of an allocation method, like the Shapley value, among different municipalities (for example) can provide difficulties when the citizens are called to pay for the services provided. It is quite possible that the allocation of costs among municipalities brings to differences in the tariffs for the final users that may be difficult to justify (at least, from the point of view of gathering enough *political* consensus at the local elections). Notice that there could be restrictions according to which the tariff is to be proportional (apart some fixed part) to some specific measurable quantity (water consumption, or an estimate of the amount of wastewater). This tension is clearly visible from the interesting interviews of the local decision makers that are mentioned in Loehman et al. (1979).

Anyway, to transfer an allocation (e.g., the Shapley value) from the level of municipalities to the level of the single citizen, one is forced to make a non-trivial choice between a tariff that is consistent with the Shapley value used at the upper (aggregated) level, and the need for a simple and manageable method to assess the fee at the level of the final user.

Another issue that is raised (by Dinar et al. 1986) is about the mix between players that are significantly different (e.g., towns and "big" farms). This heterogeneity of the players makes, for example, the use of the symmetry axiom questionable. One reason for asymmetry could be, for example, a different exposure to risk between "players" of different kinds. In their contribution, Dinar et al. point out the fact that there is a much higher exposure to risk of farmers, compared with a town, concerning the profits (or savings) obtainable from the facility for the treatment of wastewater. An answer to this issue could be to use a more sophisticated model than a classical TU-game, like a stochastic TU-game (see the references mentioned at the beginning of Sect. 5). One must take into account, however, that a more detailed model requires (usually) a higher number of data. Added to this, it could happen that the extension of a solution for TU-games to this richer model is not obvious or unique; this is precisely the case for the Shapley value, for which Timmer et al. (2004) show that three approaches that coincide in the classical TU-model will provide, in general, three *different* answers, when applied to a stochastic cooperative game.

9 Applications to biology

In the last two decades, the advent of new technologies in molecular biology and epidemiology allowed for the assessment of a large number of biological and environmental markers and other relevant factors at once. The increasing importance of results for medical research on the causes of disease and their implications for public health decisions was accompanied by a dynamic conceptual development of research strategies, study designs, measures of effects, and statistical methods of inference. The encoding of real-world problems in these fields requires many variables, representing the factors that jointly produce an effect of interest, and involves the refinement of multifactorial statistical models that realistically represent their complex interrelations. In this direction, the Shapley value of a properly defined coalitional game has been used in several applications as an index which attributes to each factor/variable a measure of its relevance in producing a certain biological or epidemiological effect.

9.1 Epidemiology and risk analysis

In many epidemiological situations, multiple influential factors affect the risk of a disease for the individuals of a given population. Consequently, it is necessary to quantify the impact on the disease load in the population that can be attributed to having been exposed to certain risk factors. Suppose, for instance, a uranium miner, exposed to some amount of occupational radiation, smokes cigarettes and has finally developed lung cancer. In the case of the miner taking legal actions in order to fight for compensation, a juridical problem arises: both exposures are known to be strong risk factors for lung cancer, but the employer is only responsible for radiation, whereas the worker himself is responsible for smoking. So, how can the compensation for him be determined? This kind of problem in Epidemiology is known as the risk attribution problem and can be formulated in the following way. Consider a *binary random variable* D with $D=1$ in case of a diseased individual and $D=0$, otherwise. In addition, consider n *categorical exposure variables* $E_1,\dots,E_n$ describing n distinct risk factors, where the ith variable has k_i categories and is equal to zero (the low risk level) in case of no exposure to the ith risk factor. In this multifactorial situation, epidemiological risk analysis can be directed towards answering the following questions for each subset $S\subseteq N$ of exposure variables, with $N=\{E_1,\dots,E_n\}$: to what maximum extent can the probability of disease be lowered by completely reducing exposure to the risk factors described by variables in S to low risk levels?

The attributable risk parameter that is suited in epidemiology to answer this question is the *combined attributable risk* AR, defined in (9). For any $S \subseteq N$, $AR(S)$ is defined as the relative difference between the original probability of disease among the exposed individuals and the reduced probability of disease $P_S^{\text{red}}(D=1)$, which hypothetically results from reducing exposure of selected risk factors in S to low risk levels in the exposed population.

$$AR(S) = \frac{P(D=1) - P_S^{\text{red}}(D=1)}{P(D=1)}, \tag{9}$$

where the probability distributions P and P^{red} have to be estimated from epidemiological data. Note that $P_\emptyset^{\text{red}} = P$; so $AR(\emptyset) = 0$.

Cox (1985) noticed the formal equivalence of multifactorial risk assessment in epidemiology and the mathematical formalism in cooperative game theory. Each variable can be compared to a player and any subset of variables specifying risk factors to be eliminated in the population corresponds to a coalition of players. The solution to the problem of multifactorial risk attribution can be found by interpreting the function $AR : \mathcal{P}(N) \to \mathbb{R}$ mapping each subset of variables to their combined (adjusted) attributable risk as a coalitional n-person game (Land and Gefeller 1997). According to Gefeller et al. (1998) and Land and Gefeller (1997), the Shapley value $\phi(AR)$ of this game, quantifies the shares of probability of diseases that can be attributed to having being exposed to the respective risk factors.

Example 2 Land and Gefeller (2000) show an example of application aimed at quantifying the influence of *smoking* and three types of cholesterol levels (namely, *LDL*, *HDL*, and *VLDL*) on a certain heart disease: myocardial infarction. The occurrence of infarction was related to cholesterol levels and smoking habits so that four binary exposure variables s, l, h, and v of primary interest are defined as follows:

- $s = 0$ for nonsmokers, $s = 1$ otherwise;
- $l = 0$ for LDL-cholesterol $< 160 mg/dl$, $l = 1$ otherwise;
- $h = 0$ for HDL-cholesterol $> 35 mg/dl$, $h = 1$ otherwise;
- $v = 0$ for VLDL-cholesterol $< 30 mg/dl$, $v = 1$ otherwise.

