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RESEARCH ARTICLE

Fault-tolerant hamiltonian cycles and paths embedding into locally exchanged twisted cubes

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Abstract The foundation of information society is computer interconnection network, and the key of information exchange is communication algorithm. Finding interconnection networks with simple routing algorithm and high fault-tolerant performance is the premise of realizing various communication algorithms and protocols. Nowadays, people can build complex interconnection networks by using very large scale integration (VLSI) technology. Locally exchanged twisted cubes, denoted by $(s + t + 1)$-dimensional $LeTQ_{s,t}$, which combines the merits of the exchanged hypercube and the locally twisted cube. It has been proved that the $LeTQ_{s,t}$ has many excellent properties for interconnection networks, such as fewer edges, lower overhead and smaller diameter. Embeddability is an important indicator to measure the performance of interconnection networks. We mainly study the fault tolerant Hamiltonian properties of a faulty locally exchanged twisted cube, $LeTQ_{s,t} - (f_v + f_e)$, with faulty vertices f_v and faulty edges f_e. Firstly, we prove that an $LeTQ_{s,t}$ can tolerate up to $s-1$ faulty vertices and edges when embedding a Hamiltonian cycle, for $s \geqslant 2$, $t \geqslant 3$, and $s \leqslant t$. Furthermore, we also prove another result that there is a Hamiltonian path between any two distinct fault-free vertices in a faulty $LeTQ_{s,t}$ with up to $(s - 2)$ faulty vertices and edges. That is, we show that $LeTQ_{s,t}$ is $(s-1)$-Hamiltonian and $(s-2)$-Hamiltonian-connected. The results are proved to be optimal in this paper with at most $(s - 1)$-fault-tolerant Hamiltonicity and $(s - 2)$ fault-tolerant Hamiltonian connectivity of $LeTQ_{s,t}$.

Keywords interconnection network, fault-tolerant, $LeTQ_{s,t}$, hamiltonian cycle, hamiltonian path

1 Introduction

Interconnection network is an important factor, which directly affects the performance of parallel computing system. It consists of a network of switching elements with a certain topology and control mode. It is used to realize the interconnection of multiple processors or multiple functional components within a computer system. With the gradual increase of network scale, its connection mode becomes more complex [1].

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Large-scale integrated circuit technology can be used to build complex Internet and predict the next generation of supercomputer systems. While adopting faster processors, it also achieves high speed and rapidity by increasing the number of processors [2, 3]. Therefore, how to design an excellent interconnection network to connect these processors is a technical difficulty in building supercomputer systems. Hypercube is one of the most commonly used interconnection structures. It has many good properties, such as regularity, recursive, low vertex degree and so on. Due to its powerful computing function and high efficiency, it is very important to run parallel algorithms on it. Almost all algorithms on linear arrays, cycles and trees can be effectively simulated on hypercubes with only constant factor delay.

Not all properties of hypercube are optimal, such as large diameter, which will cause large communication delay in communication. Therefore, many important variants are proposed based on hypercubes, such as crossed cubes [4], twisted cubes [5], locally twisted cubes [6, 7], alternating group graphs [8], exchanged hypercubes [9], exchanged crossed cubes [10] etc. Locally twisted cube, denoted by LTQ_n [11, 12], which is a regular graph with the same number of vertices as hypercubes, but its diameter is only half of that of hypercubes. Exchanged hypercube [13, 14], denoted by $EH_{s,t}$, with $s + t + 1 = n$. $EH_{s,t}$ is obtained by symmetrically deleting some edges of hypercubes. It has the same diameter as the hypercube, but the link overhead is only half that of the hypercube. Exchanged hypercubes have been used in the construction of P2P networks [15]. Chang et al. [16] proposed a novel interconnection network which is called locally exchanged twisted cube $LeTQ_{s,t}$. This new interconnection network combines the advantages of locally twisted cubes and exchanged hypercubes. For example, its diameter is the same as that of a locally twisted cube, which is much smaller than that of an exchanged hypercube. Moreover, its hardware overhead is the same as that of exchanged hypercubes, but much less than that of locally distorted cubes. In addition, it has good scalability, isomorphism and strong connectivity. Therefore, locally exchanged twisted cube becomes an effective logic structure for parallel computing processors.

Since interconnection network has a strong practical application, and the vertices and links of the network may fail, the fault tolerance of the network attracts a lot of researches [17–19]. Network fault tolerance means that when some components and connections fail at the same time, the remaining subnetworks still have some special functions [20–22]. Paths and cycles are two basic network topology for parallel computing. They can be used in the design of parallel algorithms, and they are suitable for designing simple and effective algorithms with low communication costs. The algebraic problems, graph problems and some parallel applications can be solved by using efficient algorithms on cycles and paths. Paths and cycles can also be used as control (or data) flow structures in parallel and distributed computing systems. What's more, if an interconnection network contains paths (or cycles) of different lengths, it can effectively simulate many algorithms designed on linear arrays (or cycles). The number of analog processors can also be adjusted in time to meet flexible requirements. In particular, the use of Hamilton paths in network multicast routing algorithms can effectively reduce or avoid deadlocks and congestion.

The ability of fault-tolerance is a crucial parameter in measuring performance of an interconnection network [23–26]. Thus, it is natural to consider how to tolerate as many faults as possible in the network. Numerous research has been studied on the Hamiltonicity in different special interconnection networks. In [27], Li et al. studied the embedding of many-to-many disjoint paths in hypercube with vertex failure. Zhou et al. [10] studied the embedding of the optimal path in the $ECQ_{s,t}$ of the switched intersection cube, and obtained the following conclusions: For any integer $s \geqslant 3$) and $t \geqslant 4$), there is a path of length l between any two different vertices in $ECQ_{s,t}$, where $\lceil \frac{s+1}{2} \rceil \lfloor \frac{t+1}{2} \rfloor + 4 \leqslant l \leqslant 2^{s+t+1} - 1$. Lu and Wang [28] studied the embedding of Hamiltonian paths in balanced hypercubes. In [29], Liu et al. proved that if the number of fault vertices or edges does not exceed $n-3$ in n-dimensional twisted hypercube H_n, there is a fault-free Hamiltonian path between any two fault-free vertices. Xu et al. [30] studied the vertex pancyclicity of locally twisted cubes under fault conditions. They proved that in LTQ_n with locally twisted cube, if f_v is the number of fault vertices in LTQ_n, and if $n > 3$ and $|F| \leqslant n-3$, LTQ_n contains fault-free cycles of arbitrary length l, of which $4 \leqslant l \leqslant 2^n - f_v$. Cheng and Hao [31] studied the cycle embedding of n-dimensional balanced hypercube BH_n in the case of edge faults. They proved that when the fault edge $|F_e| \leqslant 2n-3$, $BH_n - F_e$ contains a cycle of length l, of which $6 \leqslant L \leqslant 2^{2n}$. $EH_{1,t}$ and $EH_{2,2}$ are not even pancyclicity, but except $EH_{2,2}$, $EH_{s,t}(2 \leqslant s \leqslant t)$ are even-pancyclicity, and $EH_{s,t}(3 \leqslant s \leqslant t)$ are vertex even-pancyclicity. Cheng and Hsieh [32] studied the pancyclicity and even-pancyclicity of cartesian product graphs in the case of edge faults. Lv et al. [33] studied the embedding of Hamiltonian cycles and Hamiltonian paths in 3-ary n-cubes with structure faults $K_{1,3}$.

