J Shanghai Univ (Engl Ed), 2010, 14(3): 170–173
Digital Object Identifier(DOI): 10.1007/s11741-010-0624-2

Blow-up and global existence for the heat equation with nonlinear absorption-diffusion

ZHANG Hai-liang (张海亮)[1], GUO Xiu-lan (郭秀兰)[2]
1. Department of Mathematics, Zhejiang Ocean University, Zhoushan 316004, Zhejiang, P. R. China
2. College of Science, Henan University of Technology, Zhengzhou 450001, P. R. China

Abstract This paper deals with blow-up of positive solution of the nonlinear heat equation with absorption subject to a nonlinear boundary condition. The conditions under which the solutions may exist globally or blow-up are obtained by the comparison principles.

Keywords nonlinear heat equation, absorption, blow-up, comparison principle
2000 Mathematics Subject Classification 35K55, 35K60

Introduction

In this paper, we investigate the finite time blow-up and global existence of positive solutions of the following initial-boundary value problem

$$u_t = \nabla u \cdot (a(u)\nabla u) - f(u), \text{ in } \Omega \times (0,T), \tag{1}$$

$$\frac{\partial u}{\partial n} = b(u), \text{ on } \partial\Omega \times (0,T), \tag{2}$$

$$u(x,0) = u_0(x) > 0, \text{ in } \Omega, \tag{3}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with C^2 boundary, $\frac{\partial u}{\partial n}$ the outward normal derivative, $u_0(x) \in C(\overline{\Omega}) \cap C^2(\Omega)$ a positive function, satisfying compatibility condition, $a(\cdot), f(\cdot), b(\cdot)$ positive ($a(\cdot) \geqslant a_0 > 0$), nondecreasing and C^1.

Problems (1)–(3) have been formulated from physical models arising in various fields of the applied sciences. For example, it can be interpreted as a heat conduction problem with nonlinear diffusivity, absorption at interior of the domain and a nonlinear radiation law on the boundary of the material body.

In the case $f < 0$ with is a heat source in the interior of the domain, blow-up phenomena for the problems have been studied by many authors[1–7]. In the cases $f \equiv 0$, some necessary conditions for the global existence and blow-up of the solutions are given in [8–11]. Here we are interested in the blow-up phenomenon of the solution of the problems (1)–(3) as $f > 0$.

The local existence and uniqueness of the positive classical solutions of the problems (1)–(3) were established by Amann[12]. We say that the solution u blows up in a finite time if there exists $0 < T < +\infty$ such that $\lim\limits_{t\to T} \| \boldsymbol{u} \|_{L^\infty(\Omega)} = +\infty$.

In this paper, we obtain the conditions under which the solutions may exist globally or blow up in a finite time for the problem (1)–(3) by upper and lower solution techniques. Our results generalize and deepen the ones from [13–15].

1 Blow-up and global existence

The theorems presented here show that blow-up of solutions of (1)–(3) are related to the behavior of positive solutions $w(\sigma)$ of the ordinary problem:

$$\begin{cases} w'(\sigma) = \mu b(w(\sigma)), \quad \mu > 0, \\ w(0) = \varphi_0. \end{cases} \tag{4}$$

We first give the following hypothesis:

(H) The functions f and b satisfy $\gamma b(s) \geqslant f(s)$ for $s > 0$, where $\gamma = \frac{a_0|\partial\Omega|}{2|\Omega|}$, and $|\Omega|$ and $|\partial\Omega|$ denote the measures of Ω and $\partial\Omega$ respectively.

Theorem 1.1 Let (H) hold. If every positive solution of (4) blows up, then every positive solution of (1)–(3) blows up.

Theorem 1.2 Suppose that (4) has global positive solutions and $\frac{G(\sigma)}{w(\sigma)}$ increase or decrease monotonously,

Received Oct.11, 2008; Revised Feb.2, 2009
Project supported by the Education Department of Zhejiang Province (Grant No.Y200805137), and the Zhejiang Ocean University (Grant No.X08Z04)
Corresponding author ZHANG Hai-liang, Ph D, E-mail: hlzhang@zjou.edu.cn

where G is given by

$$G(\sigma) = a(w(\sigma))w'(\sigma) + (a(w(\sigma))w'(\sigma))'.$$

Then, (i) if (H) holds and

$$\int^{+\infty} \frac{w'(\sigma)}{G(\sigma)} < +\infty,$$

then every positive solution of (1)–(3) blows up.