From estimates from epidemiological data of 6029 male industrial workers aged 40-60 years, who were observed for myocardial infarction during a five year follow-up period in Gottingen, the game $(\{s, l, h, v\}, AR)$ was computed (see Table 2). For instance, it has been calculated that when all four factors are absent, the incidence rate of the disease is about 77 percent lower than the incidence rate in the total population. The Shapley attribution for the game AR is about 0.211, 0.405, 0.074, and 0.08 for smoking, LDL-, HDL-, and VLDL-cholesterol, respectively.

To justify the use of the Shapley value in risk attribution, Gefeller et al. (1998) proposed an interpretation of SMON, ANON, EFF in an epidemiological context, which axiomatically characterize the Shapley value (see Sect. 4). It should be mandatory

Table 2 The Gottingen game

S:	$\emptyset$	$\{s\}$	$\{l\}$	$\{h\}$	$\{v\}$	$\{s,l\}$	$\{s,h\}$	$\{s,v\}$
$AR(S)$:	0	0.3697	0.5773	0.1722	0.1669	0.7109	0.4380	0.4414
S:	$\{l,h\}$	$\{l,v\}$	$\{h,v\}$	$\{s,l,h\}$	$\{s,l,v\}$	$\{s,h,v\}$	$\{l,h,v\}$	$\{s,l,h,v\}$
$AR(S)$:	0.6397	0.6272	0.2619	0.7496	0.7432	0.4918	0.6715	0.7698

Table 3 Urban and rural combined attributable risks in Hordaland county

S	AR^{urban}	AR^{rural}
$\emptyset$	0	0
$\{s\}$	0.3621	0.4871
$\{o\}$	0.2104	0.1627
$\{s,o\}$	0.4851	0.5584

that a method to assess those shares of the disease in the population that can be attributed to each one of several exposure of interest (i.e., the exposure-specific values) is not influenced by the enumeration of exposure variables or by any ordering among them, which is exactly the property required by ANON. Now, consider two combined attributable risks AR^{I} and AR^{II} on the same set $\{E_1, \dots, E_n\}$ of exposure variables, but with the difference that AR^{I} is evaluated on subpopulation I, and AR^{II} is evaluated on subpopulation II (for instance, I means urban population and II means rural population). Then, the interpretation of SMON is the following: whenever the additional effect of eliminating a specific exposure is at least as high in subpopulation I as in subpopulation II (independently from choice of exposures that have been eliminated before), the assigned exposure-specific value ought to be also at least as high in I as in II. Finally, EFF means that the sum of all exposure-specific values equals the combined attributable risk $AR(N)$ of all n factors.

Example 3 Gefeller et al. (1998) show an example of study from the literature where the relationship between several exposure variables and the presence of asthma and respiratory disorders was investigated based on a sample of 4469 persons from the general population of the Norwegian Hordaland county. To illustrate SMON on a numerical example, we consider only the following two binary exposure variables: *smoking* (s), i.e., ever daily smoking vs. never daily smoking; *occupational exposure* (o), i.e., ever being exposed to gas or dust vs. never being exposed. In addition, the combined attributable risks on this set of two variables $N = \{s, o\}$ was evaluated on two sub-populations, *urban* (AR^{urban}) *vs. rural* (AR^{rural}). Table 3 represents the estimates of the two combined attributable risks. Table 4 summarizes the estimates of proportionate reductions of the chronic cough prevalence achievable by eliminating smoking according to two different scenarios: eliminating smoking first ($AR^{\text{Region}}(\{s\})$) or eradicating smoking after the elimination of occupational exposure ($AR^{\text{Region}}(\{s,o\}) - AR^{\text{Region}}(\{o\})$). It is clear that the maximum potential effect of eradicating smoking is higher in the rural than in the urban region of Horda-

Table 4 Marginal contribution of variable smoking under two different sub-populations

Region	$AR^{\text{Region}}(\{s\})$	$AR^{\text{Region}}(\{s,o\}) - AR^{\text{Region}}(\{o\})$
Urban	0.3621	0.2747
Rural	0.4871	0.3957

land, no matter whether occupational exposure has previously been eliminated or not. A method used to assess the exposure-specific values which satisfies SMON is required to assign to variable s in AR^{rural} not less than the share assigned to s in AR^{urban}.

In Land and Gefeller (2000), a new parameter based on a multiplicative analogue of the Shapley value is introduced and its application and the interpretation of the results are illustrated by epidemiologic data. An interesting question about the statistical significance of the Shapley value in risk attributions has been proposed by Kargin (2005), where a measure of the uncertainty of the Shapley value based on its probabilistic interpretation is given.

9.2 Molecular biology and genetics

An example of application in molecular biology is related with microarray technology, which is a relatively new experimental methodology allowing for the simultaneous quantification of the expression (i.e., the amount of mRNA[10]) of thousands of genes. By means of gene expression microarrays it is possible to consistently generate a matrix of gene expression data, in which the rows index the genes (i.e., the variable) and the columns index the study samples/experiments (e.g., several patients with a genetic disease), which are the observations or effects.

A method based on coalitional games and the Shapley value has been proposed in Moretti et al. (2007) for inferring, from a matrix of gene expression data, the relevance of genes keeping into account their individual behaviors and their interactions when the biological system is studied under a condition of interest (e.g., a disease state, the exposure to environmental or therapeutic agents, etc.). According to this method, the frequency of *associations* of all of the subsets of genes with a condition of interest has been described by means of a coalitional game (namely, a *microarray game*) where players are genes. The relevance of genes is assessed by means of a *relevance index*, that is a solution F defined on the class of microarray games with N as a set of genes, assigning a number to each gene in a given microarray game: the higher the number attributed by the relevance index to a certain gene in a given microarray game, the higher the relevance of that gene for the mechanisms governing the genomic effects of the condition under study.