In this paper, we focus on the robustness capability of $LeTQ_{s,t}$ in Hamiltonian properties despite the faulty vertices or edges. The results are proved to be optimal in this paper with at most $s-1$-fault-tolerant Hamiltonicity and $(s-2)$ fault-tolerant Hamiltonian connectivity of $LeTQ_{s,t}$. So far, this is the first result reported about the fault-tolerant Hamiltonian properties of $LeTQ_{s,t}$. The original results of this paper are obtained as follows.

(i) We prove that an $LeTQ_{s,t}$ can tolerate up to $s-1$ faulty vertices and edges when embedding a Hamiltonian cycle, for $s \geqslant 2, t \geqslant 3$, and $s \leqslant t$.

(ii) We prove another result that there is a Hamiltonian path between any two distinct fault-free vertices in a faulty $LeTQ_{s,t}$ with up to $(s-2)$ faulty vertices and edges, for $s \geqslant 2, t \geqslant 3$, and $s \leqslant t$.

The rest of this paper is organized as follows: Section 2 presents some useful related definitions and lemmas. Section 3 discusses the fault-tolerant Hamiltonicity of $LeTQ_{s,t}$. Eventually, our work is summarized in Section 4.

2 Preliminaries

For a simple graph $G = (V, E)$, The path from vertex x to vertex y is a vertex sequence $x = v_0 v_1 \cdots v_n = y$, where $v_k \in V (0 \leqslant k \leqslant n)$, $\langle v_{i,j-1}, v_{i,j} \rangle \in E, (1 \leqslant i, j \leqslant n)$. We also denote path P by $\langle x_0, x_1, \ldots, x_i \rangle + P_1 + \langle x_j, x_{j+1}, \ldots, x_k \rangle$, where P_1 is the subpath $\langle x_i, x_{i+1}, \ldots, x_j \rangle$ and $0 \leqslant i < j \leqslant k$. If the vertices on the path do not repeat each other, such a path is called a simple path. If the first vertex on the path coincides with the last vertex, such a path is called a cycle. In a simple connected graph G, we call a cycle that has passed every vertex once and only once as a Hamiltonian cycle. If there are Hamiltonian cycles in graph G, we call this graph Hamiltonian or Hamiltonicity. Similarly, we call the Hamilton path that passes every vertex once and only once. For any two vertices u and v in graph G, if there are Hamiltonian paths connecting u and v, then G is called Hamiltonian connected graph or Hamiltonian-connectivity. For any faulty elements $F \subset \{V(G) \cup E(G)\}$ in graph G, if and only if $|F| < f$, $G \setminus F$ is Hamiltonian graph, then we call G is f-fault-tolerant Hamiltonian. For any faulty elements $F \subset \{V(G) \cup E(G)\}$ in graph G, if and only if $|F| < f$, $G \setminus F$ is Hamiltonian connected graph, then we call G is f-fault-tolerant Hamiltonian connectivity. Graph $G_1 = (V_1, E_1)$ is a subgraph $G_2 = (V_2, E_2)$ (written by $G_1 \subseteq G_2$) if $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$. G_1 and G_2 is isomorphic if and only if there is a bijection $\Theta : V_1 \to V_2$ and $\Phi : E_1 \to E_2$.

Definition 1 [12]. For $n \geqslant 2$, the n-dimensional locally twisted cube LTQ_n, which is defined recursively as follows:

(1) LTQ_2 is a graph with four vertices, which labeled as 00, 01, 10, and 11. Four edges (00, 01), (00, 10), (01, 11), and (10, 11), which formed by these vertices.

(2) LTQ_n is constructed by two disjoint copies of LTQ_{n-1}, for $n \geqslant 3$. Let LTQ^0_{n-1} to denote the subgraph of LTQ_n, where vertex prefix of LTQ_{n-1} is 0, and let LTQ^1_{n-1} to denote the subgraph of LTQ_n, where vertex prefix of LTQ_{n-1} is 1. Connect each vertex $u = 0u_2u_3 \ldots u_n$ of LTQ^0_{n-1} to the vertex $1(u_2 + u_n)u_3 \ldots u_n$ of LTQ^1_{n-1} with one edge, where '+' denotes the modulo 2 addition.

Figures 1(a) and 1(b) demonstrate LTQ_3, LTQ_4 and LTQ_5.

Definition 2 [16]. For $s, t \geqslant 1$, the locally exchanged twisted cube, denoted by $LeTQ_{s,t}$, where the vertex set $V = \{u = u_{t+s} \ldots u_{t+1} u_t \ldots u_1 u_0 | u_i \in \{0, 1\}$ for $0 \leqslant i \leqslant t+s\}$, and the edge set E consists of three disjoint sets E_1, E_2 and E_3:

$E_1 = \{(u, v) \in V \times V | u \oplus v = 2^0\}$, where $\oplus$ is the exclusive – OR operator,

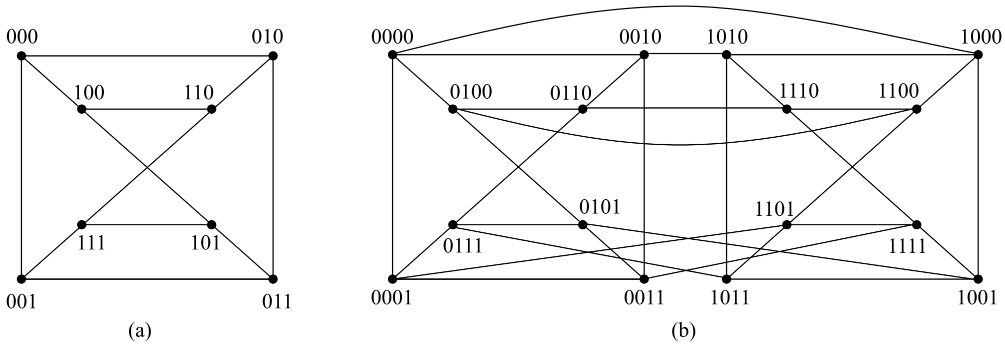


Fig. 1 Locally twisted cube (a) LTQ_3; (b) LTQ_4

$E_2 = \{(u,v) \in V \times V : u_0 = v_0 = 1, u_1 = v_1 = 0$ and $u \oplus v = 2^h$ for $h \in [3,t]\} \cup \{(u,v) \in V \times V | u_0 = v_0 = u_1 = v_1 = 1$ and $u \oplus v = 2^h + 2^{h-1}$ for $h \in [3,t]\} \cup \{(u,v) \in V \times V | u_0 = v_0 = 1$ and $u \oplus v \in \{2^1, 2^2\}\}$,

and

$E_3 = \{(u,v) \in V \times V | u_0 = v_0 = u_{t+1} = v_{t+1} = 0$ and $u \oplus v = 2^h$ for $h \in [t+3, t+s]\} \cup \{(u,v) \in V \times V | u_0 = v_0 = 0, u_{t+1} = v_{t+1} = 1$ and $u \oplus v = 2^h + 2^{h-1}$ for $h \in [t+3, t+s]\} \cup \{(u,v) \in V \times V | u_0 = v_0 = 0$ and $u \oplus v \in \{2^{t+1}, 2^{t+2}\}\}$.

By the definition of $LeTQ_{s,t}$, the number of vertex is 2^{s+t+1} and the number of edge is $(s+t+2)2^{s+t-1}$. As illustrated by Fig. 2, the 6-dimensional $LeTQ_{2,3}$, where E_1 edges are denoted by dashed lines, E_2 edges are denoted by bold lines, and E_3 are denoted by and solid lines.

