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Comparisons of coherent systems using stochastic precedence

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Abstract The comparisons of coherent systems is a relevant topic in reliability and survival studies. To this purpose, several techniques have been proposed including comparisons of expected lifetimes or comparisons using stochastic orderings. Recently, Samaniego (System Signatures and Their Applications in Engineering Reliability. International Series in Operations Research & Management Science, vol. 110. Springer, New York, 2007) proposed using stochastic precedence to compare the lifetimes of two independent coherent systems. He proved that if the components in both systems are independent and identically distributed (IID), then these comparisons are distribution free, while comparisons based on expected values are not. In the present paper, we obtain new expressions to compare systems using stochastic precedence without the assumption of IID components. In particular, we show that if the components in both systems are independent and satisfy the Cox proportional hazard rate (PHR) model with the same baseline hazard rate, then these comparisons do not depend on the baseline hazard rate function.

Keywords Coherent systems · Stochastic precedence · Signatures · Proportional hazard rate · Exponential distribution

Mathematics Subject Classification (2000) Primary 60K99 · Secondary 62N05 · 62G99

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1 Introduction

The comparisons of coherent systems is a relevant topic in reliability and survival studies. To this purpose, several techniques have been proposed including comparisons of expected lifetimes (see Navarro and Rychlik 2007; Navarro and Rubio 2010; Tokarev and Borovkov 2010) or comparisons using stochastic orderings (see Kochar et al. 1999; Shaked and Suarez-Llorens 2003; Navarro et al. 2006, 2008a, 2008b; Samaniego et al. 2009). These techniques can be applied using the concepts of signature and minimal signatures (dominance) of a coherent system (see Samaniego 1985, 2007; Navarro et al. 2007). The main disadvantage of the comparisons based on expected lifetimes is that they depend on the distribution of the components. The main disadvantage of comparisons based on partial stochastic orders is that some systems are not ordered.

Arcones et al. (2002) introduced the concept of *stochastic precedence* (SP) as an alternative approach to the notion that one random variable is smaller than another. With this concept, if X and Y are two random variables (over the same probability space), then X is SP-better than Y when $\Pr(X \le Y) \ge 1/2$. If the random variables X and Y are independent and $X \le_{ST} Y$, where $\le_{ST}$ denotes the usual stochastic order (i.e., $\Pr(X > t) \le \Pr(Y > t)$ for all t), then X is SP-better than Y (see Arcones et al. 2002, p. 171). Hence stochastic precedence is a necessary property to have the usual stochastic order.

Recently, Samaniego (2007) (see also Hollander and Samaniego 2006) proposed to use stochastic precedence to compare the lifetimes T_1 and T_2 of two independent coherent systems. Samaniego (2007) obtained some expressions to compute $\Pr(T_1 \le T_2)$ based on the signature vectors of both systems in the IID case. Therefore, these comparisons only depend on the signature vectors and hence they are distribution free. Hollander and Samaniego (2006) (see also Theorem 5.6 in Samaniego 2007) obtained similar expressions when the components in both systems are IID with distributions satisfying the proportional hazard rate model (PHR), showing that these comparisons do not depend on the baseline hazard rate function.

In the present paper, we obtain some new expressions for the probability $\Pr(T_1 \le T_2)$ used to compare the lifetimes of two coherent systems using stochastic precedence without the assumption of IID components. Our results can also be applied to mixed systems. In particular, we show that if the components of both systems are independent and satisfy the proportional hazard rate (PHR) model with the same baseline hazard rate, then these comparisons do not depend on the baseline hazard rate function. We use these properties to compare all the coherent systems with 1–5 components using stochastic precedence showing that stochastic precedence is a total order for these systems.

The rest of the paper is organized as follows. In Sect. 2, we give the main results of the paper with the expressions to compute $\Pr(T_1 \le T_2)$ under different assumptions. In Sect. 3, we give some examples of applications of these general expressions to compare coherent systems in different cases including the comparison of all the coherent systems with 1–5 IID components. A discussion is given in Sect. 4.

2 Main results

Let us assume that T_1 and T_2 are the lifetimes of two coherent systems with component lifetimes $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$, respectively. Throughout the paper we shall assume that X_i and Y_i have absolutely continuous distribution functions F_i and G_i with support contained in $[0, \infty)$ for $i = 1, 2, \ldots, n$. We shall represent their reliability functions by $\overline{F}_i = 1 - F_i$ and $\overline{G}_i = 1 - G_i$ and their probability density functions by $f_i = F_i'$ and $g_i = G_i'$. Note that the component lifetimes can be dependent. In this case, the dependence will be determined by the joint reliability function $\overline{F}$ of the random vector $(X_1, X_2, \ldots, X_n)$ of component lifetimes, defined by

$$\overline{F}(x_1, x_2, \ldots, x_n) = \Pr(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n).$$

Barlow and Proschan (1975, p. 12) proved that T_1 can be written as

$$T_1 = \psi_1(X_1, X_2, \ldots, X_n) = \max_{1 \le j \le s} \min_{i \in P_j} X_i, \tag{1}$$

where ψ_1 is the *structure function* and the sets $P_1, P_2, \ldots, P_s$ are the *minimal path sets* of the system. A *path set* P is a set of indices such that if the components in P work, then the system works. A path set is a *minimal path set* if it does not contain proper path sets. From expression (1), using the inclusion–exclusion formula, it is easy to prove (see, e.g., Agrawal and Barlow 1984 or Navarro et al. 2007) that the reliability function $\overline{F}_{T_1}(t) = \Pr(T_1 > t)$ of T_1 can be written as

$$\overline{F}_{T_1}(t) = \sum_{1 \le j \le s} \overline{F}_{P_j}(t) - \sum_{1 \le i < j \le s} \overline{F}_{P_i \cup P_j}(t) + \cdots + (-1)^{s+1} \overline{F}_{P_1 \cup P_2 \cup \cdots \cup P_s}(t), \tag{2}$$

where $\overline{F}_P(t) = \Pr(X_P > t)$ is the reliability function of the lifetime $X_P = \min_{i \in P} X_i$ of the series system with components in P. Moreover, note that $\overline{F}_P(t) = \overline{F}(\mathbf{t}_P)$, where $\mathbf{t}_P = (x_1, x_2, \ldots, x_n)$, $x_i = t$ for $i \in P$ and $x_i = 0$ for $i \notin P$. In particular, if the components are independent, then $\overline{F}_P(t) = \prod_{i \in P} \overline{F}_i(t)$. For example, the minimal path sets of the coherent system with lifetime $T_1 = \min(X_1, \max(X_2, X_3))$ are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$ and hence its reliability function can be written as

$$\overline{F}_{T_1}(t) = \overline{F}_{\{1,2\}}(t) + \overline{F}_{\{1,3\}}(t) - \overline{F}_{\{1,2,3\}}(t) = \overline{F}(t, t, 0) + \overline{F}(t, 0, t) - \overline{F}(t, t, t).$$

In particular, if the components are independent, then it can be written as

$$\overline{F}_{T_1}(t) = \overline{F}_1(t)\overline{F}_2(t) + \overline{F}_1(t)\overline{F}_3(t) - \overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t). \tag{3}$$

Note that the representation in (2) is a *generalized mixture*, that is, a linear combination of reliability functions with positive and negative coefficients which determines a proper reliability function. Some properties of generalized mixtures can be seen in Navarro and Shaked (2006), Navarro and Hernandez (2008), Navarro et al. (2009), and in the references therein. Expression (2) can also be written as

$$\overline{F}_{T_1}(t) = \sum_{\emptyset \ne I \subseteq \{1,2,\ldots,s\}} (-1)^{1+|I|} \overline{F}_{P_I}(t), \tag{4}$$

where $|I|$ is the cardinality of the non-empty set I and $P_I = \bigcup_{j\in I} P_j$ for $I \subseteq \{1,2,\ldots,s\}$.