Going into more details, a *microarray game* has been introduced as a coalitional game (N, v) where N is the set of analyzed genes and the characteristic function

[10] Proteins are the structural constituents of cells and tissues and may act as necessary enzymes for biochemical reactions in biological systems. Most genes contain the information for making a specific protein. This information is coded in genes by means of the deoxyribonucleic acid (DNA). *Gene expression* occurs when genetic information contained within DNA is *transcripted* into messenger ribonucleic acid (mRNA) molecules and then *translated* into the proteins.

v assigns to each coalition $S \subseteq N$ the frequency of *associations* between a given condition and a given *expression property* of genes realized in the coalition S. Different expression properties for genes might be considered like, e.g., under- or over-expression, strong variation, etc. A key issue for the definition of a microarray game (N, v) is the notion of *association* between a condition and a gene expression property inside a coalition $S \in \mathcal{P}_{\blacksquare}(N)$. On a single microarray experiment, a sufficient requirement to realize in a coalition $S \subseteq N$ the association between a condition and an expression property is that all the genes which present such expression property belongs to the coalition S (*sufficiency principle for groups of genes*). In other words, a group of genes $S \subseteq N$ which contains all the genes showing the expression property coded by "1" (e.g., over expression) under a certain condition y (e.g., a specific cancer disease) is said to realize the association between "1" and y. We will call the coalitions which realize the association between the expression property "1" and the condition y a *winning coalition*. For example, consider a microarray experiment on a set of genes $N = \{1, 2, \ldots, 10\}$ under condition y and suppose that only genes 1, 3, and 7 show the expression property "1". Then, all set of genes $S \subseteq N$ with $1, 3, 7 \in S$ are winning coalitions.

Moving to $k \geq 1$ microarray experiments on N, we refer to a Boolean matrix $\mathbf{B} \in \{0, 1\}^{n \times k}$, where the Boolean values $0 - 1$ represent two complementary expression properties, for example the property of normal expression (coded by "0") and the property of abnormal expression (coded by "1"). Let $\mathbf{B}_{\cdot j}$ be the jth column of $\mathbf{B}$. We define the *support* of $\mathbf{B}_{\cdot j}$, denoted by $sp(\mathbf{B}_{\cdot j})$, as the set $sp(\mathbf{B}_{\cdot j}) = \{i \in N \mid \mathbf{B}_{ij} = 1\}$. The microarray game corresponding to $\mathbf{B}$ is the coalitional game (N, v), where $v : 2^N \to \mathbb{R}$ is such that $v(T)$ is the rate of occurrences of the coalition T as a winning coalition in $\mathbf{B}$; in formula, we define $v(T)$, for each $T \in 2^N \setminus \{\emptyset\}$, as the value

$$v(T) = \frac{c(\Theta(T))}{k}, \tag{10}$$

where $\Theta(T) = \{j \in K \mid sp(\mathbf{B}_{\cdot j}) \subseteq T,\ sp(\mathbf{B}_{\cdot j}) \neq \emptyset\}$, with $K = \{1, \ldots, k\}$, and where $c(\Theta(T))$ is the cardinality of $\Theta(T)$, with the convention that $v(\emptyset) = 0$. The class of microarray games with N as a set of players is denoted by $\mathcal{M}^N$.

Example 4 Consider three hypothetical microarray experiments under condition y performed on four genes whose expression property 1 (e.g., over expression) is detected and shown in Table 5 (value 0 means that property 1 is not detected).

The microarray game $(\{1, 2, 3, 4\}, v)$ is defined associating to each coalition S the average of the values obtained in each experiment. For example, $v(\{1, 2, 3\}) = \frac{2}{3}$, since the coalition $\{1, 2, 3\}$ contains all of the genes for which 1 is detected for the

Table 5 An hypothetical collection of three microarray experiments with four genes

Gene	Microarray 1	Microarray 2	Microarray 3
1	1	0	1
2	0	1	1
3	1	1	0
4	0	0	1

cases of microarray 1 and 2, but not for microarray 3: hence, it is a winning coalition in two cases over three.

So, $v(\emptyset) = v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{4\}) = v(\{1,4\}) = v(\{2,4\}) = v(\{1,2\}) = v(\{3,4\}) = 0$; $v(\{1,3\}) = v(\{2,3\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = v(\{1,2,4\}) = \frac{1}{3}$; $v(\{1,2,3\}) = \frac{2}{3}$, $v(\{1,2,3,4\}) = 1$. The relevance of each gene, according to the Shapley value of the microarray game $(\{1,2,3,4\}, v)$ is $(\frac{5}{18}, \frac{5}{18}, \frac{1}{3}, \frac{1}{9})$.

In Moretti et al. (2007), the Shapley value has been axiomatically characterized on the class of microarray games using properties with a biological interpretation. A property used in that characterization is the NP property, i.e., a relevance index should attribute null relevance to genes that do not contribute to increase or decrease the frequency of association in any coalition.

A special version of ADD, namely the Equal Splitting (ES) property, is used with a natural interpretation of giving the same reliability to different microarray experiments. Formally, this axiom is defined as follows.

Property 14 (Equal Splitting, ES) *Let* $v_1, \ldots, v_k \in \mathcal{M}^N$, $k > 1$. *The solution* F *has the* Equal Splitting (*ES*) *property, if*

$$F\left(\frac{\sum_{i=1}^k v_i}{k}\right) = \frac{\sum_{i=1}^k F(v_i)}{k}.$$

The ES property requires that the average relevance index of genes in two or more different microarray games $v_1, \ldots, v_r \in \mathcal{M}^N$ with the same set of genes, even arising from experiments provided by different laboratories, must be equal to the relevance index of genes in the average game $\frac{\sum_{i=1}^r v_i}{r}$. There is no reference in the definition of the ES property neither to the accuracy of the relevance index in each different microarray game v_i, $i \in \{1, \ldots, r\}$, nor to the accuracy of the relevance index in the average microarray game $\frac{\sum_{i=1}^r v_i}{r}$.

Other axioms used to characterize the Shapley value on the class of microarray games are based on the definition of *partnership*, which was introduced in Kalai and Samet (1988) in a general context not involving genes. A partnership represents a coalition in a coalitional game that behaves like one individual, since all its subcoalitions are powerless.

Definition 4 (Partnership of genes) Let $v \in \mathcal{M}^N$. A coalition $S \in \mathcal{P}_\blacksquare(N)$ such that for each $T \subsetneq S$ and each $R \subseteq N \setminus S$

$$v(R \cup T) = v(R) \tag{11}$$

is a *partnership of genes* in the microarray game v.

