J Shanghai Univ (Engl Ed) (2008) 12(1): 80–84
Digital Object Identifier(DOI): 10.1007/s 11741-008-0116-1

Best performance analysis for stochastic perturbation systems with spectral factorization

WANG Jian-guo (王建国), CAO Guang-yi (曹广益), ZHU Xin-jian (朱新坚)
Department of Automation, School of Electronic, Information and Electrical Engineering, Shanghai Jiaotong University, Shanghai 200030, P. R. China

Abstract We investigate the best performance for linear feedback control systems in the case that plant uncertainty is to be considered. First, we define an average integral square criterion of tracking error over a class of stochastic model errors. By utilizing spectral factorization to minimize the performance index, we derive an optimal controller design method and further study best performance in the presence of stochastic perturbation. The results can be used to evaluate optimal performance in practical control system designs.

Keywords best tracking, plant uncertainty, stochastic perturbation, spectral factorization.

1 Introduction

Since the seminal work of Bode during the 1940s related to feedback amplifier design[1], fundamental performance limitations of feedback control loops have been a topic of interest[2,3].

There always exists plant uncertainty in practice and model error has great impact on the system performance[4,5]. Therefore optimal performance in the presence of plant uncertainty has been studied by many researchers. One of the main tasks of feedback control is to handle plant uncertainty, so the paper[6] studied the maximum uncertainty that can be dealt with by the feedback mechanism. In the spirit of the questions posed in [6], the paper[7] utilized the idea of uncertainty embedding to formulate the description of relative model errors and investigated performance limitations in the presence of plant uncertainty.

In this paper, we study the best tracking performance for systems with stochastic perturbation. Firstly, based on robust redesign, we present an average performance index of tracking error over a class of stochastic model errors. Then, applying spectral factorization to minimization of the average performance, we derive an optimal controller design method and investigate the best tracking performance in the presence of a class of stochastic model errors. The results can be used to predict the optimal average performance in practical control system designs.

2 Robust redesign

Unity feedback control system is shown in Fig.1. We consider linear SISO system with step reference input and assume that the plant is open-loop stable, the system is initially at rest and there is no output disturbance.

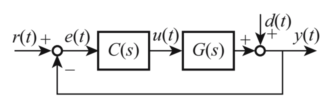

Fig.1 Unit feedback control system

Since the plant is assumed to be open-loop stable, the parameterization of all stabilizing controllers for the nominal plant $G_0(s)$ can be written in the form

$$C(s) = \frac{Q_0(s)}{1 - G_0(s)Q_0(s)}, \tag{1}$$

where $Q_0(s)$ is a stable and proper transfer function. Based on Youla parameterization, the nominal complementary sensitivity function and sensitivity function can be given by

$$T_0(s) = G_0(s)Q_0(s), \tag{2}$$

$$S_0(s) = 1 - G_0(s)Q_0(s). \tag{3}$$

As for plant uncertainty, we adopt the class of model errors described in [8] and the true plant takes the form

$$G(s) = G_0(s) + G_\varepsilon(s), \tag{4}$$

where the additive model error $G_\varepsilon(s)$ possesses the sta-

Received Jul.10, 2006; Revised Oct.26, 2006
Project supported by the National High-Technology Research and Development Program of China (Grant No.2003AA517020)
Corresponding author WANG Jian-guo, PhD Candidate, E-mail: jawang@sjtu.edu.cn

tistical properties as follows:

$$\mathrm{E}\{G_\varepsilon(\mathrm{j}\omega)\} = 0, \tag{5}$$

$$\mathrm{E}\{|G_\varepsilon(\mathrm{j}\omega)|^2\} = \mathrm{E}\{G_\varepsilon(\mathrm{j}\omega)G_\varepsilon(-\mathrm{j}\omega)\} = \alpha(\mathrm{j}\omega)\alpha(-\mathrm{j}\omega) = \bar{\alpha}^2(\omega). \tag{6}$$

Here, $\alpha(s)$ is a stable, minimum-phase spectral factor and $\bar{\alpha}(\omega)$ is a given measure of the modeling error, which can be obtained by model identification.