$LeTQ_{s,t}$ is partitioned into two disjoint subgraphs $LeTQ^0_{s,t}$ and $LeTQ^1_{s,t}$, where $V(LeTQ^\lambda_{s,t}) = \{a_{s-1} \cdots a_0 \lambda b_{t-2} \cdots b_0 d | a_j, b_k, d \in \{0,1\}, j \in [0, s-1], k \in [0, t-2]\}$, for $\omega \in \{0,1\}$. It is clear that both $LeTQ^0_{s,t}$ and $LeTQ^1_{s,t}$ are isomorphic to $LeTQ_{s,t-1}$. The edges among $LeTQ^0_{s,t}$ and $LeTQ^1_{s,t}$, which are named crossing edges, be geared to E_3. $LeTQ_{s,t}$ can also be partitioned into 2^t disjoint subgraphs isomorphic to LTQ_s, which are denoted by $LTQ_s[L]$ and 2^s disjoint subcubes isomorphic to LTQ_t, which are denoted by $LTQ_t[R]$. An edge between LTQ_s and LTQ_t, belongs to E_1.

3 Hamiltonian cycle and path embedding

Networks with Hamiltonian paths (cycles) can communicate

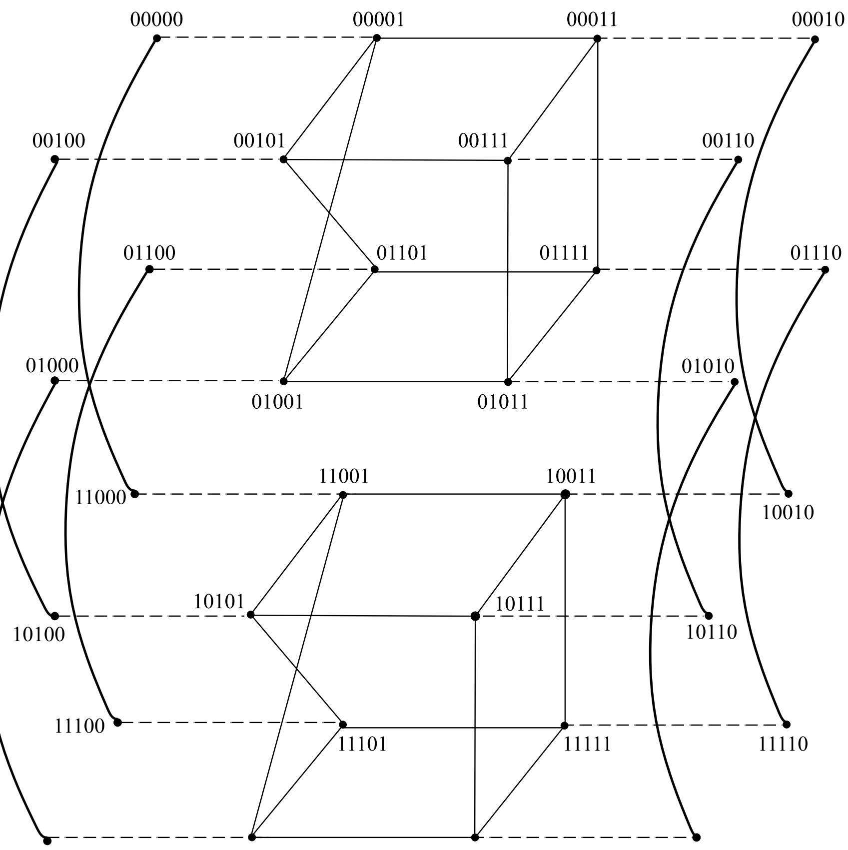


Fig. 2 Locally exchanged twisted cube $LeTQ_{1,3}$

linearly efficiently. For example, the deadlock free and additional resource free multicast routing algorithm based on Hamilton model is more efficient than the traditional multicast routing algorithm based on multicast tree. The problem of finding Hamiltonian paths or cycles is NP-Completeness [34].

In this section, we will study the fault-tolerant Hamiltonian of the locally exchanged twisted cube, $LeTQ_{s,t}$-$(f_v + f_e)$, with faulty vertices f_v and faulty edges f_e. Specifically, $LeTQ_{s,t}$ is $(s-1)$-Hamiltonian and $(s-2)$-Hamiltonian connected. To prove the main results, we first give the basic lemmas as follows.

Lemma 1 [12]. LTQ_n is Hamiltonian connected, for $n \geqslant 3$.

Lemma 2 [35]. LTQ_n is $(n-2)$-Hamiltonian and $(n-3)$-Hamiltonian connected, for $n \geqslant 3$.

Lemma 3 [16]. $LeTQ_{s,t}$ is partitioned into 2^t disjoint subcubes Q_s, which $Q_s \cong LTQ_s$ and 2^s disjoint subcubes Q_t, which $Q_t \cong LTQ_t$.

Lemma 4 [36]. For any integer $k \in \{2^{s+t+1}-2, 2^{s+t+1}-1\}$, there is an $\langle u, v\rangle$-path of length k between two arbitrary distinct vertices u and v in $LeTQ_{s,t}$, for $s \geqslant 2$ and $t \geqslant 3$.

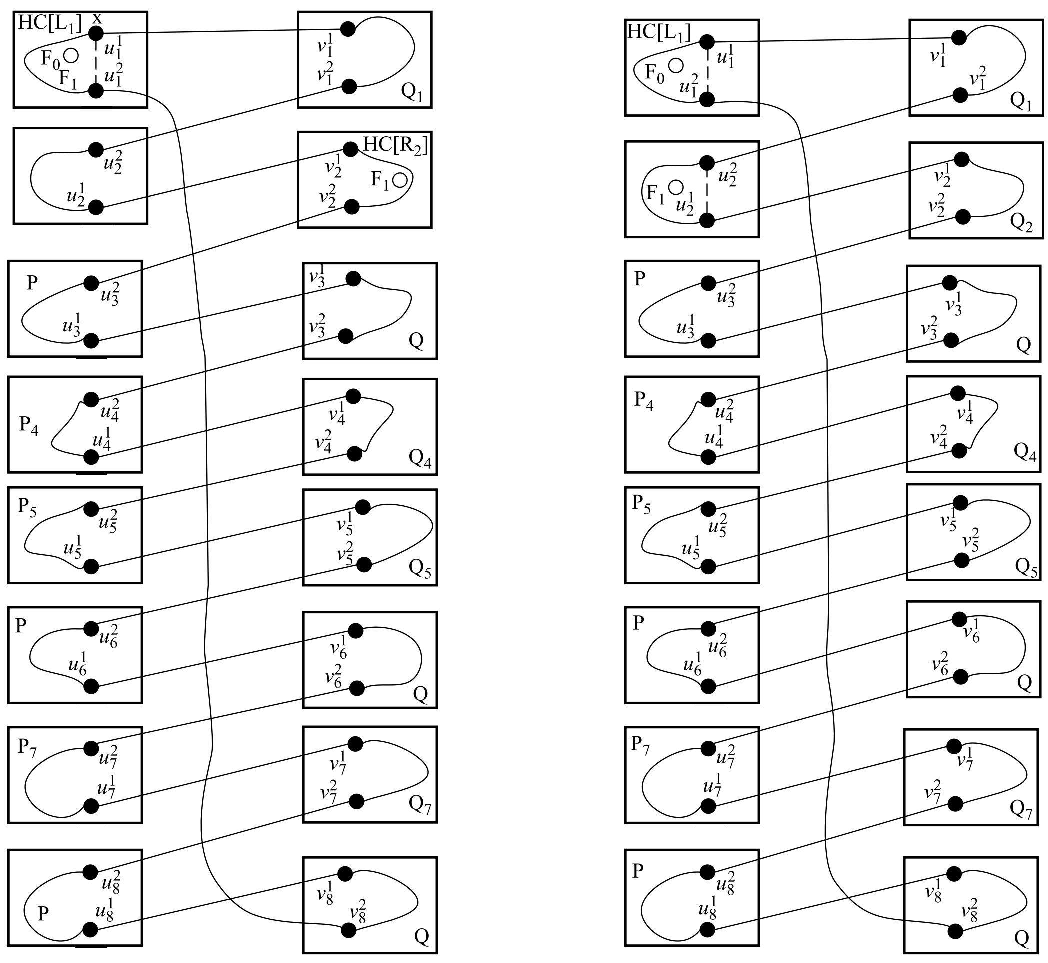
The following lemma can be obtained directly from Lemma 1.

