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PRODUCTION PROCESS

A comparison of some modified confidence intervals based on robust scale estimators for process capability index

Moustafa Omar Ahmed Abu-Shawiesh[1] · Shipra Banik[2] · B. M. Golam Kibria[3] · Hayriye Esra Akyüz[4]

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Abstract
This paper aims to compare the performances of modified confidence intervals based on robust scale estimators with classical confidence interval for process capability index (C_p) when the process has a non-normal distribution. The estimated coverage probability and the average width of the confidence intervals were obtained by a Monte-Carlo simulation under different scenarios. Simulation results showed that the modified confidence intervals performed well in terms of coverage probability and average width for all cases. Two real-life numerical examples from industry are analyzed to illustrate the performance and the implementation of the classical and modified confidence intervals for the process capability index (C_p) which also supported the results of the simulation study to some extent.

Keywords Confidence interval · Coverage probability · Average width · Process capability index · Quality engineering · Statistical process control

1 Introduction

Capability indices are widely used in practice to evaluate the performance of a process and the information about the process is used to improve the capability [20–22]. As capability indices can only be estimated in most of the cases, confidence intervals can be used to predict a range in which the capability index lies with high probability. Therefore, industry and science benefit from trustable confidence intervals for the capability indices. In recent years, process capability indices (PCIs) have drawn much attention in industries. Process capability (PC) analysis is a method of combining the statistical tools to find out how well a given process meets a set of specification limits [25]. The purpose of the process capability analysis is to find the effect of time on both the average and the spread of the process. Before evaluating the process capability, it has to be shown that the process is under the statistical process control [33]. Although there are several process capability indices such as, C_p, C_{pk}, C_{pm} and C_{pmk}, the most commonly applied process capability index is C_p [21, 36]. In this paper, we focus only on the widely used process capability index, C_p, defined by Juran [20] and Kane [21], as follows:

$$C_p = \frac{USL - LSL}{6\sigma} \quad (1)$$

where LSL and USL are the lower and upper specification limits, respectively and σ is the process standard deviation. The numerator gives the size of the range over which the process measurements is allowed to vary. The denominator gives the size of the range over which the process actually varies [23]. It follows that evaluation of the process capability can be based on the following three criteria: (i) variability in process; (ii) degree of departure of the process mean from the target value, and (iii) location of the process mean in the

✉ Moustafa Omar Ahmed Abu-Shawiesh
mabushawiesh@hu.edu.jo

Shipra Banik
banik@iub.edu.bd

B. M. Golam Kibria
kibriag@fiu.edu

Hayriye Esra Akyüz
heakyuz@beu.edu.tr

1 Faculty of Science, Department of Mathematics, The Hashemite University, Al-Zarqa 13115, Jordan

2 Department of Physical Sciences, School of Engineering and Computer Science, Bashundhara, Independent University, Dhaka 1229, Bangladesh

3 Department of Mathematics and Statistics, Florida International University, Miami, FL, USA

4 Faculty of Science and Arts, Department of Statistics, Bitlis Eren University, Bitlis 13000, Turkey

interval (LSL, USL). The process capability index, C_p, takes into account criterion (i) only, as it depends on spread for given specification limits. The larger a capability index value for a process is, the more capable the process. Most experts recommended different values for existing and new processes. The quality conditions and the corresponding C_p values are given by Chao and Lin [11] and are reported in Table 1.

The process capability index C_p greatly depends on the assumption that the underlying quality characteristic measurements are independent and normally distributed. However, the assumption of normal distribution is not always valid and thus the non-normal distribution process is also being practiced in an industrial environment. Therefore, the classic PCI may not always be available [24, 27, 30]. When the population standard deviation (σ) in Eq. (1) is unknown, it should be estimated from the sample values. Thus, the point estimate of C_p is given as follows:

$$\hat{C}_p = \frac{USL - LSL}{6S} \tag{2}$$

where S is the sample standard deviation. When normal distribution assumption is not assured or when there are extreme values in the data, robust estimators may be used [22]. An estimator, is said to be robust if it is fully efficient or nearly so for an assumed distribution but maintains high efficiency for plausible alternatives, for example normal distribution [3, 32]. Most known for robust measures are the pseudo-standard deviation (S_{ps}), the average absolute deviation from the sample median (AADM), the median absolute deviation from the sample median (MAD), the Gini's mean difference (GMD) and the Rousseeuw and Croux [29] estimators, S_n and Q_n. There are studies on the estimation of σ that show better performance of robust estimators in the literature [1, 3, 4, 26]. For estimating the population mean with a confidence interval, the coverage probability (CP) is closer to the nominal confidence level when the data are normally distributed but far from nominal level when data are from the skewed distribution [5–7, 37].

Since the normality assumption about the data is not guaranteed or may not be feasible with some real life data, in fact, it would be interesting to construct some confidence intervals (CIs) for C_p based on various robust scale estimators.

The rest of this paper is organized as follows: Sect. 2 present the classical and proposed modified CIs for the classical PCI. A comprehensive Monte-Carlo simulation study has been conducted to compare the performance of confidence intervals in Sect. 3. In Sect. 4, two real-life data sets from the industry sector are presented and analyzed to illustrate the performance and the implementation of the considered CIs for C_p. Finally, some concluding remarks are given in Sect. 5.

Table 1 Quality conditions and C_p values for centered process

Quality condition	C_p values
Supper excellent	$C_p \geq 2.00$
Excellent	$1.67 \leq C_p < 2.00$
Satisfactory	$1.33 \leq C_p < 1.67$
Capable	$1.00 \leq C_p < 1.33$
Inadequate	$0.67 \leq C_p < 1.00$
Poor	$C_p < 0.67$

2 The confidence interval (CI) and modified confidence intervals for C_p

Suppose that $X_1, X_2, \ldots, X_n$ are from a normal distribution with mean μ and standard deviation σ, then a $(1-\alpha)\,100\,\%$ confidence interval for the classical process capability index, C_p, is constructed by using a pivotal quantity: $Q = (n-1)\,S^2 / \sigma^2$ and is given as follows:

$$LCL = \frac{USL - LSL}{6S}\sqrt{\frac{\chi^2_{\left(\frac{\alpha}{2}, n-1\right)}}{(n-1)}} \text{ and}$$

$$UCL = \frac{USL - LSL}{6S}\sqrt{\frac{\chi^2_{\left(1-\frac{\alpha}{2}, n-1\right)}}{(n-1)}}$$

where LCL = lower confidence limit and UCL = upper confidence limit, $\chi^2_{\left(\frac{\alpha}{2}, n-1\right)}$ and $\chi^2_{\left(1-\frac{\alpha}{2}, n-1\right)}$ are the "α/2 and 1 − α/2" quantiles of the central Chi squared distribution with $n-1$ degrees of freedom respectively. The confidence interval for C_p shown above is to be used for normal distribution data. The underlying process distributions are non-normal in many industrial processes (e.g., Chen and Pearn [13], Bittanti and Moiraghi [9], Wu and Messimer [35], Chang et al. [10], Ding [15]). For a given nominal level of the confidence interval, a high coverage probability is desirable [7]. However, for a non-normal distribution, the coverage probability of the confidence interval is quite low (Balamurali and Kalyanasundaram [8]).