To best cope with the class of model errors, we change $Q_0(s)$ to $Q_1(s)$. Thus, the sensitivity function becomes

$$S_1(s) = \frac{1 - G_0(s)Q_1(s)}{1 + G_\varepsilon(s)Q_1(s)}. \tag{7}$$

In the same spirit as the use of additive model errors, $Q_1(s)$ can be written in terms of $Q_0(s)$ as

$$Q_1(s) = Q_0(s) + Q_\varepsilon(s). \tag{8}$$

Substituting into (7), we have

$$S_1(s) = \frac{S_0(s) - G_0(s)Q_\varepsilon(s)}{1 + G_\varepsilon(s)(Q_0(s) + Q_\varepsilon(s))}. \tag{9}$$

To preserve integral action, we assume that $Q_\varepsilon(s)$ has the form

$$Q_\varepsilon(s) = sQ'_\varepsilon(s). \tag{10}$$

Since we are considering the class of stochastic model errors with properties as in (5) and (6), the performance index should be described by the average performance. Therefore, we consider

$$\bar{J} = \mathrm{E}\left\{\int_0^\infty e(t)^2 \mathrm{d}t\right\}, \tag{11}$$

where the expectation is over the class of additive modeling errors.

Making use of Parseval's theorem and assuming sufficient regularity to interchange the integral and expectation operators, the average performance becomes

$$\bar{J} = \mathrm{E}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{|S_1(\mathrm{j}\omega)|^2}{\omega^2}\mathrm{d}\omega\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{E}\{|S_1(\mathrm{j}\omega)|^2\}}{\omega^2}\mathrm{d}\omega. \tag{12}$$

Now, we determine Q_ε that optimizes (12). Since $|T_0|$ is less than 1, closed-loop stability requires that $|G_\varepsilon(Q_0 + Q_\varepsilon)| < 1$. Thus, we make a Taylor's series expansion in $G_\varepsilon(Q_0 + Q_\varepsilon)$ as

$$\begin{aligned} S_1 &\approx (S_0 - G_0Q_\varepsilon)[1 - G_\varepsilon(Q_0 + Q_\varepsilon)] \\ &= S_0 - G_0Q_\varepsilon - S_0G_\varepsilon(Q_0 + Q_\varepsilon) \\ &\quad + G_0Q_\varepsilon G_\varepsilon(Q_0 + Q_\varepsilon). \end{aligned} \tag{13}$$

The last term in the expression depends on the product of $Q_\varepsilon, G_\varepsilon$ and $(Q_0 + Q_\varepsilon)$. We know that at those frequencies where $|G_\varepsilon|$ is small, $|Q_\varepsilon|$ will also be small and there is no need to change the nominal controller. At those frequencies where $|G_\varepsilon|$ approaches $|G_0|$, to preserve stability, it is necessary that Q_ε go to $-Q_0$ and $(Q_0 + Q_\varepsilon)$ should approach zero. Therefore, combination of factors $Q_\varepsilon, G_\varepsilon$ and $(Q_0 + Q_\varepsilon)$ ensures that the last term should be small relative to the others and we further approximate $S_1(s)$ by

$$S_1 \approx S_0 - G_0Q_\varepsilon - S_0G_\varepsilon(Q_0 + Q_\varepsilon). \tag{14}$$

Substituting the approximations into (12) and utilizing (5) and (6), the average performance becomes

$$\bar{J} \approx \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{|S_0 - G_0Q_\varepsilon|^2}{\omega^2} + \frac{|S_0|^2|Q_0 + Q_\varepsilon|^2\bar{\alpha}^2(\omega)}{\omega^2}\right]\mathrm{d}\omega. \tag{15}$$

3 Minimizing performance index

Suppose that the nominal model of the plant has one non-minimum phase zero and has no pure time delay. Then, we can write

$$G_0(s) = \frac{-s + c}{s + c}\bar{G}_0(s) = B_T(s)\bar{G}_0(s), \tag{16}$$

where $c > 0$ and $\bar{G}_0(s)$ is rational and minimum phase.