Lemma 5 $LeTQ_{2,3}$ is 1-Hamiltonian and Hamiltonian connected.

Lemma 6 $LeTQ_{3,3}$ is 2-Hamiltonian.

Proof By Lemma 3, $LeTQ_{3,3}$ can be seen as the disjoint union of 8 copies of $LTQ_3[L]$ and 8 copies of $LTQ_3[R]$. Hence, we can denote $LTQ_3[L_1]$, $LTQ_3[L_2]$, $\ldots$, and $LTQ_3[L_8]$ as 8 copies of LTQ_3 that contain the edges E_2, and $LTQ_3[R_1]$, $LTQ_3[R_2]$, $\ldots$, and $LTQ_3[R_8]$ as 8 copies of LTQ_3 that contain the edges E_3. We denote u_1^1, u_1^2, $\ldots$, and u_1^8 as 8 vertices of $LTQ_3[L_1]$, $u_2^1, u_2^2, \ldots, u_2^8$ as 8 vertices of $LTQ_3[L_2]$, $\ldots$, and u_8^1, u_8^2, $\ldots$, and u_8^8 as 8 vertices of $LTQ_3[L_8]$. And we denote v_1^1, v_1^2, $\ldots$, and v_1^8 as 8 vertices of $LTQ_3[R_1]$, v_2^1, v_2^2, $\ldots$, and v_2^8 as 8 vertices of $LTQ_3[R_2]$, $\ldots$, and $v_8^1, v_8^2, \ldots$, and v_8^8 as 8 vertices of $LTQ_3[R_8]$. What's more, each vertex of $LTQ_3[L_i]$ has only one neighbour in $LTQ_s[R_j]$ and each vertex of $LTQ_s[R_j]$ has only one neighbour in $LTQ_3[L_i]$ $(1 \leqslant i \leqslant 8, 1 \leqslant j \leqslant 8)$. Let F_0 and F_1 be the two faults in $LeTQ_{3,3}$. By the location of F_0 and F_1.

Case 1 Both F_0 and F_1 are in the same copy of LTQ_3. Suppose that $F_0, F_1 \in V(LTQ_3[L_1])$. Imaging that F_0 is fault-free, there is a fault-free Hamiltonian cycle $HC[L_1]$ in $LTQ_3[L_1]$ by Lemma 2. In fact, F_0 is faulty. Thus, there is a Hamiltonian path $HP(u_1^1, u_1^2)$ in $HC[L_1]$. Suppose that v_1^1 is the neighbour of u_1^1 and v_8^2 is the neighbour of u_1^2. Select $v_2^1 \in V(LTQ_3[R_1]) - \{v_1^1\}$



(a) (b)

Fig. 3 Illustrations for the proof of Cases 1 (a) and 2.1 (b) and $u_2^1 \in V(LTQ_3[L_2]) - \{u_2^2\}$ such that u_2^2 and v_2^1 are the neighbours of v_1^2 and u_2^1, respectively. Using the similar method, select u_3^2, v_8^1, u_4^2, v_4^1, $\ldots$, v_8^2, v_8^1 such that v_2^2, u_3^1, v_3^2, u_4^1, $\ldots$, v_7^2, v_8^1 are the neighbours of v_3^2, v_8^1, u_4^2, v_4^1, $\ldots$, u_8^2, v_8^1, respectively. By Lemma 1, there is a Hamiltonian path in each copy LTQ_3 except for $LTQ_3[L_1]$. Thus, a required fault-free Hamiltonian cycle $HP(u_2^1, u_1^1) + (u_1^1, v_1^1) + Q_1 + (v_1^2, u_2^2) + P_2 + ,\ldots, +Q_8 + (v_8^2, u_1^2)$ can be constructed by linking the Hamiltonian paths with the edges E_1 in Fig. 3(a).

Case 2 F_0 and F_1 are in different copies of LTQ_3.

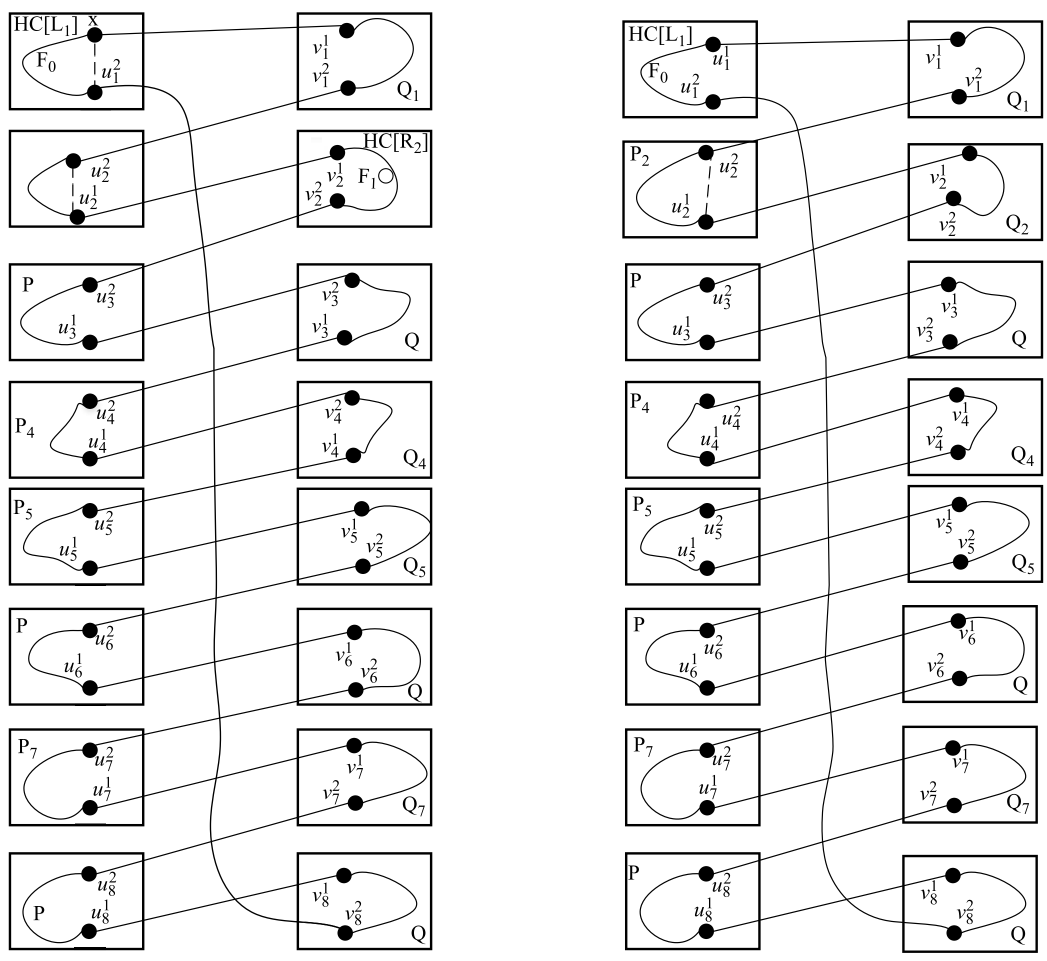
Case 2.1 F_0 and F_1 are in different copies of $LTQ_3[L]$. Suppose that $F_0 \in V(LTQ_3[L_1])$ and $F_1 \in V(LTQ_3[L_2])$. By Lemma 2, there exist fault-free Hamiltonian cycle $HC[L_1]$ in $LTQ_3[L_1]$ and $HC[L_2]$ in $LTQ_3[L_2]$, respectively. Select the edges (u_1^1, u_1^2) in $HC[L_1]$ and (u_2^1, u_2^2) in $HC[L_2]$ such that the neighbours of u_1^1 and u_2^2 (u_2^1 and u_2^1, respectively) are in the same copy of $LTQ_3[R]$. Suppose that the neighbours of u_1^1, u_2^2, u_2^1 and u_1^2 are v_1^1, v_1^2, v_2^1 and v_8^2. By Lemma 1, There exists a Hamiltonian path Q_1 between v_1^1 and v_1^2 in $LTQ_3[R_1]$. Using the similar method, it can be constructed Hamiltonian path Q_j in $LTQ_3[R_j]$ ($2 \leqslant j \leqslant 8$) and Hamiltonian path P_i in $LTQ_3[L_i]$ ($3 \leqslant i \leqslant 8$). Then, $HC[L_1] - (u_1^1, u_1^2) + (u_1^1, v_1^1) + Q_1 + (v_1^2, u_2^2) + HC[L_2] - (u_2^1, u_2^2) + (u_2^1, v_2^1) + Q_2 + (v_2^2, u_3^2) + P_3 +, \ldots, +Q_8 + (v_8^2, u_1^2)$ is a required fault-free Hamiltonian cycle in $LeTQ_{3,3}$ (refer to Fig. 3(b)).