The main goal of this paper is to obtain some modified confidence intervals for C_p by comparing the performance of confidence intervals based on the robust scale estimators as an alternative to the sample standard deviation (S) instead of bootstrap method. In this study, we proposed six modified confidence intervals for estimating the classical process capability index, C_p, for non-normal distributions based on the robust methods. Since the process capability index, C_p greatly depends on the population standard deviation (σ), we want to estimate it by robust method so that the proposed confidence intervals can be used when data are not from normal distribution. Following Wooluru et al. [34], we will propose some confidence intervals for the process capability index, C_p, in this section follow.

2.1 CI based on S_{ps}

The S_{ps} is defined as $S_{ps} = \frac{IQR}{1.349}$, where IQR stands for interquartile range. Thus, the (1 − α) 100% confidence interval for the classical process capability index, C_p, based on S_{ps} is given as follows:

$$LCL = \frac{USL - LSL}{6S_{ps}}\sqrt{\frac{\chi^2_{\left(\frac{\alpha}{2},n-1\right)}}{(n-1)}} \text{ and}$$

$$UCL = \frac{USL - LSL}{6S_{ps}}\sqrt{\frac{\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}}{(n-1)}}$$

where $\chi^2_{\left(\frac{\alpha}{2},n-1\right)}$ and $\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}$ are α/2th and 1 − α/2th quintiles of the Chi squared distribution with $n-1$ degrees of freedom.

2.2 CI based on AADM

The average absolute deviation from the sample median (AADM) is a robust scale estimator that measures the deviation of the data from the sample median, MD, which is less influenced by outliers. It is defined as $AADM = \frac{\sqrt{\pi/2}}{n}\sum_{i=1}^{n}\left|X_i - MD\right|$. The median is best known for being insensitive to outliers and has a maximal 50% breakdown point [29]. For main properties of MD, see for example Abu-Shawiesh and Kibria [2]. As stated in Gastwirth [17], AADM is a consistent estimate of σ and is asymptotically normally distributed. The (1 − α) 100% confidence interval for the classical process capability index, C_p, based on AADM is given as follows:

$$LCL = \frac{USL - LSL}{6AADM}\sqrt{\frac{\chi^2_{\left(\frac{\alpha}{2},n-1\right)}}{(n-1)}} \text{ and}$$

$$UCL = \frac{USL - LSL}{6AADM}\sqrt{\frac{\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}}{(n-1)}}$$

where $\chi^2_{\left(\frac{\alpha}{2},n-1\right)}$ and $\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}$ are α/2th and 1 − α/2th quintiles of the Chi squared distribution with $n-1$ degrees of freedom.

2.3 CI based on MAD

The mean absolute deviation from the sample median (MAD) was first introduced by Hampel [19] and is widely used in various applications as an alternative to S. The MAD is defined as $MAD = 1.4826MD\left\{\left|X_i - MD\right|\right\},\ i = 1, 2, 3, \ldots, n$. The 1.4826 factor given in MAD adjusts the scale for maximum efficiency when the data comes from a normal distribution. The $(1-\alpha)\,100\,\%$ confidence interval for the classical process capability index, C_p, based on MAD is given as follows:

$$LCL = \frac{USL - LSL}{6MAD}\sqrt{\frac{\chi^2_{\left(\frac{\alpha}{2},n-1\right)}}{(n-1)}} \text{ and}$$

$$UCL = \frac{USL - LSL}{6MAD}\sqrt{\frac{\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}}{(n-1)}}$$

where $\chi^2_{\left(\frac{\alpha}{2},n-1\right)}$ and $\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}$ are α/2th and 1 − α/2th quintiles of the Chi squared distribution with $n-1$ degrees of freedom.

2.4 CI based on GMD

The Gini's mean difference (GMD) was developed by Gini [16] and defined as follows:

$$GMD = \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left|X_i - X_j\right|_{i<j}}{\binom{n}{2}} = \frac{2}{n(n-1)}\left[\sum_{i=1}^{n}\sum_{j=i+1}^{n}\left|X_i - X_j\right|\right]$$

Gini's mean difference may be more appropriate in case of a small departure from normality as it has an asymptotic relative efficiency 98% at the normal distribution [31]. It is more efficient than S if the normal distribution is contaminated by a small fraction [14, 18]. The $(1-\alpha)\,100\,\%$ confidence interval for the classical process capability index, C_p, based on GMD is given as follows:

$$LCL = \frac{USL - LSL}{6GMD}\sqrt{\frac{\chi^2_{\left(\frac{\alpha}{2},n-1\right)}}{(n-1)}} \text{ and}$$

$$UCL = \frac{USL - LSL}{6GMD}\sqrt{\frac{\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}}{(n-1)}}$$

where $\chi^2_{\left(\frac{\alpha}{2},n-1\right)}$ and $\chi^2_{\left(1-\frac{\alpha}{2},n-1\right)}$ are α/2th and 1 − α/2th quintiles of the Chi squared distribution with $n-1$ degrees of freedom.

2.5 CI based on S_n

The S_n estimator was proposed by Rousseeuw and Croux [29] and is defined as the median of the n medians of the absolute differences between values. The S_n estimator can be defined as follows:

$$S_n = 1.1926\ MD_i\left\{MD_j\left|X_i - X_j\right|\right\},$$
$$i = 1, 2, 3, \ldots, n,\ j = 1, 2, 3, \ldots, n$$

A $(1-\alpha)\,100\,\%$ confidence interval for the classical process capability index, C_p, based on S_n is given as follows:

$$LCL = \frac{USL - LSL}{6S_n}\sqrt{\frac{\chi^2_{\left(\frac{\alpha}{2}, n-1\right)}}{(n-1)}} \text{ and}$$

$$UCL = \frac{USL - LSL}{6S_n}\sqrt{\frac{\chi^2_{\left(1-\frac{\alpha}{2}, n-1\right)}}{(n-1)}}$$

where $\chi^2_{\left(\frac{\alpha}{2}, n-1\right)}$ and $\chi^2_{\left(1-\frac{\alpha}{2}, n-1\right)}$ are α/2th and 1 − α/2th quintiles of the Chi squared distribution with $n-1$ degrees of freedom.