If the random vector of components lifetimes is *exchangeable*, that is, $(X_1, X_2, \ldots, X_n)$ is equal in law to $(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)})$ for any permutation σ, then representation (2) reduces to

$$\overline{F}_{T_1}(t) = \sum_{i=1}^{n} a_i \overline{F}_{1:i}(t), \tag{5}$$

where $\overline{F}_{1:i}(t) = \Pr(X_{1:i} > t)$ for $i = 1,2,\ldots,n$. The vector of coefficients $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ (which only depends on the structure function of the system) was called *minimal signature* in Navarro et al. (2007). In particular, if the components are IID with common reliability function $\overline{F}$, then (5) reduces to

$$\overline{F}_{T_1}(t) = \sum_{i=1}^{n} a_i \overline{F}^{i}(t). \tag{6}$$

The polynomial $p(x) = \sum_{i=1}^{n} a_i x^i$ is called the domination polynomial of the system. This representation allows computing $\overline{F}_{T_1}(t)$ and $E(T_1)$ when we know $\overline{F}$ and comparing different systems using expected lifetimes. The cases of systems with 2–4 or 5 components with a common exponential distribution was studied in Navarro and Rychlik (2007) and Navarro and Rubio (2010), respectively.

Recently, Samaniego (2007, p. 68) proposed using the concept of stochastic precedence introduced in Arcones et al. (2002) to compare independent coherent systems. This concept is defined as follows.

Definition 1 If X and Y are two independent random variables over the same probability space, then X is said to *stochastically precede* Y if $\Pr(X \le Y) \ge 1/2$. X and Y are said to be *SP-equivalent* if both $\Pr(X \le Y) \ge 1/2$ and $\Pr(Y \le X) \ge 1/2$ hold.

Samaniego (2007, p. 70) gave a procedure to compute $\Pr(T_1 \le T_2)$ when $T_1 = \psi_1(X_1, X_2, \ldots, X_n)$ and $T_2 = \psi_2(Y_1, Y_2, \ldots, Y_m)$ are the lifetimes of two coherent systems, $X_1, X_2, \ldots, X_n$ are IID with common reliability $\overline{F}$, $Y_1, Y_2, \ldots, Y_m$ are IID with common reliability $\overline{F}^{\alpha}$, and X_i and Y_j are independent for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,m$. The procedure is based on the signature vectors of the systems. Samaniego (1985) defined the signature vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ of a coherent system with lifetime $T_1 = \psi_1(X_1, X_2, \ldots, X_n)$ by $s_i = \Pr(T_1 = X_{i:n})$, where $X_{i:n}$ is the ith order statistic obtained from $X_1, X_2, \ldots, X_n$, for $i = 1,2,\ldots,n$. The signature of T_2 is defined in a similar way. He proved that if the components are IID with a common absolutely continuous distribution, then this signature vector only depends on the structure function, and hence it is distribution free. An excellent review of the results on signatures obtained in the last 30 years was given in Samaniego (2007).

It is well known that if a system has signature $\mathbf{s} = (s_1, s_2, \ldots, s_n)$, then the signature of its dual system is $\mathbf{s}^D = (s_n, s_{n-1}, \ldots, s_1)$. Therefore, from the results given in Samaniego (2007), it is easy to see that if T_1 and T_2 are the lifetimes of two coherent systems, and T_1^D and T_2^D are the lifetimes of the corresponding dual systems, then T_1 stochastically precedes T_2 if and only if T_2^D stochastically precedes T_1^D. This property is not true for the ordering based on the expected lifetimes, that is, $E(T_1) \le E(T_2)$ does not necessarily imply $E(T_1^D) \le E(T_2^D)$.

The signature vector $\mathbf{s}$ of a coherent system can be obtained from its minimal signature as

$$\mathbf{s} = \mathbf{a}A_n, \tag{7}$$

where A_n is a triangular non-singular triangular matrix (see Navarro et al. 2008b), and vice versa, $\mathbf{a} = \mathbf{s}A_n^{-1}$, where A_n^{-1} is the inverse of the matrix A_n.

The main result of the paper is given in the following theorem where we show how to compute $\Pr(T_1 \le T_2)$ in the general case by using minimal path set representation given in (4).

Theorem 1 *If $T_1 = \psi_1(X_1, X_2, \ldots, X_n)$ and $T_2 = \psi_2(Y_1, Y_2, \ldots, Y_m)$ are the lifetimes of two independent coherent systems with minimal path sets $P_1, P_2, \ldots, P_s$ and $Q_1, Q_2, \ldots, Q_r$, respectively, then*

$$\Pr(T_1 \le T_2) = \sum_{I\subseteq\{1,2,\ldots,s\}} \sum_{J\subseteq\{1,2,\ldots,r\}} (-1)^{1+|I|+|J|} \int_0^\infty \overline{G}_{Q_J}(t)\, d\overline{F}_{P_I}(t), \tag{8}$$

where $|I|$ and $|J|$ are the cardinalities of the non-empty sets I and J, $P_I = \bigcup_{j\in I} P_j$ for $I \subseteq \{1, 2, \ldots, s\}$, $Q_J = \bigcup_{j\in J} Q_j$ for $J \subseteq \{1, 2, \ldots, r\}$, $\overline{F}_P(t) = \Pr(X_P > t)$ and $X_P = \min_{i\in P} X_i$ for $P \subseteq \{1, 2, \ldots, n\}$, and $\overline{G}_Q(t) = \Pr(Y_Q > t)$ and $Y_Q = \min_{i\in Q} Y_i$ for $Q \subseteq \{1, 2, \ldots, m\}$.

Proof From (4) the reliability functions of T_1 and T_2 can be written as

$$\overline{F}_{T_1}(t) = \sum_{I\subseteq\{1,2,\ldots,s\}} (-1)^{1+|I|}\overline{F}_{P_I}(t), \tag{9}$$

and

$$\overline{F}_{T_2}(t) = \sum_{J\subseteq\{1,2,\ldots,r\}} (-1)^{1+|J|}\overline{G}_{Q_J}(t), \tag{10}$$

respectively. Moreover, using that T_1 and T_2 are independent, we have

$$\Pr(T_1 \le T_2) = \int_0^\infty \left(\int_x^\infty d\overline{F}_{T_2}(y) \right) d\overline{F}_{T_1}(x) = -\int_0^\infty \overline{F}_{T_2}(x)\, d\overline{F}_{T_1}(x). \tag{11}$$

Then, replacing (9) and (10) in (11), we obtain (8). □

Recall that if $\overline{F}$ is the joint reliability function of $(X_1, X_2, \ldots, X_n)$ and $P \subseteq \{1, 2, \ldots, n\}$, then $\overline{F}_P(t) = \overline{F}(\mathbf{t}_P)$, where $\mathbf{t}_P = (x_1, x_2, \ldots, x_n)$, $x_i = t$ for $i \in P$ and $x_i = 0$ for $i \notin P$. Analogously, if $\overline{G}$ is the joint reliability function of $(Y_1, Y_2, \ldots, Y_m)$ and $Q \subseteq \{1, 2, \ldots, m\}$, then $\overline{G}_Q(t) = \overline{G}(\mathbf{t}_Q^*)$, where $\mathbf{t}_Q^* = (y_1, y_2, \ldots, y_m)$, $y_i = t$ for $i \in Q$ and $y_i = 0$ for $i \notin Q$. Some examples are given in the next section.