Case 2.2 F_0 and F_1 are in different copies of $LTQ_3[R]$. The proof is the same as Case 2.1.

Case 2.3 F_0 is in $LTQ_3[L]$ and F_1 is in $LTQ_3[R]$. Suppose that $F_0 \in V(LTQ_3[L_1])$ and $F_1 \in V(LTQ_3[R_2])$. By Lemma 2, there is a fault-free Hamiltonian cycle $HC[L_1]$ in $LTQ_3[L_1]$ and $HC[R_2]$ in $LTQ_3[R_2]$, respectively. Select the edges (u_1^1, u_1^2) in $HC[L_1]$ and (v_2^1, v_2^2) in $HC[R_2]$ such that the neighbours of u_1^1 or u_1^2 are not in $LTQ_3[R_2]$ and the neighbours of v_2^1 or v_2^2 are not in $LTQ_3[L_1]$. Suppose that the neighbours of u_1^1, u_1^2, v_2^1 and v_2^2 are v_1^1 in $LTQ_3[R_1]$, v_8^2 in $LTQ_3[R_8]$, u_3^1 in $LTQ_3[L_3]$ and u_4^2 in $LTQ_3[L_4]$, respectively. The desired fault-free Hamiltonian cycle $HC[L_1]$ - $(u_1^1, u_1^2 + (u_1^1, v_1^1) + Q_1 + (v_1^2, u_2^2) + P_2 +, \ldots, +Q_8 + (v_8^2, u_1^2)$ can be obtained by Case 2.1 in Fig. 4(a).

Case 3 Both F_0 and F_1 are in E_1. Since the number of edges E_1 is $64 > 2$, we can select 16 fault-free edges E_1 between $LTQ_3[L_i]$ and $LTQ_3[R_j]$ ($1 \leqslant i \leqslant 8$, $1 \leqslant j \leqslant 8$). By Lemma 1, there is a Hamiltonian path in each copy LTQ_3. Thus, a desired fault-free Hamiltonian cycle $P_1 + (u_1^1, v_1^1) + Q_1 +, (v_1^2, u_2^2) + P_2 + (u_2^1, v_2^1) + Q_2 + \ldots, + Q_8 + (v_8^2, u_1^2)$ can be constructed by linking the Hamiltonian paths with the edges E_1.

Case 4 F_0 is in $LTQ_3[L]$ and F_1 is in E_1. Suppose that $F_0 \in V(LTQ_3[L_1])$. By Lemma 2, there is a fault-free Hamiltonian cycle $HC[L_1]$ in $LTQ_3[L_1]$. Select an edge (u_1^1, u_1^2) in



(a) (b)

Fig. 4 Illustrations for the proof of Cases 2.3 (a) and 4 (b)

$HC[L_1]$ such that the edges who are composed by u_1^1, u_1^2 and their neighbours in $LTQ_3[R_j]$ ($1 \leqslant j \leqslant 8$) are all fault-free. Suppose that the neighbours of u_1^1 and u_1^2, v_2^1 are v_1^1 and v_8^2, respectively. Using the similar method, it can be found other 14 fault-free edges E_1 between $LTQ_3[L_i]$ and $LTQ_3[R_j]$ ($1 \leqslant i \leqslant 8$, $1 \leqslant j \leqslant 8$). By Lemma 1, There exists a fault-free Hamiltonian path in every copies of LTQ_3 except for $LTQ_3[L_1]$. A desired fault-free Hamiltonian cycle $HC[L_1]$ - (u_1^1, u_1^2) + (u_1^1, v_1^1) + Q_1 + (v_1^2, u_2^2) + P_2 +, …, +Q_8 + (v_8^2, u_1^2) can be obtained by Case 2.1 in Fig. 4(b). □

Lemma 7 For $s \geqslant 4$, $LeTQ_{s,s}$ is $(s-1)$-Hamiltonian

Proof By Lemma 3, $LeTQ_{s,s}$ can see as the disjoint union of 2^s copies of $LTQ_s[L]$ and 2^s copies of $LTQ_s[R]$. Hence, we can denote $LTQ_s[L_1]$, $LTQ_s[L_2]$, …, and $LTQ_s[L_{2^s}]$ as 2^s copies of LTQ_s that contain the edges E_2, and $LTQ_s[R_1]$, $LTQ_s[R_2]$, …, and $LTQ_s[R_{2^s}]$ as 2^s copies of LTQ_s that contain the edges E_3. Let F be a faulty set of $LeTQ_{s,s}$ with $F_l = F \cap LTQ_s[L]$, $F_r = F \cap LTQ_s[R]$, and $F_1 = F \cap E_1$. Among them, let $f_l = |F_l|$, $f_r = |F_r|$, and $f_1 = |F_1|$. By the location of faults, we have the following cases.

Case 1 All faults are located in the the same copy of LTQ_s. Suppose that all of the faults are located in $LTQ_s[L_1]$, Then, $f_l = s-1$. Since LTQ_s is $(s-2)$-Hamiltonian by Lemma 2, there exist two vertices u_1^1 and u_1^2 such that there is a Hamiltonian path P_1 between u_1^1 and u_1^2 in $LTQ_s[L_1]$.