2.6 CI based on Q_n

The Q_n estimator was proposed by Rousseeuw and Croux [29], which is also a powerful alternative to the MAD. It is defined as follows:

$$Q_n = 2.2219\left\{\left|X_i - X_j\right|;\ i < j\right\}_{(g)},$$
$$i = 1, 2, 3, \ldots, n;\ j = 1, 2, 3, \ldots, n$$

where $g = \binom{h}{2} = \frac{h(h-1)}{2}$, $h = \left[\frac{n}{2}\right] + 1$ and $\left[\frac{n}{2}\right]$ is the integer part of the fraction $\frac{n}{2}$. Here the symbol (.) represents the combination. The Q_n estimator is 2.2219 times the g-th order statistic of the $\binom{n}{2}$ distances between data points. A $(1-\alpha)\,100\,\%$ confidence interval for the classical process capability index, C_p, based on Q_n is given as follows:

$$LCL = \frac{USL - LSL}{6Q_n}\sqrt{\frac{\chi^2_{\left(\frac{\alpha}{2}, n-1\right)}}{(n-1)}} \text{ and}$$

$$UCL = \frac{USL - LSL}{6Q_n}\sqrt{\frac{\chi^2_{\left(1-\frac{\alpha}{2}, n-1\right)}}{(n-1)}}$$

where $\chi^2_{\left(\frac{\alpha}{2}, n-1\right)}$ and $\chi^2_{\left(1-\frac{\alpha}{2}, n-1\right)}$ are α/2th and 1 − α/2th quintiles of a Chi squared distribution with $n-1$ degrees of freedom.

3 Simulation study

In this section, a Monte-Carlo simulation study is conducted using the statistical software MATLAB to compare performances of the classical and proposed confidence intervals given in this paper. Since, our obejctive is to find some good intervals for non-normal cases, we consider normal, t, Chi square, Exponnetial, Gamma, Lognorma and Beta distributions, which cover a wide range of non-normal distributions. The simulation study was designed as follows:

(i) Normal distribution, N(50, 1)
(ii) Student-t distribution, t(5).
(iii) Chi Square distribution, $\chi^2_{(1)}$.
(iv) Exponential distribution, Exp (2).
(v) Gamma distribution, G(1, 6).
(vi) Lognormal distribution, LN(0, 1).
(vii) Beta distributions, Beta (3, 3) and Beta (1, 10).

The number of simulation replications was M = 50,000 for each case. Random samples were generated from each of the above mentioned distributions with $C_p = 1.0$ and samples sizes n = 10, 25, 50 and 100. Coverage probability (CP) and average width (AW) of the selected confidence intervals were measured for each case. The most common 95% confidence interval (α = 0.05) is used for measuring coverage probability and average width of the confidence intervals. When (α = 0.05), an interval has perfect performance in terms of coverage probability that will capture the true process capability index, C_p, between the lower and upper limits 95% of the times. The estimated coverage probability and estimated average width for this simulation study are respectively given as:

$$\widehat{CP} = \frac{\#\left(L \le C_p \le U\right)}{M} \text{ and } \widehat{AW} = \frac{\sum_{i=1}^{M}\left(U_i - L_i\right)}{M}$$

where $\#\left(L \le C_p \le U\right)$ denotes the number of simulation runs for which C_p lies within confidence interval. The simulated coverage probabilities and average widths for each of the distributions described above are presented in Tables 2 and 3, respectively. For clear understanding, coverage probabilities and average widths for considered sample sizes are presented in Figs. 1a, 2b, respectively. In the Table 2, we have reported coverage probabilities for selected confidence intervals of C_p for the various distributions and also for all considered values of n. For clear understanding, in Fig. 1a, b, we have presented coverage probabilities and average widths values for sample size n = 10 to observe effects on all considered distributions. Our observations from Fig. 1a are follows:

(i) Coverage probability for the classical interval, the AADM interval, the S_n interval and the Q_n interval are close to the nominal level 0.95, while others are not when data are generated from the distributions for skewness 0.
(ii) Coverage probability for the classical interval, the AADM interval and the MAD interval are close to the nominal level 0.95 compared to other considered intervals when data are generated from the Chi square distribution with df 1 and skewness 2.8284.
(iii) Coverage probability only for the GMD interval is close to the nominal level 0.95 compared to other considered intervals when data are generated from exponential distribution for skewness 2.
(iv) Coverage probability only for the GMD interval is very close to the nominal level 0.95 compared to other considered intervals when data are generated from the gamma distribution for skewness 0.8165.
(v) Coverage probability for the classical interval, the AADM interval and the GMD interval is close to the nominal level 0.95 compared to other considered intervals when data are generated from the lognormal distribution for skewness 8.1074. It is very interesting to note that the GMD confidence interval performed best compared to other intervals.
(vi) Coverage probability for the classical interval, the AADM interval, the GMD interval, the Sn interval

Table 2 Estimated coverage probabilites of 95% CI for $C_p = 1.00$

Distribution	n	CP for C_p						
		Existing and proposed methods						
		CI	CI_{Sps}	CI_{AADM}	CI_{MAD}	CI_{GMD}	CI_{Sn}	CI_{Qn}
N(50,1)	10	0.9440	0.8444	0.9198	0.8100	0.9988	0.9080	0.9126
	25	0.9484	0.8460	0.9244	0.8192	0.9968	0.9168	0.9258
	50	0.9506	0.8550	0.9282	0.8266	0.9906	0.9174	0.9266
	100	0.9500	0.8526	0.9246	0.8342	0.9808	0.9048	0.9276
t(5)	10	0.9704	0.8450	0.9264	0.8160	1.0000	0.9146	0.9234
	25	0.9626	0.8592	0.9366	0.8216	0.9974	0.9160	0.9260
	50	0.9594	0.8600	0.9428	0.8336	0.9934	0.9190	0.9330
	100	0.9558	0.8604	0.9526	0.8400	0.9836	0.9234	0.9484
$\chi^2_{(1)}$	10	0.9066	0.7554	0.8538	0.6944	0.9946	0.7474	0.7444
	25	0.9088	0.7758	0.8656	0.7290	0.9872	0.8048	0.8066
	50	0.9188	0.8096	0.8828	0.7576	0.9796	0.8426	0.8596
	100	0.9234	0.8144	0.8896	0.7656	0.9686	0.8730	0.8796
Exp(2)	10	0.7846	0.2708	0.4002	0.2490	0.8510	0.3416	0.2322
	25	0.7850	0.5484	0.6390	0.3292	0.8510	0.3622	0.2458
	50	0.7860	0.5650	0.6492	0.3434	0.8608	0.3680	0.2566
	100	0.8014	0.6768	0.7352	0.5208	0.8686	0.6096	0.5934
Gamma(1,6)	10	0.9002	0.7244	0.8326	0.6464	0.9616	0.6762	0.6490
	25	0.9028	0.7564	0.8494	0.6904	0.9784	0.7644	0.7492
	50	0.9068	0.7896	0.8624	0.7188	0.9836	0.8096	0.8282
	100	0.9116	0.8162	0.8728	0.7528	0.9952	0.8598	0.8596
LN(0,1)	10	0.9364	0.8128	0.9100	0.4970	0.9618	0.4976	0.3574
	25	0.9742	0.8286	0.9482	0.5716	0.9868	0.6024	0.5258
	50	0.9970	0.8318	0.9846	0.6342	0.9982	0.6826	0.6548
	100	0.9994	0.8736	0.9966	0.6862	1.0000	0.7636	0.7866
Beta(3,3)	10	0.9696	0.8972	0.9500	0.8590	0.9874	0.9288	0.9384
	25	0.9740	0.9274	0.9626	0.8992	0.9960	0.9634	0.9624
	50	0.9764	0.9444	0.9734	0.9322	0.9994	0.9726	0.9776
	100	0.9808	0.9688	0.9848	0.9636	1.0000	0.9872	0.9898
Beta(1,10)	10	0.8760	0.6886	0.7358	0.4060	0.9404	0.3904	0.4932
	25	0.8774	0.7382	0.7876	0.5376	0.9480	0.5430	0.6134
	50	0.8810	0.7678	0.8160	0.5974	0.9614	0.6538	0.7352
	100	0.8928	0.7812	0.8482	0.6628	0.9694	0.7496	0.9440