In Moretti et al. (2007), the notion of partnership of genes is used as a good representation in the microarray game context of a Gene Regulatory Pathway (GRP), i.e., a collection of genes in a cell which interact with each other, dynamically orchestrating the level of expression of the genes in the collection. An interesting axiom used to bring smaller GRPs into prominence is that if two disjoint partnerships of genes have the same frequency of associations, then genes in the smaller partnership should receive a higher relevance index than genes in the bigger one (this property is called partnership monotonicity).

Property 15 (Partnership Monotonicity, PM) *Let $v \in \mathcal{M}^N$. The solution F has the* Partnership Monotonicity (*PM*) *property, if*

$$F_i(v) \geq F_j(v)$$

for each $i \in S$ and each $j \in T$, where $S, T \in \mathcal{P}_{\blacksquare}(N)$ are partnerships of genes in v such that $S \cap T = \emptyset$, $v(S) = v(T)$, $v(S \cup T) = v(N)$, $|S| \leq |T|$.

The PM property is very intuitive: consider two disjoint partnerships of genes enforcing the same average number of cases of tumor in the set of samples. If the genes outside the union of those two partnerships are irrelevant, that is, they do not contribute in increasing the average number of tumors, then genes in the smaller partnership should receive a higher relevance index than genes in the bigger one.

Finally, two properties for relevance indices, related to the concept of partnership of genes, are the following.

Property 16 (Partnership Rationality, PR) *Let $v \in \mathcal{M}^N$. The solution F has the* Partnership Rationality (*PR*) *property, if*

$$\sum_{i \in S} F_i(v) \geq v(S)$$

for each $S \in \mathcal{P}_{\blacksquare}(N)$ such that S is a partnership of genes in the game v.

The PR property determines a lower bound of the power of a partnership, i.e., the total relevance of a partnership of genes in determining the onset of the tumor in the individuals should not be lower than the average number of cases of tumor enforced by the partnership itself.

Property 17 (Partnership Feasibility, PF) *Let $v \in \mathcal{M}^N$. The solution F has the* Partnership Feasibility (*PF*) *property, if*

$$\sum_{i \in S} F_i(v) \leq v(N)$$

for each $S \in \mathcal{P}_{\blacksquare}(N)$ such that S is a partnership of genes in the game v.

On the contrary of PR, the PF property determines an upper bound of the power of a partnership, i.e., the total relevance of a partnership of genes in determining the tumor onset in the individuals should not be greater than the average number of cases of tumor enforced by the grand coalition.

Theorem 4 (Moretti et al. 2007) *Let be given a finite set N. There is a unique solution Ψ defined on $\mathcal{M}^N$ that satisfies the properties PR, PF, PM, ES, and NP. Such a Ψ coincides with the Shapley value.*

The Shapley value has been used in molecular applications also in the works by Kaufman (2004a, 2004b) as an application of the *Multi-perturbation Shapley value Analysis* (MSA) Keinan et al. (2004), a novel method for causal function localization. Kaufman et al. (2004a, 2004b) applied MSA to identify the importance in terms of causal responsibility of some genes in performing a certain function in yeast cells. In their approach, the value of each coalition of genes is a measure of the biological system's performance for a certain function (e.g., the ability of the system to survive the UV irradiation). In order to obtain such a value for each coalition, Kaufman et al. carried out a series of experiments where genes of each different subset of n genes were perturbed concomitantly; on each experiment the performance score was also measured and the score assigned to the corresponding subset of perturbed genes, finally obtaining a coalitional game. Since 2^n experiments were needed to obtain a coalitional game, implying the impossibility to deal with the complete structure of the game both for practical and computational reasons, the authors suggested to use mathematical predictors on the available data set to predict the missing performance scores.

Using a predictor, the outcomes of all multi-perturbation experiments may be extracted, and a predicted Shapley value can be calculated on these data according to (1). When the space of multi-perturbations is too large to enumerate all configurations in a tractable manner, according to MSA one can sample orderings and calculate an unbiased estimator of each gene's contribution as its average marginal contribution over all sampled orderings. Additionally, an estimator of the standard deviation of this Shapley value estimator can be obtained, allowing the construction of confidence intervals for the contribution of each of the elements, as well as the testing of statistical hypotheses on whether the contribution of a certain element equals a given value (e.g., zero or $1/n$).

An extension of the MSA method to the analysis of perturbations in networks where the importance of an element strongly depends on the state (perturbed or intact) of other elements is also introduced in Keinan et al. (2004). For example, when two genes exhibit a high degree of functional overlap, that is, redundancy, it is necessary to capture this interaction, aside from the average importance of each element. The Shapley interaction index (Grabisch and Roubens 1999; Grabisch 2000) may be useful to describe this two-dimensional interaction (Kaufman et al. 2004a, 2004b). Given a game $G = (N, v)$, the Interaction index takes into account not only the marginal contributions of players $i, j \in N$ but correlates them with the marginal contribution of the coalition $\{i, j\}$. More precisely, it considers the increasing (or decreasing) of the marginal contribution of the coalition of the two players w.r.t. the sum of the marginal contributions of the two players. For each coalition $S \subseteq N \setminus \{i, j\}$, the Interaction index takes into account the quantity $[v(S \cup \{i, j\}) - v(S)] - [v(S \cup \{i\}) - v(S)] - [v(S \cup \{j\}) - v(S)] = v(S \cup \{i, j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S)$. In doing this, it is possible to consider all the possible orderings of the players in N in which i and j are neighbors, obtaining the Interaction index *à la Shapley*, or to consider the 2^{n-2} coalitions included in $S \subseteq N \setminus \{i, j\}$, obtaining the interaction index *à la Banzhaf* (Banzhaf 1965).