Next we obtain some particular results from Theorem 1. First, we give the result for the case of independent not necessarily identically distributed (INID) components. The proof is immediate from Theorem 1.

Theorem 2 *If* $T_1 = \psi_1(X_1, X_2, \ldots, X_n)$ *and* $T_2 = \psi_2(Y_1, Y_2, \ldots, Y_m)$ *are the lifetimes of two independent coherent systems with INID components and minimal path sets* $P_1, P_2, \ldots, P_s$ *and* $Q_1, Q_2, \ldots, Q_r$*, respectively, then*

$$\Pr(T_1 \le T_2) = \sum_{I \subseteq \{1,2,\ldots,s\}} \sum_{J \subseteq \{1,2,\ldots,r\}} (-1)^{1+|I|+|J|} \times \int_0^\infty \prod_{j \in Q_J} \overline{G}_j(t)\, d\left(\prod_{i \in P_I} \overline{F}_i(t)\right) \tag{12}$$

where $|I|$ *and* $|J|$ *are the cardinalities of the non-empty sets* I *and* J, $P_I = \bigcup_{j \in I} P_j$ *for* $I \subseteq \{1, 2, \ldots, s\}$, $Q_J = \bigcup_{j \in J} P_j$ *for* $J \subseteq \{1, 2, \ldots, r\}$, $\overline{F}_i(t) = \Pr(X_i > t)$ *for* $i = 1, 2, \ldots, n$, *and* $\overline{G}_j(t) = \Pr(Y_j > t)$ *for* $j = 1, 2, \ldots, m$.

The random variables $Z_1, Z_2, \ldots, Z_k$ satisfy the Cox *proportional hazard rate* (PHR) model if $\Pr(Z_i > t) = \overline{H}^{\theta_i}(t)$, where $\overline{H}$ is the baseline reliability function and $\theta_i > 0$ for $i = 1, 2, \ldots, k$. This property holds if and only if the hazard rate functions of Z_i are proportional. This property is also known as *Lehmann alternatives*. There are several parametric models which satisfy the PHR model as, for example, the exponential, the Pareto or the Weibull with fixed shape parameter distributions. The following theorem shows that if the component lifetimes in two independent coherent systems are independent and satisfy the same PHR rate model with baseline reliability function $\overline{H}$, then $\Pr(T_1 \le T_2)$ does not depend on $\overline{H}$.

Theorem 3 *If* $T_1 = \psi_1(X_1, X_2, \ldots, X_n)$ *and* $T_2 = \psi_2(Y_1, Y_2, \ldots, Y_m)$ *are the lifetimes of two independent coherent systems with INID components which satisfy the same PHR model and minimal path sets* $P_1, P_2, \ldots, P_s$ *and* $Q_1, Q_2, \ldots, Q_r$*, respectively, then*

$$\Pr(T_1 \le T_2) = \sum_{I \subseteq \{1,2,\ldots,s\}} \sum_{J \subseteq \{1,2,\ldots,r\}} (-1)^{|I|+|J|} \frac{\alpha_{P_I}}{\alpha_{P_I} + \beta_{Q_J}}, \tag{13}$$

where $|I|$ *and* $|J|$ *are the cardinalities of the non-empty sets* I *and* J, $P_I = \bigcup_{j \in I} P_j$ *for* $I \subseteq \{1, 2, \ldots, s\}$, $Q_J = \bigcup_{j \in J} P_j$ *for* $J \subseteq \{1, 2, \ldots, r\}$, $\Pr(X_i > t) = \overline{H}^{\alpha_i}(t)$ *for* $i = 1, 2, \ldots, n$, $\Pr(Y_j > t) = \overline{H}^{\beta_i}(t)$ *for* $j = 1, 2, \ldots, m$, $\alpha_P = \sum_{i \in P} \alpha_i$ *and* $\beta_Q = \sum_{i \in Q} \beta_i$.

Proof If the components $X_1, X_2, \ldots, X_n$ are independent and satisfy the PHR model with baseline reliability function $\overline{H}$, then

$$\overline{F}_P(t) = \prod_{i \in P} \overline{F}_i(t) = \overline{H}^{\alpha_P}(t)$$

with $\alpha_P = \sum_{i \in P} \alpha_i$. Analogously,

$$\overline{G}_Q(t) = \prod_{i \in Q} \overline{G}_i(t) = \overline{H}^{\beta_Q}(t)$$

with $\beta_Q = \sum_{i \in Q} \beta_i$. Therefore, we have

$$\begin{aligned}\int_0^\infty \overline{G}_Q(t)\, d(\overline{F}_P(t)) &= \alpha_P \int_0^\infty \overline{H}^{\beta_Q + \alpha_P - 1}(t)\, d\overline{H}(t) \\ &= -\frac{\alpha_P}{\beta_Q + \alpha_P}.\end{aligned}$$

Using this expression in (8), we obtain (13). □

The expressions given in the preceding theorems can be simplified using representation (5) when the components are exchangeable or IID. These results are given in the following theorems.

Theorem 4 *If* $T_1 = \psi_1(X_1, X_2, \ldots, X_n)$ *and* $T_2 = \psi_2(Y_1, Y_2, \ldots, Y_m)$ *are the lifetimes of two independent coherent systems with minimal signatures* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{a}^* = (a_1^*, a_2^*, \ldots, a_m^*)$, *respectively, and* $(X_1, X_2, \ldots, X_n)$ *and* $(Y_1, Y_2, \ldots, Y_m)$ *are exchangeable random vectors, then*

$$\Pr(T_1 \le T_2) = -\sum_{i=1}^n \sum_{j=1}^m a_i a_j^* \int_0^\infty \overline{G}_{1:j}(t)\, d\overline{F}_{1:i}(t), \tag{14}$$

where $\overline{F}_{1:i}(t) = \Pr(X_{1:i} > t)$ *for* $i = 1, 2, \ldots, n$, *and* $\overline{G}_{1:j}(t) = \Pr(Y_{1:j} > t)$ *for* $j = 1, 2, \ldots, m$.

Theorem 5 *If* $T_1 = \psi_1(X_1, X_2, \ldots, X_n)$ *and* $T_2 = \psi_2(Y_1, Y_2, \ldots, Y_m)$ *are the lifetimes of two independent coherent systems with IID components which satisfy the same PHR model and minimal signatures* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *and* $\mathbf{a}^* = (a_1^*, a_2^*, \ldots, a_m^*)$, *respectively, then*

$$\Pr(T_1 \le T_2) = \sum_{i=1}^n \sum_{j=1}^m a_i a_j^* \frac{i}{i + j\theta}, \tag{15}$$

where $\Pr(X_i > t) = \overline{H}(t)$ *for* $i = 1, 2, \ldots, n$, *and* $\Pr(Y_j > t) = \overline{H}^\theta(t)$ *for* $j = 1, 2, \ldots, m$.