(ii) If

$$\int^{+\infty} \frac{w'(\sigma)}{G(\sigma)} = +\infty,$$

then every positive solution of (1)–(3) exists globally.

Examples We can apply the above theorems to the following cases:

(i) $a(u) \equiv 1,\ b(u) = Au^r + Bu,\ f(u) = Cu^q$ with $1 > q > 0,\ r > 0$, and A and B are some large enough constants.

(ii) $a(u) = u^p,\ b(u) = Au^r,\ f(u) = Bu^q + Cu$ with $p > 0,\ r \geqslant q \geqslant 1$, and A is a large enough constant.

Before giving the proofs of the above theorems, we first state a comparison theorem.

Comparison principle Let $\overline{u}(x,t), \underline{u}(x,t)$ be two positive smooth functions satisfying

$$\begin{aligned}&\overline{u}_t - \mathrm{div}(a(\overline{u})\nabla\overline{u}) + f(\overline{u}) \geqslant 0 \geqslant \underline{u}_t - \mathrm{div}(a(\underline{u})\nabla\underline{u})\\ &\quad + f(\underline{u}), \text{in } \Omega \times (0,T),\\ &\frac{\partial\overline{u}}{\partial n} - b(\overline{u}) \geqslant 0 \geqslant \frac{\partial\underline{u},}{\partial n} - b(\underline{u}), \text{ on } \partial\Omega \times (0,T),\\ &\overline{u}(x,0) \geqslant u_0(x) \geqslant \underline{u}(x,0), \text{ in } \Omega.\end{aligned}$$

Then $\overline{u}(x,t) \geqslant u(x,t) \geqslant \underline{u}(x,t)$ for all $x \in \overline{\Omega}, t \in (0,T)$, where $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are respectively called a super solution and a lower solution of (1)–(3).

The proof of the comparison principle is similar to those given in [12], therefore we omit it here.

Proof of Theorem 1.1 We will construct a smooth positive lower solution of the form $\underline{u}(x,t) = w(t + \alpha(x)) \equiv w(\sigma)$, where w and α are some functions to be determined later. Direct computation yields

$$\underline{u}_t = w'(\sigma), \tag{5}$$

$$\begin{aligned}&\mathrm{div}(a(\underline{u})\nabla\underline{u}) = a'(w(\sigma))(w'(\sigma))^2 \mid \nabla\alpha(x) \mid^2\\ &\quad + a(w(\sigma))(w'(\sigma)\Delta\alpha(x) + w''(\sigma) \mid \nabla\alpha(x) \mid^2),\end{aligned} \tag{6}$$

$$\frac{\partial\underline{u}}{\partial n} = w'(\sigma)\frac{\partial\alpha(x)}{\partial n}. \tag{7}$$

Hence, the following conditions insure that $\underline{u}$ is a lower solution:

$$\begin{aligned}&(a'(w')^2 + aw'') \mid \nabla\alpha \mid^2 + aw'\Delta\alpha \geqslant w' + f,\\ &\quad \text{in } \Omega \times (0,T),\end{aligned} \tag{8}$$

$$w'\frac{\partial\alpha}{\partial n} \leqslant b, \text{ on } \partial\Omega \times (0,T), \tag{9}$$

$$u_0(x) \geqslant w(\alpha(x)), \text{ in } \Omega. \tag{10}$$

We find that (8) is true if both $w'' \geqslant 0$ and the following holds:

$$w'(a\Delta\alpha - 1) \geqslant f, \text{ in } \Omega \times (0,T). \tag{11}$$

Let w be the solution of

$$w'(\sigma) = \gamma b(w(\sigma)), \text{ for } \sigma > 0 \text{ and } w(0) = w_0 > 0,$$

where w_0 is a constant. Define $\alpha(x)$ as the solution of the following problem:

$$\begin{aligned}&a_0\Delta h = 2, \text{ in } \Omega,\\ &\frac{\partial h}{\partial n} = \gamma^{-1} = \frac{2|\Omega|}{a_0|\partial\Omega|}, \text{ on } \partial\Omega.\end{aligned} \tag{12}$$

If $h(x) \in C^2(\overline{\Omega})$ is a solution of (12), it is easy to see that $h(x) \pm c$ is also a solution of (12) for any constant $c > 0$. Thus we may take $\alpha(x) > 0$ which satisfies $u_0(x) \geqslant w(\alpha(x))$.