On the framework of MSA is also based the *Contribution-Selection algorithm* (CSA) introduced in Cohen et al. (2005) which estimates the usefulness of features and selects them accordingly, using either forward selection or backward elimination. A coalitional game on a set of features (e.g., genes) is constructed, where the players are mapped to the features of a dataset and the payoff is represented by a real-valued function $v(S)$, which measures the performance of a classifier generated using the set of features S. The Shapley value of such a game is used for feature selection, selecting input variables, otherwise called features, that are relevant to predicting a target value for each instance in a dataset. Again, the MSA approach by Keinan et al. (2004) is used to build an unbiased estimator for the Shapley value by uniformly sampling permutations of features. Such an estimator of the Shapley value is then used heuristically to estimate the contribution value of a feature for the task of feature selection.

An interesting interpretation of the ADD property in the feature selection framework introduced by Cohen et al. (2005)is given in the following terms: ADD allows for applications to a combination of two different payoffs based on the same set of features. For a classification task these may be, for example, accuracy and area under the *receiver operator characteristic* (ROC) curve or false positive rate and false negative rate. In such case, the Shapley value of a feature which measures its contribution to the combined performance measure is just the sum of the corresponding Shapley values.

We conclude this section with a reference to a recent application of the Shapley value to the economic theory of biodiversity preservation. The Noah's ark problem (NAP) introduced by Weitzman (1998) asks how to prioritize species in a population if only some limited number can be saved. Given a set of taxa, each of which has a particular survival probability that can be increased at some cost, the NAP seeks to allocate limited funds to conserving these taxa so that the future expected biodiversity is maximized (Hartmann and Steel 2006). Various simple indices have been developed that give an indication of the distinctiveness of a taxon or of its importance to the future conservation of biodiversity.

In this context, evolutionary relationships between species are frequently represented by a *phylogenetic tree*. Phylogenetic trees are usually binary trees in which each internal node represents a bifurcation in some characteristic and the leaves are the species. Each edge has a weight that represents some unit of distance between the nodes at its endpoints (for instance, it could be the time between speciation events).

Given a phylogenetic tree constructed on a set of species N, Haake et al. (2005) define an associated coalitional game (N, v) called a *phylogenetic tree game*, where players are the n species on the leaves and $v(S)$ represents a measure of the diversity within S. More precisely, for any coalition $S \subseteq N$, the minimal spanning subtree containing the members in S is considered and $v(S)$ is computed as the sum of the edge weights of such a spanning tree on S. The Shapley value of a phylogenetic tree game is suggested to provide a natural ranking criterion to prioritize species in the NAP problem.

10 Reliability theory and the Shapley value as importance measure for components in a complex device

Consider a complex structure, for instance, an electronic circuit which is made of several components. Components may fail due to causes which are hard to anticipate and practically impossible to prevent. Failure of component may lead to the failure of the entire structure itself. We refer to the probability that a structure will perform the task for which it was designed as the *reliability* of the structure. This definition of reliability is based on the simplifying assumption that a structure can either perform or fail. In order to study the relationship between the reliability of the components of a structure and the reliability of the structure itself, one has to know how the performance or failure of various components affect the performance or failure of the structure. A measure of the extent to which the functioning or non-functioning of a certain component affects the functioning or non-functioning of the system is provided by the Shapley–Shubik power index of a particular simple game to be introduced in the following. An exhaustive discussion of the connections between the analysis of simple games and the *reliability theory* is presented in Ramamurthy (1990); Freixas and Puente (2002) analyze some extensions, discussing the appropriate probabilistic models to be used. Here we summarize the essential framework of reliability theory directly dealing with the application of the Shapley value.

A *system* or *structure* is assumed to consist of n components that we denote throughout this section by $N = \{1, \ldots, n\}$, i.e., the *set of components*. To indicate the state of the ith component, we assign a binary indicator variable x_i to component i and define

$$x_i = \begin{cases} 1, & \text{if component } i \text{ is in the functioning state;} \\ 0, & \text{if component } i \text{ is in failed state.} \end{cases}$$

Similarly, the binary variable y indicates the state of the structure, that is,

$$y = \begin{cases} 1, & \text{if the structure is functioning;} \\ 0, & \text{if the structure is failed.} \end{cases}$$

The assumption that the state of the system is completely determined by the states of its components implies the existence of a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ such that $y = f(\mathbf{x})$ for each $\mathbf{x} \in \{0, 1\}^n$. Such a function in the terminology of reliability theory is called a *structure function*.

In the following $\mathbf{x}^S \in \{0, 1\}^n$, for each $S \in 2^N$, is such that $x_i = 1$ for each $i \in S$ and $x_i = 0$ for each $i \in N \setminus S$ (if $S = \{i\}$ we will simply denote it as $\mathbf{x}^i$). Moreover, for any $\mathbf{x} \in \mathbb{R}^n$ and $i \in N$, we define the vectors $(1_i, \mathbf{x})$ and $(0_i, \mathbf{x})$ by

$$(1_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n),$$

$$(0_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n).$$

Example 5 One of the frequently encountered systems in practice is the so-called *series structure*. Figure 1 represents a series structure of n components. Here the

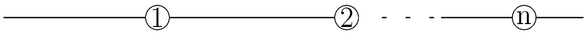


Fig. 1 Series structure

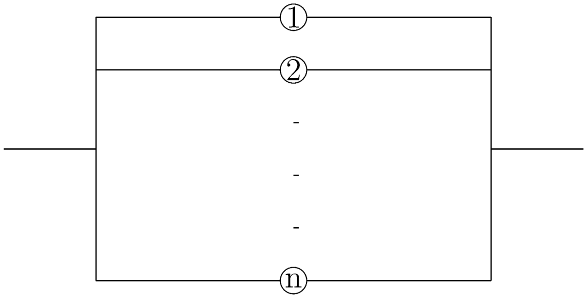


Fig. 2 Parallel structure

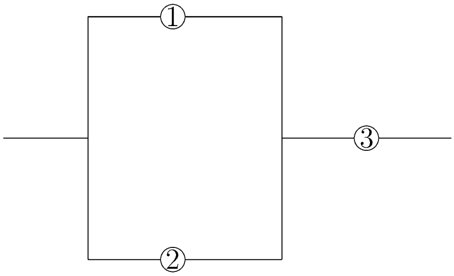


Fig. 3 Semi-coherent structure of Example 7

structure function is 1 if and only if every component function is 1. We note that the structure function is given by