The nominal design for optimal tracking can be given by

$$Q_0(s) = [\bar{G}_0(s)]^{-1}\frac{1}{(\beta s + 1)^n}, \tag{17}$$

where $\beta \to 0$ and the degree n should ensure that $Q_0(s)$ be bi-proper.

Substituting $G_0(s) = B_T(s)\bar{G}_0(s)$, $S_0(s) \approx 1 - B_T(s)$, and $Q_0(s)$ into (15) and making use of $Q_\varepsilon(s) = sQ'_\varepsilon(s)$, $\bar{J}$ can be written as

$$\bar{J} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{|2 - (-\mathrm{j}\omega + c)\bar{G}_0(\mathrm{j}\omega)Q'_\varepsilon|^2}{|\mathrm{j}\omega + c|^2} + \frac{4|Q_0(\mathrm{j}\omega) + \mathrm{j}\omega Q'_\varepsilon|^2\bar{\alpha}^2(\omega)}{|\mathrm{j}\omega + c|^2}\right]\mathrm{d}\omega. \tag{18}$$

For any complex function $A(\mathrm{j}\omega)$, there exists the property $|A(\mathrm{j}\omega)|^2 = A(\mathrm{j}\omega)A(-\mathrm{j}\omega) = |A(-\mathrm{j}\omega)|^2$. Thus, completing the operation of square for (18), $\bar{J}$ can be written as

$$\bar{J}=\int_{-\infty}^{\infty}\left[|\bar{G}_0(\mathrm{j}\omega)|^2+4\bar{\alpha}^2(\omega)\frac{\omega^2}{|\mathrm{j}\omega+c|^2}\right]|Q'_\varepsilon(\mathrm{j}\omega)^2|\mathrm{d}\omega$$
$$+\int_{-\infty}^{\infty}\left[\frac{-2(\mathrm{j}\omega+c)\bar{G}_0(-\mathrm{j}\omega)}{|\mathrm{j}\omega+c|^2}+4\bar{\alpha}^2(\omega)\frac{-\mathrm{j}\omega Q_0(\mathrm{j}\omega)}{|\mathrm{j}\omega+c|^2}\right]Q'_\varepsilon(-\mathrm{j}\omega)\mathrm{d}\omega$$
$$+\int_{-\infty}^{\infty}\left[\frac{-2(\mathrm{j}\omega+c)\bar{G}_0(\mathrm{j}\omega)}{|\mathrm{j}\omega+c|^2}+4\bar{\alpha}^2(\omega)\frac{\mathrm{j}\omega Q_0(-\mathrm{j}\omega)}{|\mathrm{j}\omega+c|^2}\right]Q'_\varepsilon(\mathrm{j}\omega)\mathrm{d}\omega$$
$$+\int_{-\infty}^{\infty}\frac{4(1+\bar{\alpha}^2(\omega)|Q_0(\mathrm{j}\omega)|^2)}{|\mathrm{j}\omega+c|^2}\mathrm{d}\omega. \tag{19}$$

The coefficient item of $|Q'_\varepsilon(\mathrm{j}\omega)|^2$ in the first portion can be written as

$$C_{\mathrm{OEF}}(s)=\bar{G}_0(s)\bar{G}_0(-s)+4\frac{s(-s)\alpha(s)\alpha(-s)}{(s+c)(-s+c)}. \tag{20}$$

It can be shown that the coefficient item has a spectral factor, which we label H. Utilizing the regularity between the second and the third portion in (19), we represent $\frac{-2(\mathrm{j}\omega+c)\bar{G}_0(-\mathrm{j}\omega)-4\mathrm{j}\omega\bar{\alpha}^2(\omega)Q_0(\mathrm{j}\omega)}{|\mathrm{j}\omega+c|^2}$ as $F(\mathrm{j}\omega)$ and the fourth portion as $D(\mathrm{j}\omega)$, then

$$\bar{J}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{|Q'_\varepsilon(\mathrm{j}\omega)|^2|H(\mathrm{j}\omega)|^2+[Q'_\varepsilon(-\mathrm{j}\omega)F(\mathrm{j}\omega)+Q'_\varepsilon(\mathrm{j}\omega)F(-\mathrm{j}\omega)]+D(\mathrm{j}\omega)\}\mathrm{d}\omega. \tag{21}$$