Table 3 Estimated average widths of 95% CI for $C_p = 1.00$

Distribution	n	AW for C_p						
		Existing and proposed methods						
		CI	CI_{Sps}	CI_{AADM}	CI_{MAD}	CI_{GMD}	CI_{Sn}	CI_{Qn}
N(50,1)	10	0.9875	1.0914	1.0455	1.1701	0.8524	0.9887	0.7951
	25	0.5797	0.5963	0.5945	0.6155	0.5085	0.5747	0.7660
	50	0.4010	0.4075	0.4061	0.4125	0.3537	0.3983	0.5725
	100	0.2804	0.2829	0.2822	0.2844	0.2479	0.2792	0.2659
t(5)	10	0.9785	1.0831	1.0385	1.0385	1.1565	0.8448	0.9819
	25	0.5698	0.5940	0.5866	0.6142	0.5009	0.5717	0.5115
	50	0.3920	0.4051	0.3989	0.4094	0.3466	0.3931	0.3871
	100	0.2747	0.2801	0.2777	0.2819	0.2434	0.2761	0.2583
$\chi^2_{(1)}$	10	1.0264	1.1436	1.0951	1.2505	0.8926	1.0688	0.9371
	25	0.5881	0.6198	0.6126	0.6497	0.5234	0.6132	0.6534
	50	0.4037	0.4230	0.4172	0.4355	0.3626	0.4236	0.4263
	100	0.2816	0.2922	0.2897	0.2993	0.2540	0.2968	0.2910
Exp(2)	10	0.2860	0.3491	0.3266	0.4016	0.2818	0.4032	0.4235
	25	0.6242	0.7534	0.7036	0.8916	0.5942	0.8600	0.8481
	50	0.6203	0.7448	0.6984	0.8837	0.5898	0.8544	0.7469
	100	1.1563	1.3997	1.2836	1.7196	1.0421	1.5187	0.8966
Gamma(1,6)	10	0.2598	0.2892	0.2775	0.3186	0.2262	0.2727	0.2412
	25	0.1476	0.1572	0.1547	0.1665	0.1321	0.1572	0.1430
	50	0.1012	0.1071	0.1053	0.1110	0.0915	0.1080	0.0976
	100	0.0702	0.0738	0.0729	0.0761	0.0638	0.0756	0.0773
LN(0,1)	10	0.7462	1.0777	0.8688	1.3612	0.6915	1.1903	0.8865
	25	0.3667	0.5719	0.4568	0.7088	0.3733	0.6783	0.4326
	50	0.2262	0.3845	0.2985	0.4695	0.2450	0.4614	0.6291
	100	0.1481	0.2630	0.2041	0.3205	0.1679	0.3195	0.3755
Beta(3,3)	10	0.9687	1.0006	1.0073	1.0800	0.8310	0.9259	1.9185
	25	0.5734	0.5410	0.5709	0.5588	0.4974	0.5360	0.5332
	50	0.3980	0.3697	0.3903	0.3747	0.3464	0.3726	0.4034
	100	0.2795	0.2565	0.2720	0.2580	0.2437	0.2623	0.2862
Beta(1,10)	10	1.0517	1.1891	1.1423	1.4274	0.9349	1.2567	1.0153
	25	0.5963	0.6387	0.6377	0.7319	0.5471	0.7056	0.9441
	50	0.4061	0.4301	0.4310	0.4794	0.3767	0.4766	0.6018
	100	0.2823	0.2979	0.2990	0.3296	0.2635	0.3306	0.4265

and the Q_n interval are very close to the nominal level 0.95 compared to other considered intervals when data are generated from the Beta (3,3) distribution for skewness 0. When data are generated from the Beta (1, 10) distribution for skewness 1.5170, it is noted that only the GMD interval has coverage probability very close to the nominal value 0.95 compared to other intervals.

Figure 1b shows that AWs for various confidence intervals and distributions for small sample size n = 10. From Fig. 1b, we noted the following performances:

(i) The Q_n interval has the lowest average width compared to other intervals, followed by the S_n interval, the classical interval and so on when random samples were generated from the N(50, 1) distribution.

(ii) The Sn interval has the lowest average width compared to other intervals, followed by the Q_n interval, the classical interval and so on when random samples were generated from the t(5) distribution.

(iii) The GMD interval has the lowest average width compared to other intervals, followed by the Q_n interval, the classical interval and so on when random samples were generated from the $\chi^2_{(1)}$ distribution.