$$f(x) = \prod_{i=1}^{n} x_i \quad \text{for all } x \in \{0, 1\}^n.$$

Example 6 A *parallel structure* function is 1 if and only if at least one component function is 1: see Fig. 2. We note the structure function is given by

$$f(x) = 1 - \prod_{i=1}^{n} (1 - x_i) \quad \text{for all } x \in \{0, 1\}^n.$$

Let f be a structure on N. We say that f is *monotonic* if $\mathbf{x}, \mathbf{y} \in \{0, 1\}^n$ and $\mathbf{x} \geq \mathbf{y}$ imply $f(\mathbf{x}) \geq f(\mathbf{y})$. A monotonic structure f is called *semi-coherent* if $f(\mathbf{0}) = 0$ and $f(\mathbf{1}) = 1$. It is easy to check that the game (N, λ), where f is a semi-coherent structure on N and $\lambda(S) = f(\mathbf{x}^S)$ for each $S \in 2^N$, is a monotonic simple game.

Example 7 Consider the semi-coherent structure f on $\{1, 2, 3\}$ defined by

$$f(x_1, x_2, x_3) = (1 - (1 - x_1)(1 - x_2))x_3$$

for all $(x_1, x_2, x_3) \in \{0, 1\}^3$. Figure 3 represents the corresponding semi-coherent structure f. The following table gives the values of f for all possible 2^3 Boolean vectors in $\{0, 1\}^3$.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

The corresponding simple game $(\{1, 2, 3\}, \lambda)$ is shown in the following table.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
λ	0	0	0	0	0	1	1	1

Now, let G_i denote the distribution function of the lifetime of component $i \in N$. At a given instant of time t, component i has the probability $p_i = 1 - G_i(t)$ of being in the functioning state and the complementary probability $1 - p_i$ of having failed. We also note that $p_i = E(X_i)$, where $E(X_i)$ is the expected value of the random variable X_i defined below. We call the p_i's the *component reliabilities* and let $\mathbf{p} = (p_1, \ldots, p_n)$. Let $X_i(t)$ be the random variable representing the state of component i at given instant of time t, that is,

$$X_i(t) = \begin{cases} 1, & \text{if the component } i \text{ is in the functioning state;} \\ 0, & \text{if the component } i \text{ is in failed state,} \end{cases}$$

and also let $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))$ (in the following, we omit time variable t and simply denote it as $\mathbf{X} = (X_1, \ldots, X_n)$). The random variable $f(\mathbf{X})$ represents the state of the system at a given instant of time t, that is,

$$f(\mathbf{X}) = \begin{cases} 1, & \text{if the structure is in the functioning state;} \\ 0m, & \text{if the structure is in the failed state.} \end{cases}$$

The *reliability function* of a structure f on N is the function $\hat{f} : [0, 1]^n \to [0, 1]$ defined by

$$\hat{f}(\mathbf{p}) = \text{Prob}\{f(\mathbf{X}) = 1\} = E(f(\mathbf{X})). \tag{12}$$

Example 8 It is easy to verify the expressions for the reliability functions for the following structures.

(1) Series structure: $\hat{f}(\mathbf{p}) = p_1 \cdot p_2 \cdots p_n$.
(2) Parallel structure: $\hat{f}(\mathbf{p}) = 1 - (1 - p_1)(1 - p_2) \cdots (1 - p_n)$.
(3) Semi-coherent structure of Example 7: $\hat{f}(\mathbf{p}) = (1 - (1 - p_1)(1 - p_2))p_3$.

Let f be a semi-coherent structure on N. An *index of relative importance* of a component is a measure of the extent to which the functioning or non-functioning of the component affects the functioning or non-functioning of the system. Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$ be the random vector representing the states of the components. For any $i \in N$ we define the index $\pi_i(f)$ by

$$\begin{aligned}\pi_i(f) &= \text{Prob}\big\{f(1_i, \mathbf{X}) = 1 \text{ and } f(0_i, \mathbf{X}) = 0\big\} \\ &= \text{Prob}\big\{f(1_i, \mathbf{X}) - f(0_i, \mathbf{X}) = 1\big\} \\ &= E\big(f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})\big). \end{aligned} \tag{13}$$

Example 9 If f is a series structure on N, then it is easy to verify that

$$f(1_i, \mathbf{x}) = x_1 x_2 \cdots x_{i-1} x_{i+1} \cdots x_n,$$

and $f(0_i, \mathbf{x}) = 0$ for all $x \in \{0, 1\}^n$ and $i \in N$. It follows, therefore, that in this case

$$\pi_i = E(X_1 X_2 \cdots X_{i-1} X_{i+1} \cdots X_n).$$

We note that $\pi_i(f)$ is the probability that a system function is 1 when the ith component function is 1 and is failed otherwise. This index was introduced by Straffin (1976) in the context of voting (simple) games, where $\pi_i(f)$ is the probability that the vote of player i makes a difference in the outcome of the game.

Obviously, the value of $\pi_i(f)$ depends on how we specify the joint probability distribution of $X_1, X_2, \dots, X_n$. This problem was also considered by Straffin in the context of simple games, who proposed the following assumption of general homogeneity.

Definition 5 (General homogeneity) The joint probability distribution of the binary random variables $X_1, X_2, \dots, X_n$ is said to satisfy the general homogeneity assumption if there exists a random variable P with a distribution function F on $[0, 1]$ such that $X_1, X_2, \dots, X_n$ can be considered as independent and identically distributed Bernoulli variables with parameter p, conditionally given $P = p$.

Straffin gave the following interpretation in the context of voting games. Suppose a voting body which is represented by a simple game must decide to pass or reject a sequence of bills. Let each bill be characterized by a probability vector $(p_1, p_2, \dots, p_n)$ where p_i is to be interpreted as the probability that the ith player will vote "yes" on the given bill. For different bills, the p_i's are to be selected from some distribution F over $[0, 1]$ and $p_i = p$ for all $i \in N$. The number p could then be interpreted as the "level of acceptability" of the bill. Some bills are highly acceptable (p near 1), some are highly unacceptable (p near 0) and some are controversial (p near $\frac{1}{2}$).