The proofs are similar to the proofs of Theorems 1 and 3, respectively. In particular, the result for the case of systems with IID components sharing the same distribution is obtained taking $\theta = 1$ in Theorem 5. Expression (15) can be used instead of expressions (5.25) in Samaniego (2007, p. 70). In the next section, we apply (15) to compare all the coherent systems with 1–5 IID components with a common distribution. These comparisons do not depend on the common component distribution.

The *mixed systems* are stochastic mixtures of coherent systems (see Boland and Samaniego 2004). Then their reliability functions can be written as generalized mixtures of series system reliability functions. Therefore, all the preceding results can also be applied to them (with the appropriate representations in each case). In particular, Theorems 4 and 5 can be applied to mixed systems taking into account that if the signature vector of a mixed systems is $\mathbf{s}$, then (5) holds for $\mathbf{a} = \mathbf{s}A_n^{-1}$ (see Navarro et al. 2008b), where A_n is the non-singular triangular matrix given in (7). This procedure is illustrated in the next section (see Example 4). Finally, we would like to note that our results can also be applied to compare the residual lifetimes of two coherent systems using the representations given in Li and Zhang (2008), Navarro et al. (2008a) and Samaniego et al. (2009).

3 Applications

In this section, we illustrate our theoretical results with some examples of applications. The first example shows how the main result given in Theorem 1 can be applied to systems with dependent not necessarily identically distributed components.

Example 1 Let us consider the coherent system with the lifetime

$$T = \max\big(\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_4, X_5)\big).$$

Its minimal path sets are $P_1 = \{1, 2\}$, $P_2 = \{1, 3\}$ and $P_3 = \{2, 4, 5\}$. Therefore, from (2), the system reliability can be written as

$$\overline{F}_T(t) = \overline{F}_{\{1,2\}}(t) + \overline{F}_{\{1,3\}}(t) + \overline{F}_{\{2,4,5\}}(t) - \overline{F}_{\{1,2,3\}}(t) - \overline{F}_{\{1,2,4,5\}}(t).$$

Suppose that we want to compare this system with its components using stochastic precedence, that is, we want to compute $\Pr(Y_j \le T)$, where Y_j is equal in law to X_j and X_i and Y_j are independent for all $i, j = 1, \ldots, 5$. Then, from (8), we have

$$\begin{aligned}\Pr(Y_j \le T) &= \int_0^\infty \big(\overline{F}_{\{1,2\}}(t) + \overline{F}_{\{1,3\}}(t) + \overline{F}_{\{2,4,5\}}(t)\big) f_j(t)\, dt \\ &\quad - \int_0^\infty \big(\overline{F}_{\{1,2,3\}}(t) + \overline{F}_{\{1,2,4,5\}}(t)\big) f_j(t)\, dt \\ &= \int_0^\infty \big(\overline{F}(t, t, 0, 0, 0) + \overline{F}(t, 0, t, 0, 0) + \overline{F}(0, t, 0, t, t)\big) f_j(t)\, dt \\ &\quad - \int_0^\infty \big(\overline{F}(t, t, t, 0, 0) + \overline{F}(t, t, 0, t, t)\big) f_j(t)\, dt,\end{aligned}$$

for $j = 1, \ldots, 5$, where f_j is the probability density function of X_j. Now, assume that the joint distribution of the component lifetimes is a Farlie–Gumbel–Morgenstern distribution with exponential marginals, that is, suppose that the reliability function of $(X_1, X_2, X_3, X_4, X_5)$ is given by

$$\overline{F}(x_1, x_2, x_3, x_4, x_5) = e^{-\sum_{i=1}^{5} x_i/\mu_i}\left(1 + \alpha \prod_{i=1}^{5}\left(1 - e^{-x_i/\mu_i}\right)\right), \tag{16}$$

for $(x_1, x_2, x_3, x_4, x_5) \geq (0, 0, 0, 0, 0)$, where $|\alpha| \leq 1$ and $E(X_i) = \mu_i > 0$ for $i = 1, \ldots, 5$. Then, we have

$$\begin{aligned}
\Pr(Y_j \leq T) &= \lambda_j \int_0^{\infty} \left(e^{-(\lambda_1+\lambda_2)t} + e^{-(\lambda_1+\lambda_3)t} + e^{-(\lambda_2+\lambda_4+\lambda_5)t}\right) e^{-\lambda_j t}\, dt \\
&\quad - \lambda_j \int_0^{\infty} \left(e^{-(\lambda_1+\lambda_2+\lambda_3)t} + e^{-(\lambda_1+\lambda_2+\lambda_4+\lambda_5)t}\right) e^{-\lambda_j t}\, dt \\
&= \frac{\lambda_j}{\lambda_j + \lambda_1 + \lambda_2} + \frac{\lambda_j}{\lambda_j + \lambda_1 + \lambda_3} + \frac{\lambda_j}{\lambda_j + \lambda_2 + \lambda_4 + \lambda_5} \\
&\quad - \frac{\lambda_j}{\lambda_j + \lambda_1 + \lambda_2 + \lambda_3} - \frac{\lambda_j}{\lambda_j + \lambda_1 + \lambda_2 + \lambda_4 + \lambda_5},
\end{aligned}$$

where $\lambda_i = 1/\mu_i$ for $i = 1, \ldots, 5$. Note that $\Pr(Y_j \leq T)$ does not depend on α. For example, if $\lambda_i = i$ for $i = 1, \ldots, 5$, then $\Pr(Y_j \leq T)$ is equal to 0.313553, 0.494322, 0.609524, 0.688095, 0.744392, for $j = 1, \ldots, 5$, respectively. Therefore, T stochastically precedes the first and the second components and the other components stochastically precede T.

Next we compare using stochastic precedence two coherent systems with INID components and similar structure functions. These results can be used to study where the best place (using the stochastic precedence criterion) to put some components in a given system structure is.

Example 2 Let us consider the coherent system defined by the lifetime $T_1 = \min(X_1, \max(X_2, X_3))$ which has minimal path sets $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. If we assume that its components are INID then its reliability was given in (3). Let us to compare it to a system with an analogous structure, for example, the system with lifetime $T_2 = \min(Y_2, \max(Y_1, Y_3))$. The minimal path sets of T_2 are $Q_1 = \{1, 2\}$ and $Q_2 = \{2, 3\}$. If we assume that its components are INID with the same distributions as that of system T_1 (i.e., X_i is equal in law to Y_i for $i = 1, 2, 3$), then its reliability function is given by

$$\overline{F}_{T_2}(t) = \overline{F}_1(t)\overline{F}_2(t) + \overline{F}_2(t)\overline{F}_3(t) - \overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t). \tag{17}$$

Then, from (12), we have

$$\begin{aligned}
&\Pr(T_1 \leq T_2) \\
&\quad = -\int_0^{\infty} \left(\overline{F}_1(t)\overline{F}_2(t) + \overline{F}_2(t)\overline{F}_3(t) - \overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\right) d\left(\overline{F}_1(t)\overline{F}_2(t)\right)
\end{aligned}$$

$$-\int_0^\infty \left(\overline{F}_1(t)\overline{F}_2(t)+\overline{F}_2(t)\overline{F}_3(t)-\overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\right)d\left(\overline{F}_1(t)\overline{F}_3(t)\right)$$
$$+\int_0^\infty \left(\overline{F}_1(t)\overline{F}_2(t)+\overline{F}_2(t)\overline{F}_3(t)-\overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\right)d\left(\overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\right)$$

and, in particular, if the components satisfy the PHR model, that is, $\overline{F}_i(t)=\overline{H}^{\alpha_i}(t)$ for $\alpha_i>0$ and $i=1,2,3$, then from (13), we have

$$\Pr(T_1\le T_2)=\frac{\alpha_1+\alpha_2}{\alpha_1+2\alpha_2+\alpha_3}+\frac{\alpha_1+\alpha_3}{2\alpha_1+\alpha_2+\alpha_3}+\frac{\alpha_1+\alpha_3}{\alpha_1+\alpha_2+2\alpha_3}-\frac{\alpha_1+\alpha_3}{2\alpha_1+\alpha_2+2\alpha_3}-\frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+2\alpha_2+2\alpha_3}.$$