Adding plus and minus $|F(\mathrm{j}\omega)|^2/|H(-\mathrm{j}\omega)|^2$ to the preceding equation respectively, $\bar{J}$ becomes

$$\bar{J}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{|Q'_\varepsilon(\mathrm{j}\omega)|^2|H(\mathrm{j}\omega)|^2+Q'_\varepsilon(-\mathrm{j}\omega)F(\mathrm{j}\omega)+Q'_\varepsilon(\mathrm{j}\omega)F(-\mathrm{j}\omega)+\frac{|F(\mathrm{j}\omega)|^2}{|H(-\mathrm{j}\omega)|^2}+D(\mathrm{j}\omega)-\frac{|F(\mathrm{j}\omega)|^2}{|H(-\mathrm{j}\omega)|^2}\right\}\mathrm{d}\omega. \tag{22}$$

The first four portions can be combined as

$$\left[Q'_\varepsilon(\mathrm{j}\omega)H(\mathrm{j}\omega)+\frac{F(\mathrm{j}\omega)}{H(-\mathrm{j}\omega)}\right]\cdot\left[Q'_\varepsilon(-\mathrm{j}\omega)H(-\mathrm{j}\omega)+\frac{F(-\mathrm{j}\omega)}{H(\mathrm{j}\omega)}\right]. \tag{23}$$

Making use of $|A(\mathrm{j}\omega)|^2=A(\mathrm{j}\omega)A(-\mathrm{j}\omega)=|A(-\mathrm{j}\omega)|^2$, performance index $\bar{J}$ can be written further as

$$\bar{J}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\left|Q'_\varepsilon(\mathrm{j}\omega)H(\mathrm{j}\omega)+\frac{F(\mathrm{j}\omega)}{H(-\mathrm{j}\omega)}\right|^2+D(\mathrm{j}\omega)-\left|\frac{F(\mathrm{j}\omega)}{H(-\mathrm{j}\omega)}\right|^2\right\}\mathrm{d}\omega, \tag{24}$$

where

$$F(\mathrm{j}\omega)=\frac{-2(\mathrm{j}\omega+c)\bar{G}_0(-\mathrm{j}\omega)-4\mathrm{j}\omega\bar{\alpha}^2(\omega)Q_0(\mathrm{j}\omega)}{|\mathrm{j}\omega+c|^2}, \tag{25}$$

$$D(\mathrm{j}\omega)=\frac{4(1+\bar{\alpha}^2(\omega)|Q_0(\mathrm{j}\omega)|^2)}{|\mathrm{j}\omega+c|^2}. \tag{26}$$

Note that the magnitude of (24) depends only on the first term, and according to Lemma 15.2 in [8], the minimum of (24) can be achieved by making the choice

$$Q'_\varepsilon(s)=-\frac{1}{H(s)}\left\{\frac{F(s)}{H(-s)}\right\}_{\text{stable part}}. \tag{27}$$

Substituting the optimal $Q'_\varepsilon(s)$ into (18), the optimal average performance can be obtained. The redesigned controller can be obtained from (8) and (10).

A pure time delay can be treated by use of a first-order Padé approximation

$$\mathrm{e}^{-s\tau_0}\approx\left(1-\frac{s\tau_0}{2}\right)\Big/\left(1+\frac{s\tau_0}{2}\right). \tag{28}$$

Therefore, for the plant with a pure time delay in the nominal model, the optimal average performance can be obtained by substituting $c=2/\tau_0$ to the achieved results.