No such simple and appealing interpretation is possible in the context of reliability. However, we shall now show that the general homogeneity assumption can be derived as a logical consequence of certain desirable properties of the indices. To be specific, we observe that it is possible to prove that the general homogeneity assumption for the binary random variables $X_1, X_2, \dots, X_n$ is equivalent to the following two requirements:

R_1: For any given nonempty subset $S \subseteq N$, we shall denote by f_S the series structure on S. For any $S \in \mathcal{P}_\blacksquare(N)$ and $i, j \in S$, we have $\pi_i(f_S) = \pi_j(f_S)$.

R_2: Suppose we increase the number of components by an arbitrary positive number k and also let $X_{n+1}, \dots, X_{n+k}$ be the corresponding binary random variables representing the states of the newly introduced components $n+1, \dots, n+k$. If $N' = \{1, 2, \dots, n, n+1, \dots, n+k\}$, then it should be possible to extend the joint probability distribution of $X_1, X_2, \dots, X_n, X_{n+1}, \dots, X_{n+k}$ so that R_1 still holds true when we replace N by N'.

The following proposition makes clear the connection between reliability function and power indices.

Proposition 1 *Let f be a semi-coherent structure on N and let $\hat{f}$ be its reliability function. Under the assumption of general homogeneity, we have for all $i \in N$*

$$\pi_i(f) = \int_0^1 \big(\hat{f}(1_i, p) - \hat{f}(0_i, p)\big)\, dF(p) \tag{14}$$

where F is the prior distribution of the parameter P.

Proof Let $\mathbf{X} = (X_1, X_2, \dots, X_n)$. Simply note that

$$\begin{aligned}\pi_i(f) &= E\big(f(1_i, \mathbf{X}) - f(0_i, \mathbf{X})\big)\\ &= \int_0^1 E\big(f(1_i, \mathbf{X}) - f(0_i, \mathbf{X}) | P = p\big)\, dF(p)\\ &= \int_0^1 \big(\hat{f}(1_i, p) - \hat{f}(0_i, p)\big)\, dF(p).\end{aligned}$$

□

Barlow and Proschan (1975) considered the problem of a priori quantification of relative importance of component of reliability systems. They referred to this as the problem of measurement of structural importance of components. They prove that if $F(p) = p$, for each $p \in [0, 1]$, then by Relation (14) we have

$$\pi_i(f) = \int_0^1 \big(\hat{f}(1_i, p) - \hat{f}(0_i, p)\big)\, dp = \phi_i(\lambda), \tag{15}$$

where $\phi(\lambda)$ is the Shapley–Shubik index of the game (N, λ). From this result it follows that the Shapley–Shubik index ϕ_i is the probability that component i caused system failure under the assumption that the lives of the components are independent and identically distributed Bernoulli random variables with common reliability p. In fact, Barlow and Proschan rediscovered the Shapley–Shubik index in 1975. Note also that if $F(p) = 0$ for $p < \frac{1}{2}$ and $F(p) = 1$ for $p \geq \frac{1}{2}$, then by relation (14) it is possible to prove that $\pi(f)$ coincides with Banzhaf value (Birnbaum 1969).

Example 10 Consider the semi-coherent structure f of Example 7. Using relation (15) we can compute the Shapley–Shubik index of the game (N, λ) of Example 7. Written in formulas,

$$\pi_1(f) = \pi_2(f) = \int_0^1 \big(\hat{f}(1_1, p) - \hat{f}(0_1, p)\big)\, dp = \int_0^1 p - p^2\, dp = \frac{1}{2} - \frac{1}{3} = \frac{1}{6},$$

and

$$\pi_3(f) = \int_0^1 \big(\hat{f}(1_1, p) - \hat{f}(0_1, p)\big)\, dp = \int_0^1 \big(2p - p^2\big) - 0\, dp = 1 - \frac{1}{3} = \frac{2}{3}.$$

11 Theory of belief functions

The theory of *belief functions* provides a non-Bayesian way of using mathematical probability to quantify subjective judgements. Whereas a Bayesian assesses probabilities directly for the answer to a question of interest, a belief-function user assesses probabilities for related questions and then considers the implications of these probabilities for the question of interest. For a comprehensive single reference for the mathematical theory of belief functions see the book by Shafer (1976).

The *Transferable Belief Model* (TBM) (Smets and Kennes 1994) is a model for the representation of quantified beliefs held by a belief holder. In this framework one proposes the existence of a two-level mental model: the credal level, where beliefs are held and represented by belief functions, and the *pignistic*[11] level, where decisions are made by maximizing expected utilities. Hence, in order to compute such expectations, it is necessary to build a probability measure at the pignistic level. This probability measure is based on the agents beliefs, but should not be understood as representing the agents beliefs. It is just a probability function derived from the belief function. Such probability function is called a *pignistic probability function* and denoted as Bet$P$ to enhance its real nature, a probability measure for decision-making, for betting. Of course, Bet$P$ is just a probability measure. The problem is to derive and justify the transformation between belief functions and pignistic probabilities.

Consider beliefs held by an agent on what is the actual value of a variable ranging on a set Ω, called the *frame of discernment*. It is assumed that such beliefs can be represented by a belief function. A *basic belief assignment* (bba), or *basic belief function*, on Ω is a mapping $m : \mathcal{P}(\Omega) \rightarrow [0, 1]$ that satisfies:

$$\Sigma_{A \subseteq \Omega} m(A) = 1. \tag{16}$$

[11] 'Pignistic' is from the Latin *pignus*, meaning to bet.

The degree $m(A)$ is understood as the weight given to the fact that the agent knows only that the value of the variable of interest lies somewhere in set A, and nothing else. In other words, the probability allocation $m(A)$ is potentially shared between elements of A, but remains suspended for lack of knowledge. In the absence of conflicting information it is generally assumed that $m(\emptyset) = 0$. This is what is assumed in the following. A *belief function* Bel attached to each event (or each proposition of interest) can be bijectively associated with the basic mass function m (Shafer 1976). They are related by

$$\mathrm{Bel}(A) = \Sigma_{\emptyset \neq E \subseteq A} m(A), \tag{17}$$

for each $A \subseteq \Omega$. The belief function evaluates to what extent events are logically implied by the available evidence.