Note that it does not depend on the baseline reliability function $\overline{H}$. For example, if $\alpha_i=i$ for $i=1,2,3$, then $\Pr(T_1\le T_2)=12347/27720=0.445418$. Therefore, T_2 stochastically precedes T_1. Analogously, we can obtain that $T_3=\min(Z_3,\max(Z_1,Z_2))$ stochastically precedes T_1, whenever Z_i has the same distribution as X_i, for $i=1,2,3$. Therefore, the best option (using the stochastic precedence criterion) is T_1, that is, to place the best component at the first position (i.e., in the series structure). This agrees with the asymptotic results given Navarro and Shaked (2006) and Navarro and Hernandez (2008).

Intuitively, it is clear that series systems are improved if their components are homogeneous, while parallel systems are improved if their components are heterogeneous. However, for other systems structures, this property is not so clear. The next example shows how we can use Theorems 2 and 3 to study this property using the stochastic precedence criterion.

Example 3 Let us consider again the coherent system with the lifetime $T_1=\min(X_1,\max(X_2,X_3))$ and minimal path sets $P_1=\{1,2\}$ and $P_2=\{1,3\}$. If we assume that its components are INID, then its reliability was given in (3). Let us compare it to a system with the same structure and with the lifetime $T_2=\min(Y_1,\max(Y_2,Y_3))$ but with IID components. If $\Pr(Y_i>t)=\overline{G}(t)$ for $i=1,2,3$, then its reliability function is given by

$$\overline{F}_{T_2}(t)=2\overline{G}^2(t)-\overline{G}^3(t), \tag{18}$$

that is, its minimal signature is $\mathbf{a}=(0,2,-1)$. Then, from (12), we have

$$\Pr(T_1\le T_2)=-\int_0^\infty \left(2\overline{G}^2(t)-\overline{G}^3(t)\right)d\left(\overline{F}_1(t)\overline{F}_2(t)\right)$$
$$-\int_0^\infty \left(2\overline{G}^2(t)-\overline{G}^3(t)\right)d\left(\overline{F}_1(t)\overline{F}_3(t)\right)$$
$$+\int_0^\infty \left(2\overline{G}^2(t)-\overline{G}^3(t)\right)d\left(\overline{F}_1(t)\overline{F}_2(t)\overline{F}_3(t)\right).$$

In particular, if the components satisfy the PHR model, that is, $\overline{F}_i(t) = \overline{G}^{\alpha_i}(t)$ for $i = 1, 2, 3$, then from (13), we have

$$\Pr(T_1 \le T_2) = \frac{2(\alpha_1+\alpha_2)}{2+\alpha_1+\alpha_2} + \frac{2(\alpha_1+\alpha_3)}{2+\alpha_1+\alpha_3} + \frac{\alpha_1+\alpha_2+\alpha_3}{3+\alpha_1+\alpha_2+\alpha_3} - \frac{\alpha_1+\alpha_2}{3+\alpha_1+\alpha_2} - \frac{\alpha_1+\alpha_3}{3+\alpha_1+\alpha_3} - \frac{2(\alpha_1+\alpha_2+\alpha_3)}{2+\alpha_1+\alpha_2+\alpha_3}.$$

Note that it does not depend on the baseline reliability function $\overline{G}$. To compare both systems, we must assume that the components in both systems are, on average, *similar*. For example, we can assume that the average of the hazard rate functions of the components in T_1 is equal to the common hazard rate function of the IID components in T_2, that is, we can assume $\alpha_1+\alpha_2+\alpha_3 = 3$. In this case, the preceding expression reduces to

$$\Pr(T_1 \le T_2) = \frac{6-2\alpha_2}{5-\alpha_2} + \frac{6-2\alpha_3}{5-\alpha_3} - \frac{3-\alpha_2}{6-\alpha_2} - \frac{3-\alpha_3}{6-\alpha_3} - \frac{7}{10}.$$

Therefore, $\Pr(T_1 \le T_2)$ decreases in α_2 and α_3, and we obtain $\Pr(T_1 \le T_2) \ge 1/2$ if and only if $0 \le \alpha_i \le 1.627719$ for $i = 2, 3$ and

$$\alpha_3 \le \frac{39\alpha_2^2 - 374\alpha_2 + 675 - \sqrt{321\alpha_2^4 - 6672\alpha_2^3 + 50326\alpha_2^2 - 162000\alpha_2 + 185625}}{2(4\alpha_2^2 - 39\alpha_2 + 75)}.$$

In particular, if $\alpha_1 = 2-\lambda$ and $\alpha_2 = \alpha_3 = \lambda$, then $\Pr(T_1 \le T_2) \ge 1/2$ if and only if $0 < \lambda \le 1$. That is, if $\alpha_2 = \alpha_3$, the system with heterogeneous components is better than the system with homogeneous components (using the stochastic precedence criterion) if we place the best component in the first position (i.e., in the series structure). For example, if $\alpha_1 = 2$ and $\alpha_1 = \alpha_2 = 0.5$, then $\Pr(T_1 \le T_2) = 0.613$, and T_2 is SP-better that T_1. Comparisons using other criterions can be seen in Shaked and Suarez-Llorens (2003) and Navarro (2007).

The next example shows how our general expressions can be simplified in the case of systems with exchangeable (possibly dependent) components using the concept of *minimal signature*. It also shows how our results can be applied to mixed (non-coherent) systems with exchangeable (or IID) components.

Example 4 Let us consider again the coherent system with the lifetime

$$T_1 = \max\big(\min(X_1, X_2), \min(X_1, X_3), \min(X_2, X_4, X_5)\big),$$

and let us assume that it has exchangeable components. The signature and the minimal signature of T_1 are $\mathbf{s} = (0, 0.4, 0.4, 0.2, 0)$ and $\mathbf{a} = (0, 2, 0, -1, 0)$, respectively (see system number 72 in Table 2 of Navarro and Rubio 2010). Let us compare it with the mixed system lifetime T_2 based on coherent systems with exchangeable component lifetimes $(Y_1, Y_2, Y_3, Y_4, Y_5)$ with the signature $\mathbf{s}^* = (0.2, 0.2, 0.3, 0.3, 0)$, that is, $T_2 = Y_{j:5}$ with probabilities $0.2, 0.2, 0.3, 0.3$ for $j = 1, 2, 3, 4$, respectively. To obtain its minimal signature, we need the matrix used to obtain the minimal signature from the Samaniego's signature which is given by

$$A_5^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 10 & -15 & 6 \\ 0 & 10 & -20 & 15 & -4 \\ 5 & -10 & 10 & -5 & 1 \end{pmatrix}.$$