Suppose that v_1^1 is the neighbour of u_1^1 in $LTQ_s[R_1]$ and $v_{2^s}^2$ is the neighbour of u_1^2 in $LTQ_s[R_{2^s}]$. Select $v_1^2 \in V(LTQ_s[R_1]) - \{v_1^1\}$ such that u_2^2 is the neighbour of v_1^2 in $LTQ_3[L_2]$. Select $u_2^1 \in V(LTQ_s[L_2]) - u_2^2$, $v_2^2 \in V(LTQ_s[R_2]) - v_2^1$, u_3^1, v_3^2, …, $v_{2^{s-1}}^2$, $u_{2^s}^1 \in V(LTQ_s[L_{2^s}]) - \{u_{2^s}^2\}$ such that v_2^1, u_3^2, v_3^1, u_4^2, …, $u_{2^s}^2$, $v_{2^s}^1$ are the neighbours of $u_2^1 \in V(LTQ_s[L_2]) - u_2^2$, $v_2^2 \in V(LTQ_s[R_2]) - \{v_2^1\}$, u_3^1, v_3^2, …, $v_{2^{s-1}}^2$, $u_{2^s}^1 \in V(LTQ_s[L_{2^s}]) - u_{2^s}^2$, respectively. Since LTQ_s is Hamiltonian connected by Lemma 2, there is a Hamiltonian path in each copy fault-free LTQ_s. Thus, a required fault-free Hamiltonian cycle P_1 + (u_1^1, v_1^1) + Q_1 + (v_1^2, u_2^2) + P_2 + (u_2^1, v_2^1) + Q_2 +, …, +Q_{2^s} + $(v_{2^s}^2, u_1^2)$ can be constructed by linking the Hamiltonian paths with the edges E_1 (refer to Fig. 3(a)).

Case 2 Faults are dispersed in $LTQ_s[L]$, $LTQ_s[R]$, and E_1. Suppose that f_l is the greatest of f_l, f_r and f_1. Since at least two of f_l, f_r and f_1 are greater than zero, then $f_l \leqslant s-2$, $f_r \leqslant s-3$ and $f_r + f_1 \leqslant s-2 (s \geqslant 4)$. Without loss of generality, we assume that F_l is in $LTQ_s[L_1]$. Because LTQ_s is $(s-2)$-Hamiltonian, there is a Hamiltonian cycle $HC[L_1]$ with at least $2^s - (s-2)$ edges. Also because $2^s - (s-2) > 2(s-2)$, we can select a fault-free edge (u_1^1, u_1^2) in $HC[L_1]$ such that the edges E_1 who are composed by u_1^1, u_1^2 and their neighbours are all fault-free. Without loss of generality, we assume that v_1^1 is the neighbour of u_1^1 in $LTQ_s[R_1]$ and $v_{2^s}^2$ is the neighbour of u_1^2 in $LTQ_s[R_{2^s}]$. Since LTQ_s is $(s-3)$-Hamiltonian connected, there is a Hamiltonian path in each copy LTQ_s except for $LTQ_s[L_1]$. Then, a desired fault-free Hamiltonian cycle $HC[L_1]$ -(u_1^1, u_1^2) + (u_1^1, v_1^1) + Q_1 + (v_1^2, u_2^2) + P_2 +, …, +Q_{2^s} + $(v_{2^s}^2, u_1^2)$ can be constructed by linking the Hamiltonian paths and the path $HC[L_1] - (u_1^1, u_1^2)$ with the fault-free edges E_1 (refer to Fig. 3(b)).

Case 3 All of the faults are in the E_1. The discussion for the situation is the same as Case 3 of Lemma 6. □

Lemma 8 $LeTQ_{3,3}$ is 1-Hamiltonian connected.

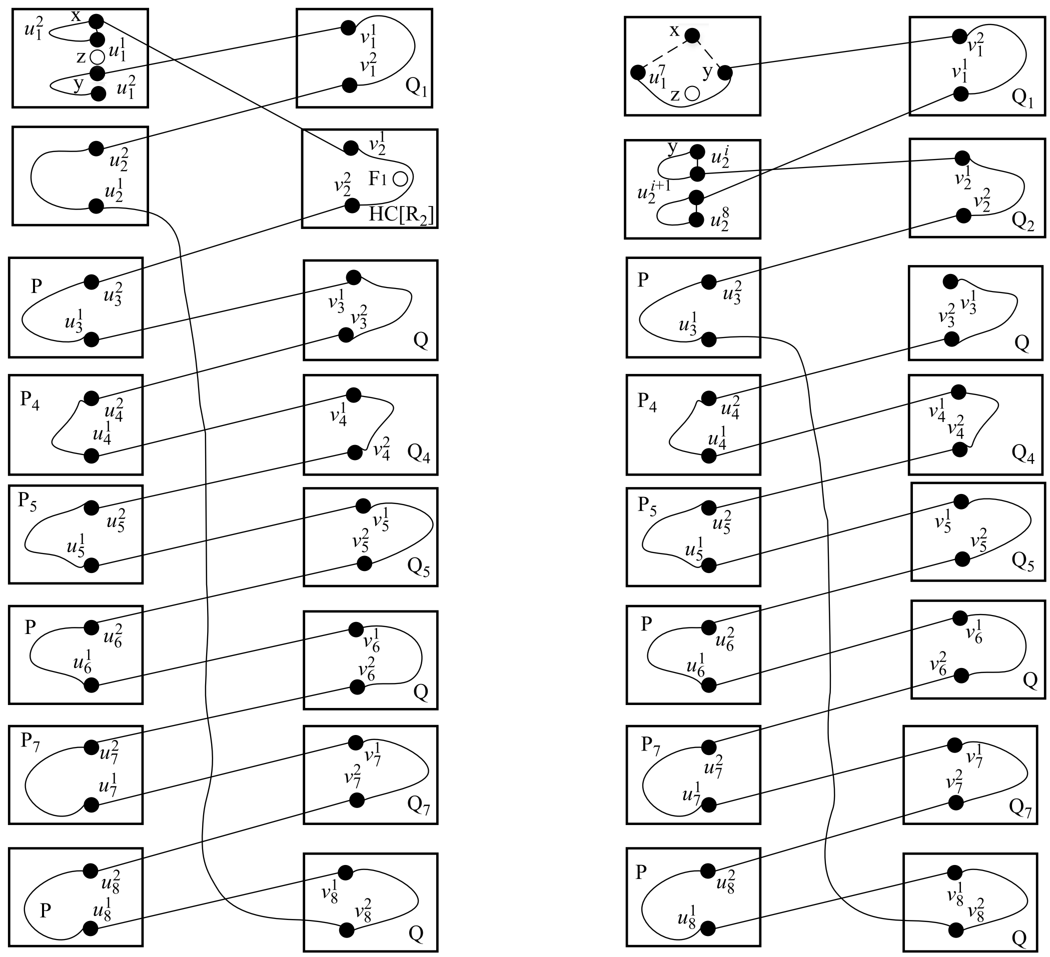
Proof By Lemma 3, $LeTQ_{3,3}$ can be divided into 8 copies of $LTQ_3[L]$ and 8 copies of $LTQ_3[R]$. Hence, we can denote $LTQ_3[L_1]$, $LTQ_3[L_2]$, …, and $LTQ_3[L_8]$ as 8 copies of LTQ_3 that contain the edges E_2, and $LTQ_3[R_1]$, $LTQ_3[R_2]$, …, and $LTQ_3[R_8]$ as 8 copies of LTQ_3 that contain the edges E_3. We denote u_1^1, u_1^2, …, and u_1^8 as 8 vertices of $LTQ_3[L_1]$, u_2^1, u_2^2, …, u_2^8 as 8 vertices of $LTQ_3[L_2]$, …, and u_8^1, u_8^2, …, and u_8^8 as 8 vertices of $LTQ_3[L_8]$. And we denote v_1^1, v_1^2, …, and v_1^8 as 8 vertices of $LTQ_3[R_1]$, v_2^1, v_2^2, …, and v_2^8 as 8 vertices of $LTQ_3[R_2]$, …, and v_8^1, v_8^2, …, and v_8^8 as 8 vertices of $LTQ_3[R_8]$. What's more, each vertex of $LTQ_3[L_i]$ has only one neighbour in $LTQ_s[R_j]$ and each vertex of $LTQ_s[R_j]$ has only one neighbour in $LTQ_3[L_i]$ ($1 \leqslant i \leqslant 8, 1 \leqslant j \leqslant 8$). According to the location of the faulty vertex z, we have the following three cases:

Case 1 $z \in LTQ_3[L]$. There are four subcases:

Case 1.1 x, y, and z are in the same copy of $LTQ_3[L]$. Suppose that $x = u_1^1$, $y = u_8^1$, and z are in $LTQ_3[L_1]$. Imaging that z is fault-free, there is a fault-free Hamiltonian path $HP[L_1]$ between x and y by Lemma 1. Find the neighbours u_1^i and $u_1^{i+2} (1 \leqslant i \leqslant 6)$ of z in $HP[L_1]$. Suppose that the neighbour of u_1^i is v_2^1 in $LTQ_3[R_2]$ and the neighbour of u_1^{i+2} is v_1^1 in $LTQ_3[R_1]$. Select $v_1^2 \in V(LTQ_3[R_1]) - \{v_1^1\}$, $v_2^2 \in V(LTQ_3[R_2]) - \{v_2^1\}$, $u_3^1 \in V(LTQ_3[L_3]) - u_3^2$, …, $v_8^2 \in V(LTQ_3[R_8]) - \{v_8^1\}$ such that u_2^2, u_3^2, v_3^1, … , u_2^1 are the neighbours of v_1^2, v_2^2, u_3^1, … , v_8^2, respectively. By Lemma 1, there is a Hamiltonian path in each copy LTQ_3. Thus, a required fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths with the edges E_1 in Fig. 5(a). That is, $\langle x, u_1^i\rangle + \langle u_1^i, v_2^1\rangle$ + Q_2 + $\langle v_2^2, u_3^2\rangle$ + P_3 +, …, +Q_8 + $\langle v_8^2, u_2^1\rangle$ + P_2 + $\langle u_2^2, v_1^2\rangle$ + Q_1 + $\langle v_1^1, u_1^{i+2}\rangle + \langle u_1^{i+2}, y\rangle$.

Case 1.2 x (or y), z are in the same copy of $LTQ_3[L]$. Suppose that $x = u_1^1$, $z \in V(LTQ_3[L_1])$ and $y = u_2^1 \in V(LTQ_3[L_2])$. By Lemma 2, there is a fault-free Hamiltonian cycle $HC[L_1] = \langle x, u_1^2, \ldots, u_1^7, x\rangle$ in $LTQ_3[L_1]$. Find $u_2^8 \in V(LTQ_3[L_2]) - \{u_2^1\}$, then, there is a fault-free Hamiltonian path $HP(u_2^8, y)$ between u_2^8 and y. Select an edge (u_2^2, u_2^{i+1}) $(2 \leqslant i \leqslant 6)$ in $HP(u_2^8, y)$ such that the neighbours of u_1^2 (u_1^7) and u_2^{i+1} are in the same copy of $LTQ_3[R]$. Suppose that the neighbours of u_1^2, u_2^{i+1}, u_2^i, and u_2^8 are v_1^2, v_1^1, v_2^1 and v_3^1, respectively. Select $v_2^2 \in V(LTQ_3[R_2]) - \{v_2^1\}$, $v_3^2 \in V(LTQ_3[R_3]) - \{v_3^1\}$, $u_8^1 \in V(LTQ_3[L_8]) - \{u_8^2\}$, …, $u_3^1 \in V(LTQ_3[L_3]) - \{u_3^2\}$ such that $u_3^2, u_4^2, v_8^1, \ldots, v_8^2$ are the neighbours of $v_2^2, v_3^2, u_8^1, \ldots, u_3^1$, respectively. By Lemma 1, there exists a Hamiltonian path in each copy $LTQ_3[L_j] (2 \leqslant j \leqslant 8)$ and $LTQ_3[R_k] (1 \leqslant k \leqslant 8)$. Thus, a desired fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths, $\langle x, u_1^2\rangle$-path, $\langle u_2^{i+1}, u_2^8\rangle$-path, and $\langle u_2^i, y\rangle$-path with the edges E_1 in Fig. 5(b). That is, $HC[L_1] - \langle x, u_2^{i+1}$ + $\langle u_1^2, v_1^2\rangle$ + Q_1 + $\langle v_1^1, u_2^{i+1}\rangle$ + $P(u_2^{i+1}, u_2^8)$ +, …, +Q_8 + $\langle v_8^2, u_3^1\rangle$ + P_3 + $\langle u_3^2, v_2^2\rangle$ + Q_2 + $P(v_2^1, y)$.

Case 1.3 x and y are in the same copy of $LTQ_3[L]$, x and z are in different copy of $LTQ_3[L]$. Suppose that $x = u_1^1$, $y = u_1^8 \in V(LTQ_3[L_1])$ and $z \in V(LTQ_3[L_2])$. By Lemma 1, there is a Hamiltonian path $HP[L_1]$ between x and y in $LTQ_3[L_1]$. By Lemma 2, there exists a fault-free Hamiltonian cycle $HC[L_2]$ in $LTQ_3[L_2]$. Select an edge (u_1^i, u_1^{i+1}) $(1 \leqslant i \leqslant 6)$ in $HP[L_1]$ and an edge (b, c) in $HC[L_2]$ such that the neighbours of u_1^i and (b, c) are in the same copy $LTQ_3[R]$. Suppose that the



(a) (b)

Fig. 5 Illustrations for the proof of Cases 1.1 (a) and 1.2 (b) of Lemma 8

neighbours of u_1^i, c, u_1^{i+1}, and b are v_2^1, v_2^2, v_3^1 and v_1^2, respectively. Select v_1^1, v_3^2, u_4^1, … , u_8^1, u_3^2 such that u_3^1, u_4^2, v_4^1, … , v_8^1, v_8^2 are the neighbours of v_1^1, v_3^2, u_4^1, …, u_8^1, u_3^2, respectively. By Lemma 1, there is a Hamiltonian path in each copy LTQ_3 except for $LTQ_3[L_2]$. Thus, a desired fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths, $\langle x, u_1^i\rangle$-path, and $\langle u_1^{i+1}, y\rangle$-path with the edges E_1 in Fig. 6(a). That is, $P(x, u_1^i) + \langle u_1^i, v_2^1\rangle + Q_2 + \langle v_2^2, c\rangle + HC[L_2]$ -$\langle b, c\rangle + \langle b, v_1^2\rangle + Q_1 + \langle v_1^1, u_3^1\rangle + P_3 + P(u_3^2, v_8^1) + Q_8 + \langle v_8^1, u_8^1\rangle$ $+ P_8 + \langle u_8^2, v_7^2\rangle + Q_7 +, \cdots, + P(v_3^1, y)$.