As we already said, Smets (1990) proposed one particular transformation of belief functions, called the *pignistic transformation*. Its justification was based on an intuitive argument that is rephrased in the following example. To be consistent with the original example in Smets (2005), the belief holder is called You. Moreover, we use intuitively the notion of conditional bba (for a formal definition of this notion see Smets 2005).

Example 11 (Buying a drink for Your friend Smets 2005) Suppose You have two friends, Glenn (G) and Judea (J). You know they will toss a fair coin and the winner will visit You tonight. You want to buy the drink Your friend would like to receive tonight: coke, wine, or beer. You can only buy one drink. Let $D = \{\text{coke, wine, beer}\}$. Let $m[G]$ be the conditional bba that represents Your belief about the drink Glenn will ask for, should You know he will come. From $m[G]$, You build the pignistic probability function $\mathrm{Bet}P[G]$ about the drink Glenn will ask by applying the (still to be defined) pignistic transformation. Similarly, You build the pignistic probability function $\mathrm{Bet}P[J]$ based on the conditional bba $m[J]$ that represents Your belief about the drink Judea will ask for, should You know he will come. The two pignistic probability functions $\mathrm{Bet}P[G]$ and $\mathrm{Bet}P[J]$ are the conditional probability functions about the drink that will be asked for, given You know which of Glenn or Judea will come, respectively. Before knowing who the visitor will be, the pignistic probability function $\mathrm{Bet}P$ about the drink that Your visitor will ask for is derived from classical probability theory

$$\mathrm{Bet}P(d) = \frac{1}{2}\mathrm{Bet}P[G](d) + \frac{1}{2}\mathrm{Bet}P[J](d),$$

for each $d \in D$ and where $\frac{1}{2}$ is the probability that the visitor is Glenn and the probability that the visitor is Judea, respectively. You will use the pignistic probability function $\mathrm{Bet}P$ to decide which drink to buy. But You might as well reconsider the whole problem and first compute m that represents Your belief about the drink Your visitor will ask for. It is possible to show that m is given by (see Smets 1997):

$$m(d) = \frac{1}{2}m[G](d) + \frac{1}{2}m[J](d),$$

where the $\frac{1}{2}$ is the basic belief mass given to the fact that the visitor is Glenn or that the visitor is Judea, respectively. These basic belief masses result from the coin tossing experiment, and the accepted assumption that the belief that results from an aleatory experiment is equal to the probability measure associated with the aleatory experiment. Given m, You could then build the pignistic probability Bet$P$ You should use to decide which drink to buy. It seems reasonable to assume that both solutions must be equal. This requirement implies that Bet$P$ must satisfy the linearity property defined in the following. It is the major requirement that will lead to the unique solution for the pignistic transformation.

In what follows, let Φ be a transformation that maps a belief function Bel over Ω into a probability function $F^{\text{Bel}} = \Phi(\text{Bel})$ over Ω (in the following, we will think to F^{Bel} as a function which assigns probability values F_ω^{Bel} to each element ω of Ω; the probability assigned to nonempty subsets $S \subseteq \Omega$ is then given by $F_S^{\text{Bel}} = \sum_{\omega \in S} F_\omega^{\text{Bel}}$).

Property 18 (Linearity, LIN) *Let* Bel_1 *and* Bel_2 *be two belief functions on the frame of discernment* Ω. *Then* F *is said to satisfy the linearity property iff, for any* $\alpha \in [0, 1]$,

$$F_\omega^{\alpha \text{Bel}_1 + (1-\alpha)\text{Bel}_2} = \alpha F_\omega^{\text{Bel}_1} + (1-\alpha) F_\omega^{\text{Bel}_2}$$

for each $\omega \in \Omega$.

Other assumptions used by Smets (1990) to derive the pignistic transformation are the following, assumed to hold for every $F = F^{\text{Bel}}$. The EFF property, i.e., the probabilities given by F to the elements of Ω add to one; the ANON property, which states that renaming the elements of Ω does not change the probabilities F; the NP property, which requires that the probability F given to the impossible event is zero.

For the class of belief functions on the frame of discernment Ω, we have that the unique transformation Φ which satisfies LIN, EFF, ANON, and NP properties is the Shapley value. Differently stated, the pignistic transformation Bet$P$ of a belief function Bel on the frame of discernment Ω is the Shapley value of game (Ω, Bel).[12]

In Smets (2005), a formal justification of the linearity axiom for pignistic transformation is given. For connections with possibility theory see also Dubois and Prade (1982, 2002).

12 Conclusions

We hope that the reader has taken some advantage from our efforts. Our goal was mainly to provide some nice examples of the diverse and clever ideas that lie behind

[12] In Smets (2005) the pignistic transformation Bet$P$ is associated to a bba m and is simply introduced as $\text{Bet}P(\omega) = \sum_{S \subseteq \Omega, \omega \in S} \frac{m(S)}{s} \frac{1}{1-m(\emptyset)}$ for each $\omega \in \Omega$. We note that by Relations (16) and (17) we have $\text{Bel}(A) = \sum_{S \subseteq \Omega} m(S) u_S(A)$ for each $A \subseteq \Omega$, where u_S is the unanimity game on S. So, if we assume that $m(\emptyset) = 0$, by relation (3), $\text{Bet}P(\omega)$ is, in fact, the Shapley value of player ω in game (Ω, Bel).

so many applications and adaptations of the Shapley value. This latitude of applications should not be surprising, since, after all, the Shapley value isn't anything more than a reasonable way to transform a (superadditive?) map defined on $\mathcal{P}_{\blacksquare}(N)$ into an additive map.

So, the variety of applications derives from the strength of mathematics in analyzing and proving properties that have (hopefully ...) universal validity. In principle, it is just a standard situation of the relationship between mathematics and the remaining (?) world, including the different various interpretations than can be given to the axioms. De facto, it is far from being easy to see mathematical structures behind the veil of sensible experience, so the bunch of examples seen here can possibly help to train our sight.

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