For example, for T_1, we have

$$\mathbf{a} = \mathbf{s}A_5^{-1} = (0, 0.4, 0.4, 0.2, 0) \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 10 & -15 & 6 \\ 0 & 10 & -20 & 15 & -4 \\ 5 & -10 & 10 & -5 & 1 \end{pmatrix} = (0, 2, 0, -1, 0).$$

Analogously, (5) holds for the mixed system lifetime T_2 with the coefficients given in the following minimal signature

$$\mathbf{a}^* = \mathbf{s}^* A_5^{-1} = \big(0.2,\ 0.2,\ 0.3,\ 0.3,\ 0\big) \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 5 & -4 \\ 0 & 0 & 10 & -15 & 6 \\ 0 & 10 & -20 & 15 & -4 \\ 5 & -10 & 10 & -5 & 1 \end{pmatrix}$$
$$= \big(0,\ 3,\ -3,\ 1,\ 0\big).$$

Now let us assume that $(X_1, X_2, X_3, X_4, X_5)$ and $(Y_1, Y_2, Y_3, Y_4, Y_5)$ have Farlie–Gumbel–Morgenstern distributions with exponential marginals, that is, suppose that their joint reliability functions are given by (16) with parameters $|\alpha_1| \le 1$ and $E(X_i) = \mu_1 > 0$, and $|\alpha_2| \le 1$ and $E(Y_i) = \mu_2 > 0$ for $i = 1, \ldots, 5$. Then, from (14), we have

$$\begin{aligned} \Pr(T_1 \le T_2) &= -\sum_{i=1}^{5}\sum_{j=1}^{5} a_i a_j^* \int_0^\infty \overline{G}_{1:j}(t)\, d\overline{F}_{1:i}(t) \\ &= \sum_{i=1}^{4}\sum_{j=1}^{4} a_i a_j^* \frac{i/\mu_1}{i/\mu_1 + j/\mu_2} \\ &= 6\frac{2/\mu_1}{2/\mu_1 + 2/\mu_2} - 6\frac{2/\mu_1}{2/\mu_1 + 3/\mu_2} + 2\frac{2/\mu_1}{2/\mu_1 + 4/\mu_2} \\ &\quad - 3\frac{4/\mu_1}{4/\mu_1 + 2/\mu_2} + 3\frac{4/\mu_1}{4/\mu_1 + 3/\mu_2} - \frac{4/\mu_1}{4/\mu_1 + 4/\mu_2}. \end{aligned}$$

Note that it does not depend on α_1 and α_2. In particular, if $\mu_1 = \mu_2$, then

$$\Pr(T_1 \le T_2) = 6\frac{2}{2+2} - 6\frac{2}{2+3} + 2\frac{2}{2+4} - 3\frac{4}{4+2} + 3\frac{4}{4+3} - \frac{4}{4+4}$$

$$= \frac{101}{210} = 0.48095.$$

Therefore, T_1 stochastically precedes T_2 for all $\mu_1 = \mu_2$ and for all α_1 and α_2. Note that T_1 and T_2 cannot be ordered in the usual stochastic order using the results given in Navarro et al. (2008b) since their signature vectors are not ST-ordered.

Next we show that the results obtained for systems with IID components are a good alternative to the results given in Samaniego (2007). In the following example, we give the alternative calculations obtained from Theorem 5 to those given in Example 5.3 of Samaniego (2007, p. 71).

Example 5 Samaniego (2007, p. 71) compared using stochastic precedence the coherent systems with IID components and common reliability function $\overline{H}$ with the lifetimes T_1 and T_2 determined by the signatures $(1/4, 1/4, 1/2, 0)$ and $(0, 2/3, 1/3, 0)$. These lifetimes can be seen in lines 13 and 16 of Table 1 in Navarro et al. (2008b). These systems cannot be ordered in the usual stochastic order using the results given in Fig. 1 of Navarro et al. (2008b). Their minimal signatures are $(0, 3, -3, 1)$ and $(0, 2, 0, -1)$, see lines 5 and 8 of Table 2 in Navarro and Shaked (2006) or in Navarro et al. (2007). Then, using (15) with $\theta = 1$, we have

$$\Pr(T_1 \le T_2) = 6\frac{2}{2+2} - 3\frac{2}{2+4} - 6\frac{3}{3+2} + 3\frac{3}{3+4} + 2\frac{4}{4+2} - \frac{4}{4+4}$$
$$= \frac{109}{210} = 0.51905,$$

which coincides with the value obtained in Example 3.5 of Samaniego (2007).

Finally, in Table 1, we compare using stochastic precedence all the coherent systems with 1–5 IID components using (15) with $\theta = 1$. Recall that these comparisons are distribution free. In Table 1, we give the signatures of order 5 of all the coherent systems with 1–5 components ordered (from the worst to the best) using stochastic precedence. The concept of signature of order k can be seen in Navarro et al. (2008b). The lifetimes of the systems in Table 1 can be seen in Navarro and Rubio (2010). In Table 1, we also give the probability $\Pr(T_N \le T_{N+1})$ for $N = 1, 2, \ldots, 93$. We have also computed $\Pr(T_N \le T_M)$ for all N and M and we observe that $\Pr(T_N \le T_M) \ge 1/2$ for $N < M$, that is, for systems with 1–5 components, if T_1 stochastically precedes T_2 and T_2 stochastically precedes T_3, then T_1 stochastically precedes T_3. Hence stochastic precedence is a total order for these systems. Of course, in the IID case, if two systems have the same signature, then they have the same law, and hence they are equivalent in stochastic precedence.

There are systems with different signatures that are equivalent in stochastic precedence. For example, the systems in lines 24 and 25 of Table 1 have signatures $(1/5, 1/5, 3/5, 0, 0)$ and $(0, 3/5, 2/5, 0, 0)$ but satisfy $\Pr(T_{24} \le T_{25}) = 1/2$. In this case, we might say that the best system is that with the least disperse signature. Thus, if we compute the variance of the random variable Z which takes the values $1, 2, \ldots, n$ with probabilities $s_1, s_2, \ldots, s_n$ for these signatures, we obtain

Table 1 Signatures of order 5 of coherent systems with 1–5 IID components ordered using stochastic precedence