(iv) The GMD interval has the lowest average width compared to other intervals, followed by the classical interval, the AADM interval and so on when random samples were generated from the exponential distri-

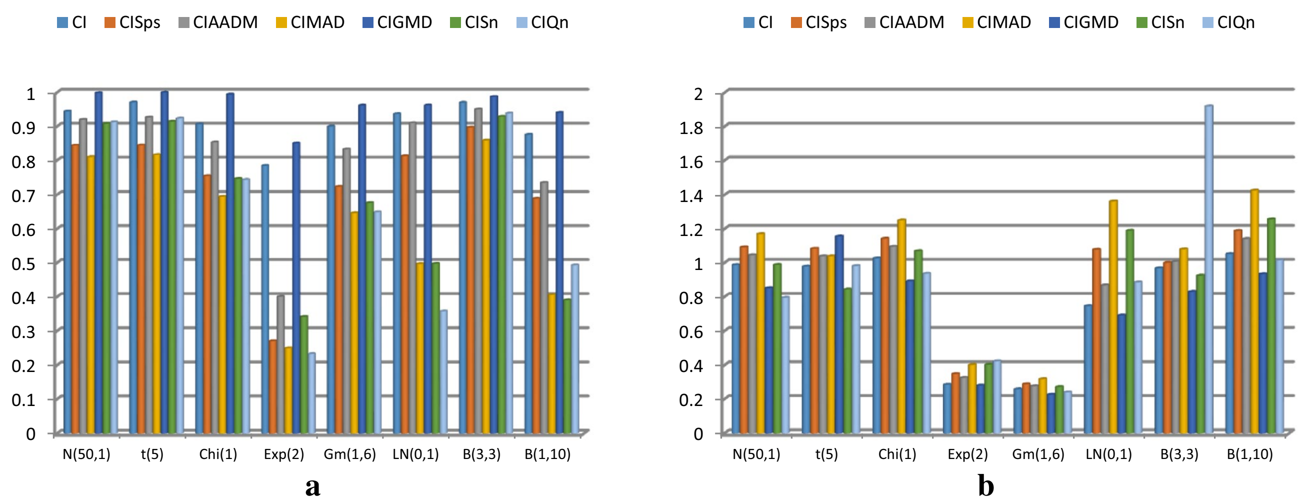

Fig. 1 **a** CP for n = 10; **b** AW for n = 10

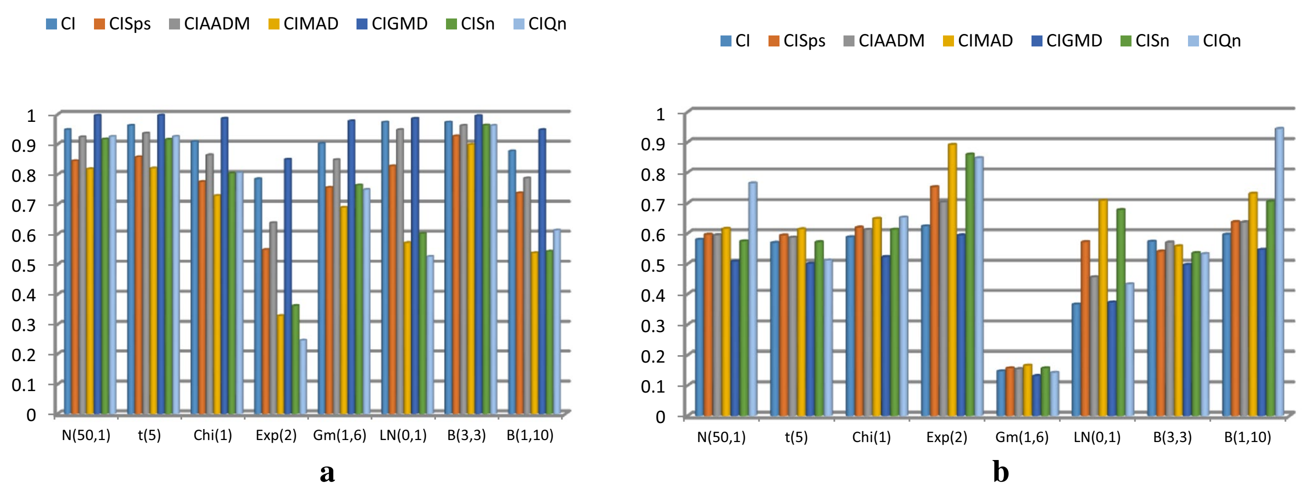

Fig. 2 **a** CP for n = 25; **b** AW for n = 25

bution. It is very interesting to note that compared to N(50, 1), t(5) and $\chi^2_{(1)}$ distributions, for all intervals have small AWs when random samples were taken from this non-normal distribution.

(v) The GMD interval has the lowest average width compared to other intervals, followed by the Q_n interval, the classical interval and so on when random samples were generated from the gamma distribution.

(vi) The GMD interval has the lowest average width compared to other intervals, followed by the classical interval, the AADM interval and so on when random samples were generated from the lognormal distribution.

(viii) The GMD interval has the lowest average width compared to other intervals, followed by the S_n interval, the classical interval and so on when random samples were generated from the Beta (3, 3) distribution. For Beta (1, 10) distribution, the GMD interval has the lowest average width compared to other intervals, followed by the Q_n interval, the classical interval and so on.

From above discussions, we may conclude that the classical interval, the AADM interval, the GMD interval, the S_n interval and the Q_n interval have performed better with respect to coverage probability for all considered distributions. In terms of measure AW, the above-mentioned confidence intervals observed smaller values compared to other considered intervals.

Figure 2a, b present CP and AW values for sample size n = 25 to see influences on all selected distributions. We observed the following:

(i) Coverage probability of all intervals are closer to the nominal level. Among them again the classical interval, the AADM interval, the S_n interval and the Q_n interval have the nominal level close to 0.95 compared to other considered intervals when data are generated from N(50, 1).
(ii) Coverage probability of classical interval, the AADM interval and the MAD interval are closer to the nominal level compared to other considered intervals when data are generated from the Chi square distribution.
(iii) Coverage probability only for the GMD interval is closer to the nominal level 0.95 compared to other considered intervals when data are generated from the exponential distribution.
(iv) We observed that with increasing size of the random samples there is a tendency of increasing coverage probability for all selected confidence intervals. Among them, only for the GMD interval is very close to the nominal level when data are generated from the gamma distribution.
(v) Coverage probability for the classical interval, the AADM interval and the GMD interval are more close to the nominal level when data are generated from the lognormal distribution for skewness 8.1074. It is very interesting to note that again the GMD interval performed best compared to other intervals.
(vi) Coverage probability for the classical interval, the AADM interval, the GMD interval, the Sn interval and the Q_n interval are more close to the nominal level compared to sample size n = 10, when data are generated from the Beta (3, 3) distribution and also the Beta (1, 10) distribution.