For general nominal plants such as the case of two non-minimum phase zeros in the nominal plant, one can obtain similar results using this methodology. It is certain that the results become more complex as the complexity of nominal model increases.

As is well known, the sensitivity function describes the responses of both system output to disturbance and tracking error to reference input. And the responses of plant input to both output disturbance and reference input can be described by the control sensitivity function[9]. Therefore, in the same way, the system will possess the optimal performance of disturbance rejection in the presence of step output disturbance.

4 Simulation

To illustrate the preceding results, we consider stable plant with a pure time delay and nominal model given by

$$G_0(s)=\frac{1}{s+1}\mathrm{e}^{-s}. \tag{29}$$

We can obtain

$$\bar{G}_0(s)=\frac{1}{s+1} \tag{30}$$

and the nominal design for optimal tracking can be given as

$$Q_0(s) = \frac{s+1}{\beta s+1}, \tag{31}$$

where $\beta \to 0$.

Now, we consider the case that plant has stochastic perturbation. Suppose that the true model takes the form

$$G(s) = \frac{K_{\rm u}}{B_{\rm u}s+1}{\rm e}^{-T_{\rm u}s}, \tag{32}$$

where $K_{\rm u}$, $B_{\rm u}$ and $T_{\rm u}$ are uniformly distributed and independent random variables, which ensure that the true model satisfy the statistical description centered at $G_0({\rm j}\omega)$ and has standard deviation of $\sqrt{0.4}$.

Substituting (30), (31), $\bar{\alpha}^2(\omega) = 0.4$ and $c = 2$ into (27), we can obtain optimal $Q'_\varepsilon(s)$ and further obtain the redesigned parameterized function, which can be simplified as

$$Q_1(s) = \frac{0.7503(s+2.577)(s+0.8788)}{s^2+2.406s+1.699}. \tag{33}$$

From (31) and (33), we can obtain the nominal controller $C_0(s)$ and redesigned controller $C_1(s)$.

(A) Performance prediction

Substituting the corresponding optimal value of $Q'_\varepsilon(s)$ into (18), we can obtain that the expected performance is 1.7489. Thus, we have presented prediction to the achievable optimal tracking performance.

Assuming that there is no plant uncertainty and the preceding nominal model is exact, we can obtain the best tracking error for the plant should be 1. Therefore, we can find that plant uncertainty has great influence on the achievable optimal performance in practice.

(B) Stability robustness

By simulation, we can verify that the redesigned controller can stabilize any realization of the uncertainty with $K_{\rm u}$, $B_{\rm u}$ and $T_{\rm u}$, which satisfy the preceding conditions, while the optimal nominal controller has poor robustness.

For instance, we take a single realization of the uncertainty as follows:

$$G_1(s) = \frac{1.3}{0.5s+1}{\rm e}^{-1.3s}. \tag{34}$$

Note the difference between the perturbed plant $G_1(s)$ and the nominal model is much larger than might be expected in any practical situation. For the nominal plant and perturbed plant, controlled by nominal and redesigned controller respectively, the different closed loop response of system output is shown in Fig.2.