N	**a**	**s**	$\Pr(T_N \le T_{N+1})$
1	$(0,0,0,0,1)$	$(1,0,0,0,0)$	0.555555556
2	$(0,0,0,1,0)$	$(\frac{4}{5},\frac{1}{5},0,0,0)$	0.555555556
3	$(0,0,0,2,-1)$	$(\frac{3}{5},\frac{2}{5},0,0,0)$	0.517857143
4	$(0,0,1,0,0)$	$(\frac{3}{5},\frac{3}{10},\frac{1}{10},0,0)$	0.535714286
5	$(0,0,0,3,-2)$	$(\frac{2}{5},\frac{3}{5},0,0,0)$	0.519841270
6	$(0,0,1,1,-1)$	$(\frac{2}{5},\frac{1}{2},\frac{1}{10},0,0)$	0.519841270
7	$(0,0,2,-1,0)$	$(\frac{2}{5},\frac{2}{5},\frac{1}{5},0,0)$	0.511904762
8	$(0,0,0,4,-3)$	$(\frac{1}{5},\frac{4}{5},0,0,0)$	0.509920635
9	$(0,0,3,-3,1)$	$(\frac{2}{5},\frac{3}{10},\frac{3}{10},0,0)$	0.509920635
10	$(0,0,1,2,-2)$	$(\frac{1}{5},\frac{7}{10},\frac{1}{10},0,0)$	0.504761905
11	$(0,1,0,0,0)$	$(\frac{2}{5},\frac{3}{10},\frac{1}{5},\frac{1}{10},0)$	0.514285714
12	$(0,0,2,0,-1)$	$(\frac{1}{5},\frac{3}{5},\frac{1}{5},0,0)$	0.507936508
13	$(0,0,0,5,-4)$	$(0,1,0,0,0)$	0.515873016
14	$(0,0,3,-2,0)$	$(\frac{1}{5},\frac{1}{2},\frac{3}{10},0,0)$	0.505952381
15	$(0,0,1,3,-3)$	$(0,\frac{9}{10},\frac{1}{10},0,0)$	0.510714286
16	$(0,1,0,1,-1)$	$(\frac{1}{5},\frac{1}{2},\frac{1}{5},\frac{1}{10},0)$	0.504761905
17	$(0,0,4,-4,1)$	$(\frac{1}{5},\frac{2}{5},\frac{2}{5},0,0)$	0.503968254
18	$(0,0,2,1,-2)$	$(0,\frac{4}{5},\frac{1}{5},0,0)$	0.513492063
19	$(0,1,1,-1,0)$	$(\frac{1}{5},\frac{2}{5},\frac{3}{10},\frac{1}{10},0)$	0.503968254
20	$(0,0,5,-6,2)$	$(\frac{1}{5},\frac{3}{10},\frac{1}{2},0,0)$	0.501984127
21	$(0,0,3,-1,-1)$	$(0,\frac{7}{10},\frac{3}{10},0,0)$	0.516269841
22	$(0,1,2,-3,1)$	$(\frac{1}{5},\frac{3}{10},\frac{2}{5},\frac{1}{10},0)$	0.501587302
23	$(0,1,0,2,-2)$	$(0,\frac{7}{10},\frac{1}{5},\frac{1}{10},0)$	0.503174603
24	$(0,0,6,-8,3)$	$(\frac{1}{5},\frac{1}{5},\frac{3}{5},0,0)$	0.5
25	$(0,0,4,-3,0)$	$(0,\frac{3}{5},\frac{2}{5},0,0)$	0.514285714
26	$(0,2,-1,0,0)$	$(\frac{1}{5},\frac{3}{10},\frac{3}{10},\frac{1}{5},0)$	0.503571429
27	$(0,1,1,0,-1)$	$(0,\frac{3}{5},\frac{3}{10},\frac{1}{10},0)$	0.500396825
28	$(0,1,3,-5,2)$	$(\frac{1}{5},\frac{1}{5},\frac{1}{2},\frac{1}{10},0)$	0.501984127
29	$(0,0,5,-5,1)$	$(0,\frac{1}{2},\frac{1}{2},0,0)$	0.517857143
30	$(0,2,0,-2,1)$	$(\frac{1}{5},\frac{1}{5},\frac{2}{5},\frac{1}{5},0)$	0.500793651
31	$(0,1,2,-2,0)$	$(0,\frac{1}{2},\frac{2}{5},\frac{1}{10},0)$	0.503174603
32	$(0,0,6,-7,2)$	$(0,\frac{2}{5},\frac{3}{5},0,0)$	0.517460317
33	$(0,2,-1,1,-1)$	$(0,\frac{1}{2},\frac{3}{10},\frac{1}{5},0)$	0.500396825
34	$(0,3,-3,1,0)$	$(\frac{1}{5},\frac{1}{5},\frac{3}{10},\frac{3}{10},0)$	0.500396825
35	$(0,1,3,-4,1)$	$(0,\frac{2}{5},\frac{1}{2},\frac{1}{10},0)$	0.502380952
36	$(0,0,7,-9,3)$	$(0,\frac{3}{10},\frac{7}{10},0,0)$	0.519047619
37	$(0,2,0,-1,0)$	$(0,\frac{2}{5},\frac{2}{5},\frac{1}{5},0)$	0.501587302
38	$(0,1,4,-6,2)$	$(0,\frac{3}{10},\frac{3}{5},\frac{1}{10},0)$	0.501587302
39	$(0,0,8,-11,4)$	$(0,\frac{1}{5},\frac{4}{5},0,0)$	0.501587302
40	$(0,4,-6,4,-1)$	$(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{2}{5},0)$	0.517063492

Table 1 (*continued*)

N	**a**	**s**	$\Pr(T_N \leq T_{N+1})$
41	$(0, 3, -3, 2, -1)$	$(0, \frac{2}{5}, \frac{3}{10}, \frac{3}{10}, 0)$	0.500793651
42	$(0, 2, 1, -3, 1)$	$(0, \frac{3}{10}, \frac{1}{2}, \frac{1}{5}, 0)$	0.500793651
43	$(0, 1, 5, -8, 3)$	$(0, \frac{1}{5}, \frac{7}{10}, \frac{1}{10}, 0)$	0.500793651
44	$(0, 0, 9, -13, 5)$	$(0, \frac{1}{10}, \frac{9}{10}, 0, 0)$	0.516666667
45	$(1, 0, 0, 0, 0)$	$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.5
46	$(0, 4, -6, 5, -2)$	$(0, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}, 0)$	0.5
47	$(0, 3, -2, 0, 0)$	$(0, \frac{3}{10}, \frac{2}{5}, \frac{3}{10}, 0)$	0.5
48	$(0, 2, 2, -5, 2)$	$(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0)$	0.5
49	$(0, 1, 6, -10, 4)$	$(0, \frac{1}{10}, \frac{4}{5}, \frac{1}{10}, 0)$	0.5
50	$(0, 0, 10, -15, 6)$	$(0, 0, 1, 0, 0)$	0.523809524
51	$(0, 1, 7, -12, 5)$	$(0, 0, \frac{9}{10}, \frac{1}{10}, 0)$	0.500793651
52	$(0, 2, 3, -7, 3)$	$(0, \frac{1}{10}, \frac{7}{10}, \frac{1}{5}, 0)$	0.500793651
53	$(0, 3, -1, -2, 1)$	$(0, \frac{1}{5}, \frac{1}{2}, \frac{3}{10}, 0)$	0.500793651
54	$(0, 4, -5, 3, -1)$	$(0, \frac{3}{10}, \frac{3}{10}, \frac{2}{5}, 0)$	0.517063492
55	$(1, 0, 0, 1, -1)$	$(0, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$	0.501587302
56	$(0, 2, 4, -9, 4)$	$(0, 0, \frac{4}{5}, \frac{1}{5}, 0)$	0.501587302
57	$(0, 3, 0, -4, 2)$	$(0, \frac{1}{10}, \frac{3}{5}, \frac{3}{10}, 0)$	0.501587302
58	$(0, 4, -4, 1, 0)$	$(0, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0)$	0.519047619
59	$(0, 3, 1, -6, 3)$	$(0, 0, \frac{7}{10}, \frac{3}{10}, 0)$	0.502380952
60	$(0, 4, -3, -1, 1)$	$(0, \frac{1}{10}, \frac{1}{2}, \frac{2}{5}, 0)$	0.500396825
61	$(1, 0, 1, -1, 0)$	$(0, \frac{3}{10}, \frac{3}{10}, \frac{1}{5}, \frac{1}{5})$	0.500396825
62	$(0, 5, -7, 4, -1)$	$(0, \frac{1}{5}, \frac{3}{10}, \frac{1}{2}, 0)$	0.517460317
63	$(0, 4, -2, -3, 2)$	$(0, 0, \frac{3}{5}, \frac{2}{5}, 0)$	0.503174603
64	$(0, 5, -6, 2, 0)$	$(0, \frac{1}{10}, \frac{2}{5}, \frac{1}{2}, 0)$	0.500793651
65	$(1, 0, 2, -3, 1)$	$(0, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})$	0.517857143
66	$(0, 5, -5, 0, 1)$	$(0, 0, \frac{1}{2}, \frac{1}{2}, 0)$	0.501984127
67	$(1, 0, 3, -5, 2)$	$(0, \frac{1}{10}, \frac{1}{2}, \frac{1}{5}, \frac{1}{5})$	0.500396825
68	$(0, 6, -9, 5, -1)$	$(0, \frac{1}{10}, \frac{3}{10}, \frac{3}{5}, 0)$	0.503571429
69	$(1, 1, -1, 0, 0)$	$(0, \frac{1}{5}, \frac{3}{10}, \frac{3}{10}, \frac{1}{5})$	0.514285714
70	$(0, 6, -8, 3, 0)$	$(0, 0, \frac{2}{5}, \frac{3}{5}, 0)$	0.5
71	$(1, 0, 4, -7, 3)$	$(0, 0, \frac{3}{5}, \frac{1}{5}, \frac{1}{5})$	0.503174603
72	$(0, 7, -12, 8, -2)$	$(0, \frac{1}{10}, \frac{1}{5}, \frac{7}{10}, 0)$	0.501587302
73	$(1, 1, 0, -2, 1)$	$(0, \frac{1}{10}, \frac{2}{5}, \frac{3}{10}, \frac{1}{5})$	0.516269841
74	$(0, 7, -11, 6, -1)$	$(0, 0, \frac{3}{10}, \frac{7}{10}, 0)$	0.501984127
75	$(1, 1, 1, -4, 2)$	$(0, 0, \frac{1}{2}, \frac{3}{10}, \frac{1}{5})$	0.503968254
76	$(1, 2, -3, 1, 0)$	$(0, \frac{1}{10}, \frac{3}{10}, \frac{2}{5}, \frac{1}{5})$	0.513492063
77	$(0, 8, -14, 9, -2)$	$(0, 0, \frac{1}{5}, \frac{4}{5}, 0)$	0.503968254
78	$(1, 2, -2, -1, 1)$	$(0, 0, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})$	0.504761905
79	$(1, 3, -6, 4, -1)$	$(0, \frac{1}{10}, \frac{1}{5}, \frac{1}{2}, \frac{1}{5})$	0.510714286
80	$(0, 9, -17, 12, -3)$	$(0, 0, \frac{1}{10}, \frac{9}{10}, 0)$	0.505952381
81	$(1, 3, -5, 2, 0)$	$(0, 0, \frac{3}{10}, \frac{1}{2}, \frac{1}{5})$	0.515873016