It is easy to check that (9)–(11) hold if we use w and $\alpha(x)$ chosen as above and the assumption (H) are true. In addition, one can see that $w'' = \gamma b'w' = \gamma^2 b'b \geqslant 0$. Therefore, $\underline{u}$ is a lower solution. Since $\underline{u} = w(t + \alpha(x))$ blow up hence the solution u of the problems (1)–(3) also blows up by the comparison theorem. Theorem 1.1 is proved.

Proof of Theorem 1.2 (i) Analogous with the proof of the Theorem 1.1, we look for suitable functions $\beta(t), \alpha(x)$ such that $\underline{u}(x,t) = w(\beta(t) + \alpha(x)) \equiv w(\sigma)$ is a smooth positive lower solution of (1)–(3).

Let w be the solution of

$$w'(\sigma) = \gamma b(w(\sigma)), \text{ for } \sigma > 0 \text{ and } w(0) = w_0 > 0$$

where w_0 is a constant. Then the function $w(\sigma)$ is positive global and $w'' = \gamma b'w' \geqslant 0$.

Let $\alpha(x)$ be a positive solution of the problem (12). We can assume that $0 < \alpha(x) \leqslant w^{-1}(u_0(x))$, *i.e.*, $u_0(x) \geqslant w(\alpha(x)) > 0$, and there exists a constant $\varepsilon_0 > 0$ such that $|\nabla\alpha(x)| \geqslant \varepsilon_0$.

Let $\beta(t)$ be the solution of the ordinary differential equation

$$\begin{aligned}&\beta'(t) = \delta(a(w(\beta)) + a(w(\beta))b'(w(\beta))\\ &\qquad + a'(w(\beta))b(w(\beta))),\\ &\beta(0) = 0,\end{aligned} \tag{13}$$

where $t > 0$ $\delta = \min\{\varepsilon_0, \varepsilon_0\gamma, \frac{2}{a_0}\}$.

Now we will validate that $\underline{u}(x,t) = w(\beta(t) + \alpha(x))$ is a lower solution of (1)–(3). Direct computation yields

$$\underline{u}_t = w'(\sigma)\beta'(t), \tag{14}$$

$$
\begin{aligned}
&\nabla\cdot(a(\underline{u})\nabla\underline{u})-f(\underline{u})\\
&=(a'(w)(w')^2+a(w)w'')\mid\nabla\alpha\mid^2+a(w)w'\Delta\alpha-f(w)\\
&=(a'(w)(w')^2+a(w)w'')\mid\nabla\alpha\mid^2+\frac{1}{2}a(w)w'\Delta\alpha\\
&\quad+\Big(\frac{1}{2}a(w)w'\Delta\alpha-f(w)\Big)\\
&\geqslant\varepsilon_0\Big(a'(w)(w')^2+a(w)w''\Big)+\frac{2}{a_0}a(w)w'\\
&\quad+(w'-f(w)\Big)\\
&=\varepsilon_0(a'(w)(w')^2+\gamma a(w)b'w')+\frac{2}{a_0}a(w)w'\\
&\quad+(\gamma b(w)-f(w))\\
&\geqslant\delta\Big(a(w)+a'(w)b(w)+a(w)b'(w)\Big)w'. \qquad (15)
\end{aligned}
$$

Since $\beta(t)$ is a solution of problem (13), from (14) and (15) we have that

$$\underline{u}_t-\nabla\cdot(a(\underline{u})\nabla\underline{u})+f(\underline{u})\leqslant 0.$$

On the other hand,

$$
\begin{aligned}
\frac{\partial\underline{u}}{\partial n}-b(\underline{u})&=w'\frac{\partial\alpha}{\partial n}-b(w)\\
&=\gamma b(w)\gamma^{-1}-b(w)\leqslant 0, \qquad (16)
\end{aligned}
$$

$$\underline{u}(x,0)=w(\beta(0)+\alpha(x))=w(\alpha(x))\leqslant u_0(x). \qquad (17)$$