Figure 2b shows that average widths for various intervals and for various considered distributions for small sample size n = 25. We noted that with increasing size of the samples, AWs decrease and observed almost same performances, discussed as follows:

(i) The Q_n interval has the lowest average width compared to other intervals, followed by the S_n interval, the classical interval and so on when random samples were generated from the N(50, 1) distribution.
(ii) The S_n interval has the lowest average width compared to other intervals, followed by the Q_n interval, the classical interval and so on when random samples were generated from the t(5) distribution.
(iii) The GMD interval has the lowest average width compared to other intervals, followed by the Q_n interval, the classical interval and so on when random samples were generated from the $\chi^2_{(1)}$ distribution.
(iv) The GMD interval has the lowest average width compared to other intervals, followed by the classical interval, the AADM interval and so on when random samples were generated from the exponential distribution. It is very interesting to note that compared to N(50, 1), t(5) and $\chi^2_{(1)}$ distributions, for all intervals have small AWs when random samples were taken from this non-normal distribution.
(v) The GMD interval has the lowest average width compared to other intervals, followed by the Q_n interval, the classical interval and so on when random samples were generated from the gamma distribution. It is also very interesting to note that compared to all considered distributions, for all intervals observed smallest AWs when random samples were taken from this non-normal distribution.
(vi) The GMD interval has the lowest average width compared to other intervals, followed by the classical interval, the AADM interval and so on when random samples were generated from the lognormal distribution.
(vii) The GMD interval has the lowest average width compared to other intervals, followed by the S_n interval, the classical interval and so on when random samples were generated from the Beta (3, 3) distribution. For Beta (1, 10) distribution, the GMD interval has the lowest AW compared to other intervals, followed by the Q_n interval, the classical interval and so on.

From the above discussions, again we may conclude that the classical interval, the AADM interval, the GMD interval, the S_n interval and the Q_n interval have performed better with respect to coverage probability for all considered distributions. In terms of measure AW, we noted again the above-mentioned confidence intervals observed smaller values compared to n = 10 and other confidence intervals.

In Figs. 3a, 4b, we presented coverage probabilities and AWs for n = 50 and n = 100 under our simulation flowchart. We observed that with increasing samples sizes, coverage probabilities are closer to the nominal level compared to small sample sizes. The average widths observed smaller for large sample sizes compared to small sample sizes for all selected confidence intervals and also under all considered distributions.

4 Applications

In this section, two real-life data examples from the industry sector are presented to illustrate the implementation and performance of the classical and modified confidence intervals for the classical process capability index, C_p.

4.1 Example I

The first data set was obtained from Rezaie et al. [28]. Their case study involved a manufacturer and supplier of audio-speaker components. The data represents the weight (*in grams*) of the rubber edge, which is one of the key components that reflect the sound quality of drive unit, has been studied (Table 4).

The company decided that the process has upper and lower specifications at USL = 8.94 g and LSL = 8.46 g. If the weight of the rubber edge falls outside the specification limits, it is unacceptable. A summary of the location and scale statistics values are calculated and given in Table 5.

The normality has been examined by using the Kolmogorov–Smirnov (K–S) goodness-of-fit test. The histogram, density plot, and normal probability plot from the data was obtained using Minitab® Release 14 (Minitab Inc., 2012) and shown below in Fig. 5.

As it can be observed, the Kolmogorov–Smirnov (K–S) goodness-of-fit test for normality have a p value greater than $\alpha = 0.05$. In addition, the histogram and the normal probability plot show a normal distribution. Thus, it may be concluded that the sample data can be regarded as taken from a normal process. The estimated capability indices, the resulting 95% confidence interval and the corresponding