Table 1 (*continued*)

N	**a**	**s**	$\Pr(T_N \le T_{N+1})$
82	$(0, 10, -20, 15, -4)$	$(0, 0, 0, 1, 0)$	0.507936508
83	$(1, 4, -8, 5, -1)$	$(0, 0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5})$	0.514285714
84	$(2, -1, 0, 0, 0)$	$(0, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5})$	0.504761905
85	$(1, 5, -11, 8, -2)$	$(0, 0, \frac{1}{10}, \frac{7}{10}, \frac{1}{5})$	0.509920635
86	$(2, -1, 1, -2, 1)$	$(0, 0, \frac{3}{10}, \frac{3}{10}, \frac{2}{5})$	0.509920635
87	$(1, 6, -14, 11, -3)$	$(0, 0, 0, \frac{4}{5}, \frac{1}{5})$	0.511904762
88	$(2, 0, -2, 1, 0)$	$(0, 0, \frac{1}{5}, \frac{2}{5}, \frac{2}{5})$	0.519841270
89	$(2, 1, -5, 4, -1)$	$(0, 0, \frac{1}{10}, \frac{1}{2}, \frac{2}{5})$	0.519841270
90	$(2, 2, -8, 7, -2)$	$(0, 0, 0, \frac{3}{5}, \frac{2}{5})$	0.535714286
91	$(3, -3, 1, 0, 0)$	$(0, 0, \frac{1}{10}, \frac{3}{10}, \frac{3}{5})$	0.517857143
92	$(3, -2, -2, 3, -1)$	$(0, 0, 0, \frac{2}{5}, \frac{3}{5})$	0.555555556
93	$(4, -6, 4, -1, 0)$	$(0, 0, 0, \frac{1}{5}, \frac{4}{5})$	0.555555556
94	$(5, -10, 10, -5, 1)$	$(0, 0, 0, 0, 1)$	

$\mathrm{Var}(Z_{24}) = 0.64$ and $\mathrm{Var}(Z_{25}) = 0.24$. Therefore, T_{24} is written before T_{25} in Table 1. We use this criterion to order the systems in lines 45–50 of Table 1 which are SP-equivalent. However, we do not use this criterion for the systems in lines 70 and 71 to maintain the symmetry of Table 1 which leads to the following property. In Table 1, the dual system T_N^D of T_N is T_{95-N} for $N = 1, 2, \ldots, 44$. The dual system T_N^D of T_N is itself for $N = 45, 46, \ldots, 50$. Note that if T_N stochastically precedes T_{N+1} if and only if T_{N+1}^D stochastically precedes T_N^D. Actually, $\Pr(T_N \le T_{N+1}) = \Pr(T_{N+1}^D \le T_N^D)$. Of course, this is a general property of dual systems in the IID case. The stochastic precedence order does not imply that the expected lifetimes are ordered. For example, in the IID case, X_1 stochastically precedes the system with lifetime $T = \min(\max(X_1, X_2), \max(X_3, X_4))$ and signature $(0, 1/3, 2/3, 0)$, while if the component lifetimes have a common exponential distribution with mean 1, then $E(T) = 11/12 < 1 = E(X_1)$ (see Tables 1–2 in Navarro and Rubio 2010). These systems are not ordered in the usual stochastic order (see Fig. 1 in Navarro et al. 2008b).

4 Discussion

Stochastic precedence is a good option to compare systems. If the two systems have IID components with the same baseline distribution, then these comparisons are distribution free, that is, they do not depend on the baseline distribution while other comparisons do (as happens, for example, with the comparisons based on expected values). Moreover, stochastic precedence is a necessary condition for the stochastic (reliability) ordering and, from the expressions given in Samaniego (2007), if the components are IID, the comparisons of coherent or mixed systems can be computed using system signatures. The main advantage of stochastic precedence is that if T_1

and T_2 are the lifetimes of two systems, then T_1 stochastically precedes T_2 or vice versa, while there exist systems that are not ordered in the usual stochastic order.

In the present paper, we obtain alternative expressions for the IID case based on the notion of minimal signatures (domination). Moreover, we obtain analogous expressions for coherent systems with dependent components and INID components using minimal path sets representations while the results in Samaniego (2007) only apply to systems with IID components. Our results can also be applied to mixed systems. Moreover, we show how to apply these theoretical results in different cases and, in particular, we compare (using stochastic precedence) all the coherent systems with 1–5 IID components. For these coherent systems, we have seen that the transitive property holds. Hence, stochastic precedence is a total ordering for them. We conjecture that this property holds for the set of coherent systems with less than n components for arbitrary n.

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