This shows that $\underline{u}(x,t)=w(\beta(t)+\alpha(x))$ is a lower solution of (1)–(3); therefore, $u(x,t)\geqslant\underline{u}(x,t)$. Since $\beta(t)$ is a solution of problem (13) and $\int^{+\infty}\frac{w'(\sigma)}{G(\sigma)}<+\infty$, then $\beta(t)$ blows up in finite time and hence $\underline{u}(x,t)$ blows up. So does $u(x,t)$.

(ii) Now we look for a global upper solution. Let $\overline{u}(x,t)=w(\beta(t)+\alpha(x))\equiv w(\sigma)$, where $w(\sigma)$ is the solution of

$$w'(\sigma)=b(w(\sigma)),\ \text{for } \sigma>0 \text{ and } w(0)=w_0>0.$$

Then the function $w(\sigma)$ is positive global and $w''=b'w'\geqslant 0$.

We choose $\alpha(x)$ as a C^2 function such that $\frac{\partial\alpha}{\partial n}\geqslant 1$ at $\partial\Omega$ (for instance a smooth extension of the distance to $\partial\Omega$). We can assume that $\alpha(x)>0$ in $\overline{\Omega}$ (just add a constant). With this α we have that

$$\frac{\partial\overline{u}}{\partial n}-b(\overline{u})=b(w)\frac{\partial\alpha}{\partial n}-b(w)\geqslant 0.$$

To satisfy the initial condition $\overline{u}(x,0)\geqslant u_0(x)$, we take w_0 big enough. For the equation, we choose $\beta(t)$ to be a solution of the Cauchy problem

$$\beta'(t)=L\Big(\frac{G(\beta(t+k))}{w'(\beta(t+k))}\Big),\ \beta(0)=\beta_1>0,$$

if $\frac{G}{w'}$ is increasing, or the Cauchy problem

$$\beta'(t)=L\Big(\frac{G(\beta(t))}{w'(\beta(t))}\Big),\ \beta(0)=\beta_2>0,$$

if $\frac{G}{w'}$ is decreasing. Here $k=\max_{x\in\overline{\Omega}}\alpha(x)$, $L=\max(1,\Delta\alpha,|\nabla\alpha|^2)$. We observe that $\beta(t)$ is global because of our hypothesis.

Since w and β exist globally, we know that $\overline{u}(x,t)$ exists for all $t>0$. So does $u(x,t)$. This concludes the proof of Theorem 1.2.

In Theorems 1.1 and 1.2, if $f(u)\equiv 0$, then the following conclusion holds.

Corollary 1.1 Let $v(x,t)$ be a smooth solution of the following problem:

$$
\begin{cases}
v_t=\nabla\cdot(a(v)\nabla v),\ \text{in } \Omega\times(0,T),\\
\frac{\partial v}{\partial n}=b(v),\ \text{on } \partial\Omega\times(0,T),\\
v(x,0)=v_0>0,\ \text{in } \Omega.
\end{cases}
$$

Then it holds:

(i) If $\int^{+\infty}\frac{ds}{b(s)}<+\infty$, then $v(x,t)$ blows up in finite time.

(ii) Let $\int^{+\infty}\frac{ds}{b(s)}=+\infty$ and $a(s)+(a(s)b(s))'$ be nondecreasing.

(a) If $\int^{+\infty}\frac{ds}{a(s)+(a(s)b(s))'}<+\infty$, then $v(x,t)$ blows up in finite time.

(b) If $\int^{+\infty}\frac{ds}{a(s)+(a(s)b(s))'}=+\infty$, then $v(x,t)$ exists globally.

This is the case of [10].

2 Concluding remarks and applications

Problems (1)–(3) arises in the nonlinear diffusion process, in which $\nabla u\cdot(a(u)\nabla u)$ denotes the nonlinear diffusion effect, $f(u)$ absorption in the interior of the domain, and $\frac{\partial u}{\partial n}$ the boundary flux along the outward normal to the domain. With this model, all the results obtained in the preceding sections are physically meaningful.