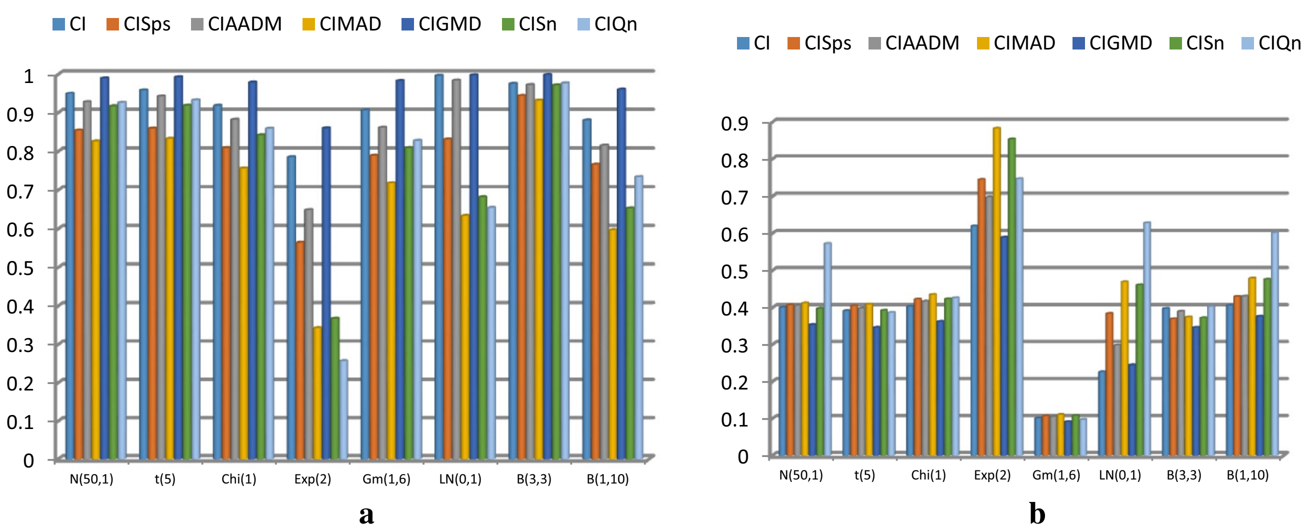

Fig. 3 **a** CP for n = 50; **b** AW for n = 50

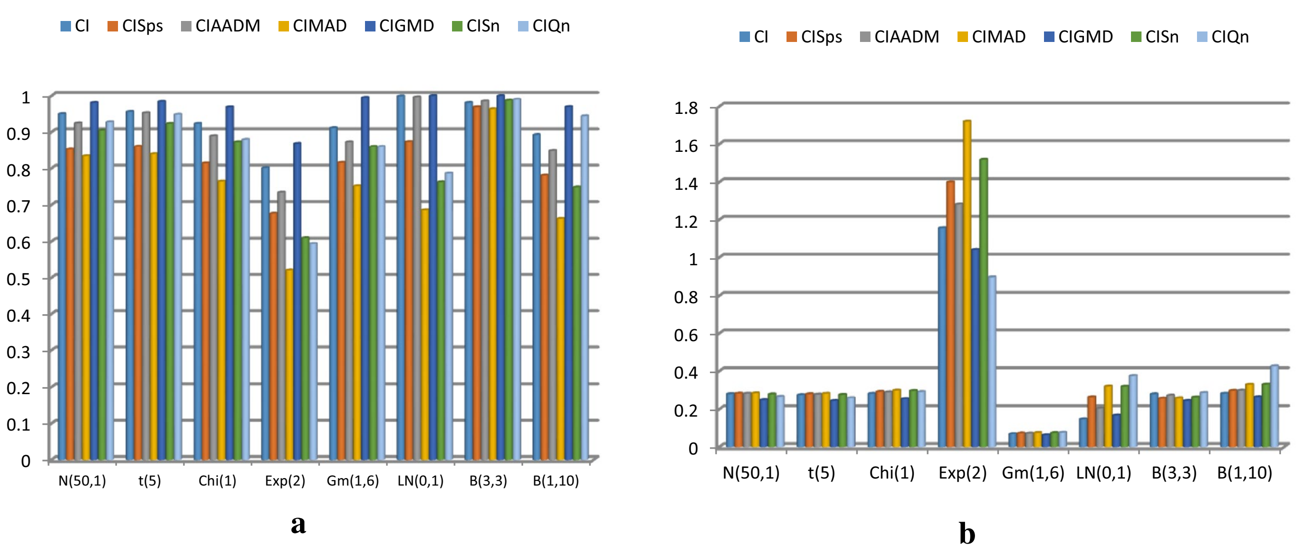

Fig. 4 **a** CP for n = 100; **b** AW for n = 100

Table 4 The weight (in gram) of the rubber edge data

8.63	8.65	8.57	8.57	8.54	8.69	8.63	8.64	8.59	8.61
8.60	8.66	8.65	8.50	8.61	8.61	8.63	8.67	8.54	8.62
8.65	8.58	8.65	8.67	8.67	8.65	8.69	8.66	8.62	8.63
8.59	8.65	8.64	8.64	8.52	8.69	8.66	8.66	8.61	8.55
8.57	8.64	8.63	8.57	8.61	8.59	8.56	8.71	8.53	8.51
8.72	8.58	8.64	8.69	8.64	8.75	8.59	8.61	8.58	8.65
8.73	8.70	8.65	8.56	8.66	8.65	8.66	8.68	8.62	8.54
8.67	8.62	8.54	8.62	8.66	8.56	8.60	8.62	8.61	8.66

Table 5 Summary statistics for the weight of the rubber edge data

Statistics	Abbreviation	Value
Sample mean	$\bar{X}$	8.6234
Sample median	MD	8.6300
Sample standard deviation	S	0.0522
Inter-quartile range	IQR	0.0700
Pseudo-standard deviation	S_{ps}	0.0519
Average absolute deviation from sample median	AAMD	0.0519
Median absolute deviation from sample median	MAD	0.0445
Gini's mean difference	GMD	0.0591
Rousseeuw and Croux S_n	S_n	0.0477
Rousseeuw and Croux Q_n	Q_n	0.0444

confidence interval width for all confidence intervals of C_p are reported in Table 6.

From Table 6, we observe that the CI_{GMD} interval has the smallest width followed by the classical confidence interval, CI_{Sps} and CI_{AAMD}. The CI_{Qn} interval has the highest width. Both CI_{MAD} and CI_{Sn} have the shorter widths compared to the corresponding interval CI_{Qn}. Thus the CI_{GMD} interval performs the best in the sense of having smaller width than the classical confidence interval and the other modified confidence intervals. In addition, according to the quality conditions for C_p value given in Table 1, we observe from Table 6 that when S, S_{ps}, AAMD and GMD are used to estimate σ, then the process is satisfactory capable of meeting the given specifications, but when MAD, S_n and Q_n are used to estimate σ, then the process is excellent capable of meeting the given specifications. The Q_n gives the best value followed by MAD and S_n. The results of this example supported the simulation study results to some extent. The results showed that the process is being capable and the normal distribution is adequate for modeling this data.

4.2 Example II

The second data set was obtained from Chen and Ding [12], which represents the inner diameter (*in mm*) for roller bearing and presented in Table 7.

According to the results, the upper specification limit is 60.004 mm and lower specification limit is 59.981 mm. A summary of location and scale statistics values are given in Table 8.

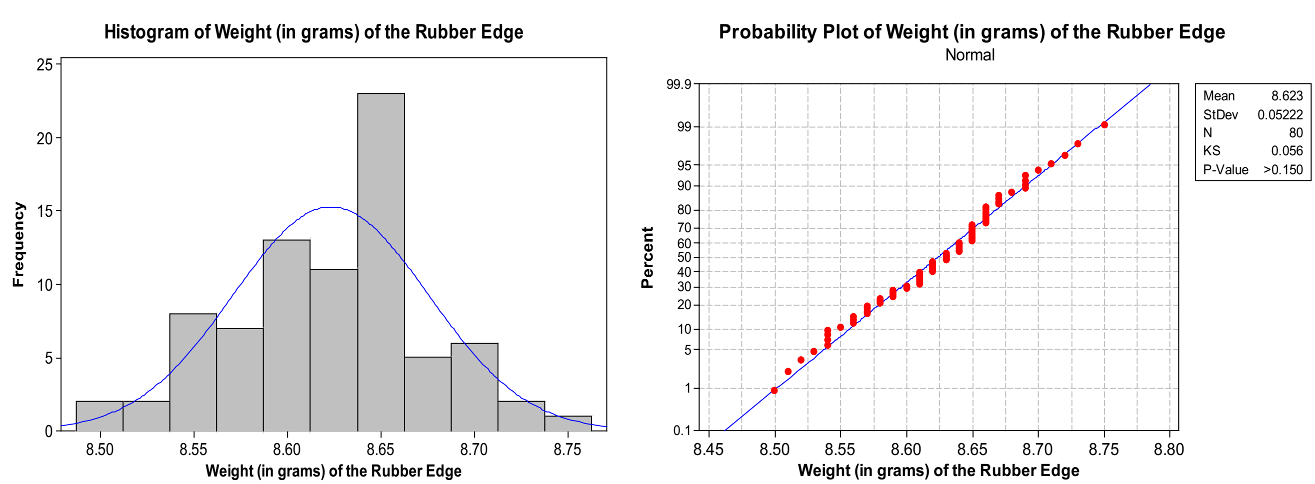

Fig. 5 Plots for the weight of the rubber edge data

Table 6 The 95% CIs for C_p for the weight rubber edge data

Method	$\hat{C}_p$	CI limits		
		Lower limit	Upper limit	Width
CI	1.533	1.294	1.771	0.477
CI_{Sps}	1.541	1.301	1.781	0.480
CI_{AAMD}	1.541	1.301	1.781	0.480
CI_{MAD}	1.798	1.518	2.077	0.559
CI_{GMD}	1.354	1.143	1.564	0.421
CI_{Sn}	1.677	1.416	1.938	0.522
CI_{Qn}	1.802	1.521	2.082	0.561

The normality has been examined by using the Kolmogorov–Smirnov (K–S) goodness-of-fit test. The histogram, density plot, and normal probability plot are shown in Fig. 6. The Kolmogorov–Smirnov (K–S) goodness-of-fit test has a p value less than $\alpha = 0.01$, which indicates that the data do not follow normality assumption. The histogram and the normal probability plot show a non-normal distribution. Thus, it may be concluded that the inner diameter of roller bearing data can be regarded as taken from a non-normal process.