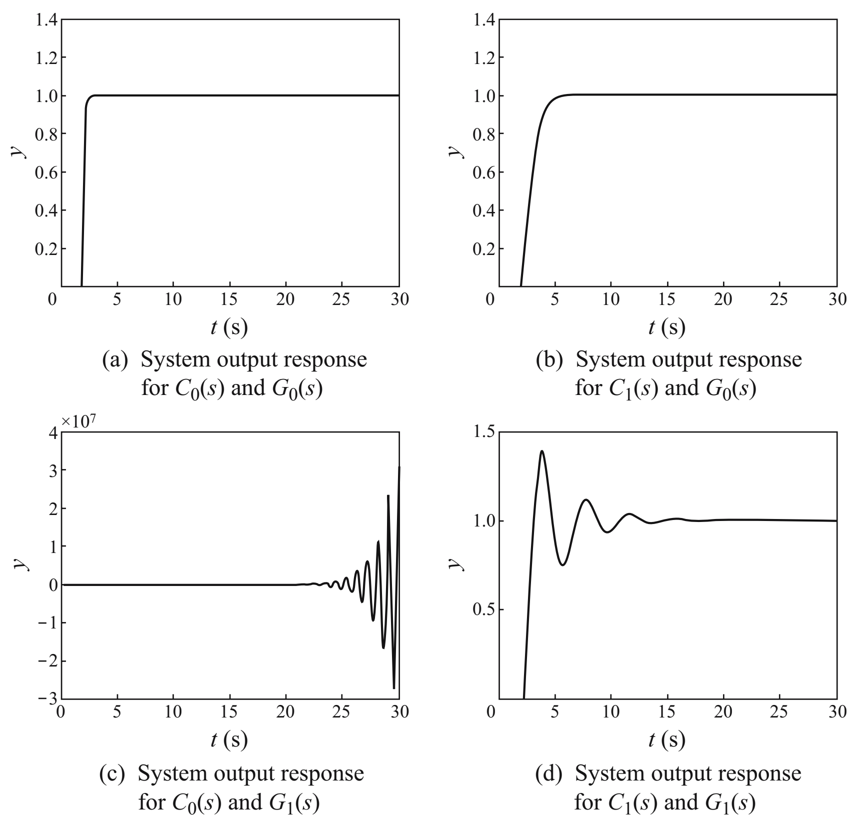

Fig.2 Time response contrast for different combination of plant and controller

We note that nominal controller could control nominal plant ideally, while the nominal controller could not stabilize the perturbed plant and the corresponding system output is not converged. As for the redesigned controller, there is a price paid in terms of the nominal performance to achieve robust performance, and, as we expect, the redesigned nominal controller could stabilize the perturbed plant commendably.

It is obvious that optimal nominal controller has better tracking performance for the nominal plant. However, the perturbed plant could not be stabilized by the optimal nominal controller, which is designed to achieve the limiting performance in the absence of uncertainty. In contrast, redesigned controller has good stability robustness to the large plant perturbation.

5 Conclusion

In practice, there exists the case that plant uncertainty is to be considered. In this paper, we have derived an optimal controller design method and investigated best tracking performance for feedback control systems with stochastic perturbation, which can be used to predict the optimal average performance and give a significant guide for practical control system designs.

One extension is to study similar issues for plants having unstable poles in nominal model. It is also desirable to study similar issues under control energy constraint. These should be future research directions.

References

[1] Bode H W. *Network Analysis and Feedback Amplifier Design* [M]. New York: Van Nostrand, 1945.

[2] Seron M M, Braslavsky J H, Goodwin G C. *Fundamental Limitations in Filtering and Control* [M]. London: Springer-Verlag, 1997.

[3] Åström K J. Fundamental limitations of control system performance [C]//Paulraj A, Roychowdhury V, Schaper C D. *Communication, Computation, Control and Signal Processing: A Tribute to Thomas Kailath*. Boston, MA: Kluwer, 1997: 355–363.

[4] Doyle J C. Synthesis of robust controller and filter [C]//*Proceedings of the* 22nd *IEEE Conference Decision and Control*, San Antonio. 1983.

[5] Morari M, Zafiriou E. *Robust Process Control* [M]. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[6] Xie L L, Guo L. How much uncertainty can be dealt with by feedback [J]. *IEEE Transactions Automatic Control*, 2000, **45**: 2203–2217.

[7] Goodwin G C, Salgado M E, Yuz J I. Performance limitations for linear feedback systems in the presence of plant uncertainty [J]. *IEEE Transactions Automatic Control*, 2003, **48**: 1312–1319.

[8] Goodwin G C, Graebe S F, Salgado M E. *Control System Design* [M]. Upper Saddle River, NJ: Prentice-Hall, 2002.

[9] Freudenberg J, Middleton R, Stefanopoulou A. A survey of inherent design limitations [C]// *Proceedings of the American Control Conference*, Chicago. 2000: 2987–3001.

(Editor CHEN Ai-ping)