Our results show that the strength of the boundary flux plays a key role in the blow-up properties of the problems (1)–(3). If the boundary flux is sufficiently strong, then it may cause blow-up in a finite time. If the boundary flux is not sufficiently strong, it is probable that the solution will never blow up.

As an, application of theorems, now we consider the following porous medium problem:

$$
\begin{aligned}
&u_t=\Delta u^p-u^q,\ \text{in } \Omega\times(0,T),\\
&\frac{\partial u}{\partial n}=u^r,\ \text{on } \partial\Omega\times(0,T),\\
&u(x,0)=u_0(x)>0,\ \text{in } \Omega,
\end{aligned}
$$

where $p\geqslant 1,\ q,\ r>0$.

By our theorems, it is easy to obtain: if $r \leqslant 1 \leqslant p \leqslant 2$, every positive solution $u(x,t)$ of the problem exists globally. If $r > p \geqslant 1$, $r > q$, every positive solution $u(x,t)$ of the problem blows up in a finite time.

References

[1] ACOSTA G, ROSSI J D. Blow-up *vs* global existence for quasilinear prabolic systems with a nonlinear boundary condition [J]. Journal of Applied Mathematics and Physics, 1997, 48(5): 711–724.

[2] DING J, LI S. Blow-up solutions and global solutions for a class of quasilinear parabolic equations with Robin boundary conditions [J]. Computers and Mathematics with Applications, 2005, 49(5): 689–701.

[3] FRIEDMAN A, MCLEOD J B. Blow-up of positive solutions of semi-linear heat equations [J]. Indiana University Mathematics Journal, 1985, 34(2): 425–447.

[4] FUJITA H. On the blowing up of solutions of the Cauchy problem for $u_t = \triangle u + u^{1+\alpha}$ [J]. Journal of the Faculty of Science, 1966, 16(1): 105–113.

[5] PINSKY R. Existence and noexistence of global solutions for $u_t = \Delta u + a(x)u^p$ in $\mathbf{R}^d$ [J]. Journal of Differential Equations, 1997, 133(1): 152–177.

[6] YU Wei, WANG Yuan-di. Properties of parabolic partial differential equation of the first kind boundary condition problems [J]. Journal of Shanghai University (Natural Science Edition), 2005, 11(4): 391–401 (in Chinese).

[7] ZHANG H, LIU Z, ZHANG W. Growth estimates and blow-up in quasilinear parabolic problems [J]. Applicable Analysis, 2007, 86(2): 261–268.

[8] HU B, YIN H M. The profile near blowup time for solution of the heat equation with a nonlinear boundary condition [J]. Transactions of the American Mathematical Society, 1994, 346(1): 117–135.

[9] LEVINE H, PAYNE L. Nonexistce theorems for the heat equation with nonlinear boundary conditions and for the porous medium equation backward in time [J]. Journal of Differential Equations, 1974, 16(2): 319–334.

[10] WANG M, WU Y. Global existence and blow-up problems for quasilinear parabolic equations with nonlinear boundary conditions [J]. SIAM Journal on Mathematical Analysis, 1993, 24(6): 1515–1521.

[11] YING H M. Blow-up *versus* global solvability for a class of nonlinear parabolic equations [J]. Nonlinear Analysis, 1994, 23(7): 911–924.

[12] AMANN A. Quasilinear parabolic systems under nonlinear boundary conditions [J]. Archive for Rational Mechanics and Analysis, 1986, 92(2): 153–192.

[13] BONI T K. Sur i'explosion et le comportement asymptotique de la solution d'une equation parabolique semilinaire du second ordre [J]. Comptes Rendus de UAcadémie des Scuebces, Série I, 1998, 326(3): 317–322.

[14] GALAKTIONOV V A, VAZQUEZ J L. Asymptic behaviour of nonlinear parabolic equations with critical exponents [J]. Journal of Functional Ananlysis, 1991, 100(2): 435–462.

[15] QUITTNER P. On global existence and stationary solutions for two classes of semilinear parabolic problems [J]. Commentationes Mathematicae Universitatis, 1993, 34(1): 105–124.

(Editor HONG Ou)