The estimated capability indices, the resulting 95% confidence interval and the corresponding confidence interval width for all confidence intervals of C_p are presented in Table 9. From Table 9, we observe that the CI_{Sps} has the smallest width followed by the CI_{Qn}, CI_{MAD}, CI_{Sn}, CI_{AAMD}, CI_{GMD} and classical confidence interval. The classical confidence interval has the highest width. Also, widths for all modified CIs are very close to each other while the width for the classical confidence interval is very far from other widths and has a large value. Thus all the modified confidence intervals perform better than the classical confidence interval in the sense of having smaller width. Thus, according to the quality conditions for the C_p value given in Table 1, we observe from Table 9 that all values based on S and the other robust estimators are less than 0.67. However, the C_p value based on S is the largest one of them, then the process is poor capable of meeting the given specifications. The results are not surprising because the data is skewed. The results of this example supported the simulation study results to some extent. The results of data set under non-normality showed that the process is not capable and the normal distribution is not adequate for modeling this data.

Table 8 Summary statistics for inner diameter of roller bearing data

Statistics	Abbreviation	Value
Sample mean	$\bar{X}$	59.990
Sample median	MD	59.988
Sample standard deviation	S	0.0084
Inter-quartile range	IQR	0.0170
Pseudo-standard deviation	S_{ps}	0.0126
Average absolute deviation from sample median	AAMD	0.0093
Median absolute deviation from sample median	MAD	0.0096
Gini's mean difference	GMD	0.0092
Rousseeuw and Croux S_n	S_n	0.0095
Rousseeuw and Croux Q_n	Q_n	0.0097

5 Conclusions

Most of the standing confidence intervals for C_p (process capability index) are based on the confidence interval for the standard deviation, which is based on a normal distribution. However, the underlying distribution of the observed data may or may not be normally distributed. In the presented study, we proposed some interval estimators for the classical process capability index, C_p, based on some robust estimators for scale parameter. A Monte-Carlo simulation study has been conducted to compare the performance of our proposed confidence intervals with the classical existing confidence interval based on coverage probability and average width. Random samples were generated from a variety of symmetric and non-symmetric distributions. Two real-life data

Table 7 The inner diameter (in mm) for roller bearing data

59.984	59.981	59.981	60.003	59.982	60.005	60.004	59.983	59.981	59.980
60.000	59.998	59.982	59.983	59.981	59.982	59.999	60.001	59.982	59.988
59.995	59.998	59.982	59.983	59.981	59.994	60.002	59.988	59.980	59.982
59.982	59.983	59.981	59.986	59.987	60.001	59.982	60.003	60.001	59.984
59.985	59.979	59.987	59.990	59.998	59.984	59.989	59.999	59.985	60.003
60.004	60.001	60.000	59.982	59.981	59.984	59.998	59.983	59.999	59.987
59.991	59.992	59.992	59.983	59.981	59.996	59.997	60.000	60.000	59.991
60.002	60.001	59.990	59.987	59.982	60.006	59.981	59.982	59.984	59.985
60.003	60.004	59.992	59.991	59.986	59.992	59.991	59.981	59.998	59.985
60.001	59.980	59.993	59.984	59.981	59.984	59.988	59.999	60.000	60.001

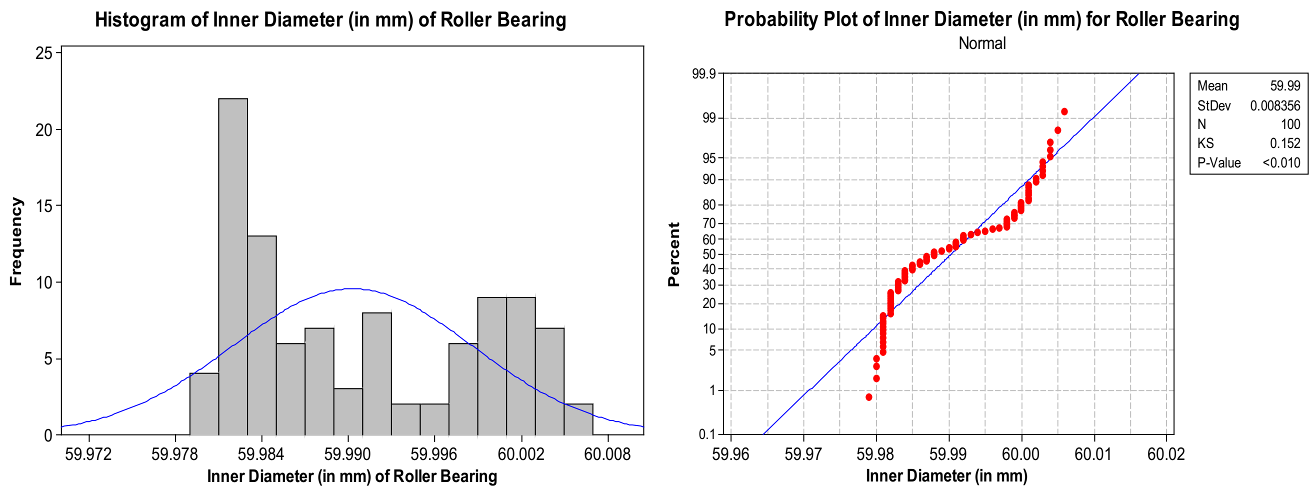

Fig. 6 Plots for the inner diameter of roller bearing data

Table 9 95% CIS for C_p for the inner diameter of roller bearing data

Method	$\hat{C}_p$	CI limits		
		Lower limit	Upper limit	Width
CI	0.456	0.393	0.520	0.127
CI_{Sps}	0.304	0.262	0.347	0.085
CI_{AAMD}	0.412	0.355	0.469	0.114
CI_{MAD}	0.399	0.344	0.455	0.111
CI_{GMD}	0.417	0.359	0.475	0.116
CI_{Sn}	0.404	0.347	0.460	0.113
CI_{Qn}	0.395	0.340	0.450	0.110

sets have been analyzed which supported the findings of our simulation study to some extent. From our study, it may be concluded that the classical confidence interval, the AADM confidence interval, the GMD confidence interval, the S_n confidence interval and the Q_n confidence interval have performed very well with respect to high coverage probability for all considered distributions. In terms of measure of average width, the above mentioned confidence intervals also performed very well compared to other confidence intervals. The findings of this paper are consistent with the results of Rousseeuw and Croux [29], Abu-Shawiesh and Kibria [2] and Piña-Monarrez et al. [27] among others.

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