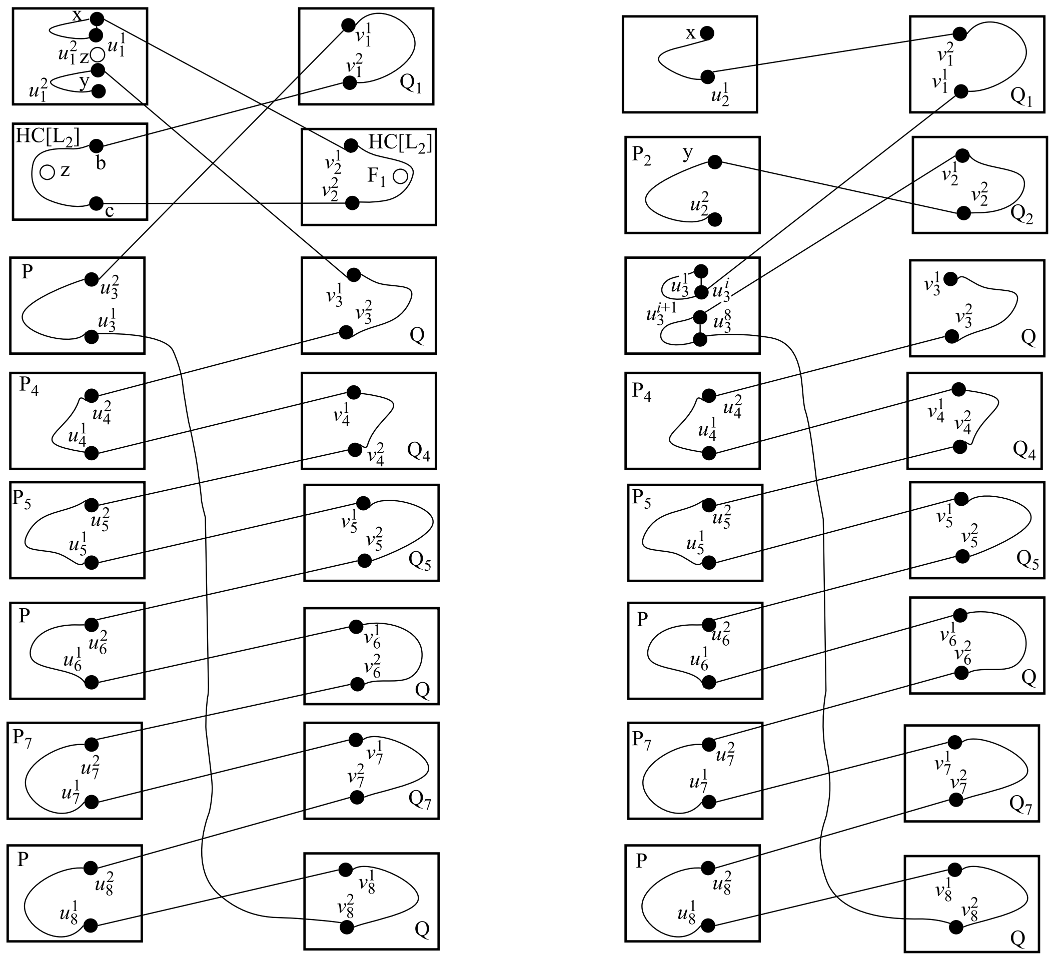
Case 1.4 x, y, and z are in different copy of $LTQ_3[L]$. Suppose that $x = u_1^1 \in V(LTQ_3[L_1])$, $y = u_2^1 \in V(LTQ_3[L_2])$, and $z \in V(LTQ_3[L_3])$. Imaging that z is fault-free. Since LTQ_3 is Hamiltonian connected, it can be selected a Hamiltonian path $\langle u_3^1, u_3^2, \ldots, u_3^8\rangle$ in $LTQ_3[L_3]$ such that $z = u_3^i$ ($3 \leqslant i \leqslant 6$).

Suppose that the neighbours of u_3^1, $u_3^i - 1$, u_3^{i+1} and u_3^8 are v_3^1, v_1^1, v_2^1, and v_8^2, respectively. Select $u_1^2 \in V(LTQ_3[L_1]) - \{x\}$ in $LTQ_3[L_1]$, $u_2^2 \in V(LTQ_3[L_2]) - \{y\}$ in $LTQ_3[L_2]$, u_4^2, v_4^1, …, v_8^1 such that v_1^2, v_2^2, v_3^2, u_3^1, …, u_8^1 are the neighbours of $u_1^2, u_2^2, u_4^2, v_3^1, \ldots, v_8^1$, respectively. (if the neighbours of x or y are in $LTQ_3[R_1]$ or $LTQ_3[R_8]$, we can choose other Hamiltonian path which meet the condition $z = u_3^i$ ($3 \leqslant i \leqslant 6$)).

By Lemma 1, there is a Hamiltonian path in each copy LTQ_3 except for $LTQ_3[L_3]$. Thus, a desired fault-free Hamiltonian path between u and v can be constructed by linking the Hamiltonian paths, $\langle u_3^1, u_3^{i-1}\rangle$-path, and $\langle u_3^{i+1}, u_3^8\rangle$-path with the edges E_1 in Fig. 6(b). That is, $P_1 + \langle u_1^2, v_1^2\rangle + Q_1 + \langle v_1^1, u_3^{i-1}\rangle + P(u_3^{i-1}, u_3^1)$ $+ \langle u_3^1, v_3^1\rangle + Q_3 +, \cdots, + Q_8 + \langle v_8^2, u_3^8\rangle + P(u_3^8, u_3^{i+1}) + \langle u_3^{i+1}, v_2^1\rangle +$ $Q_2 + P(v_2^2, y)$.

Case 1.5 x and z are in different copy of $LTQ_3[L]$, $y \in V(LTQ_3[R])$. Without loss of generality, suppose that $x = u_1^1 \in V(LTQ_3[L_1])$, $y = v_1^8 \in V(LTQ_3[R_1])$, and $z \in V(LTQ_3[L_2])$. By Lemma 2, there exists a fault-free Hamiltonian cycle $HC[L_2] = \langle u_2^1, u_2^2, \ldots, u_2^7, u_2^1\rangle$ in $LTQ_3[L_2]$. Select a vertex u_2^1 in $HC[L_2]$ and a vertex u_1^2 in $V(LTQ_3[L_2]) - \{x\}$ such that the neighbours of u_2^1 and u_1^2 are in the same copy of $LTQ_3[R_i]$ ($2 \leqslant i \leqslant 8$). Suppose that the neighbours of u_2^1, u_1^2, and u_2^2 are v_2^2, v_2^1, and v_3^1, respectively. Select $v_1^1, u_3^2, u_4^2, v_4^1, \ldots, u_8^1$ such that $u_3^1, v_8^2, v_3^2, u_4^1 \ldots, v_8^1$ are the neighbours of $v_1^1, u_3^2, u_4^2, v_4^1, \ldots, u_8^1$, respectively. By Lemma 1, there is a Hamiltonian path in each copy LTQ_3 except for $LTQ_3[L_2]$. Thus, a desired fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths and $u_2^1, u_2^{2^i}$-path with the edges E_1 in Fig. 7(a). That is, $P_1 + \langle u_1^2, v_2^2\rangle + Q2 + \langle v_2^1, u_2^1\rangle + HC[L_2]$-$\langle u_2^1, u_2^2\rangle + hu_2^2, v_3^1\rangle + \ldots,$ $+ Q_8 + \langle v_8^2, u_3^2\rangle + P_3 + P(u_3^1, y)$.

Case 2 $z \in LTQ_3[R]$. There are four subcases:



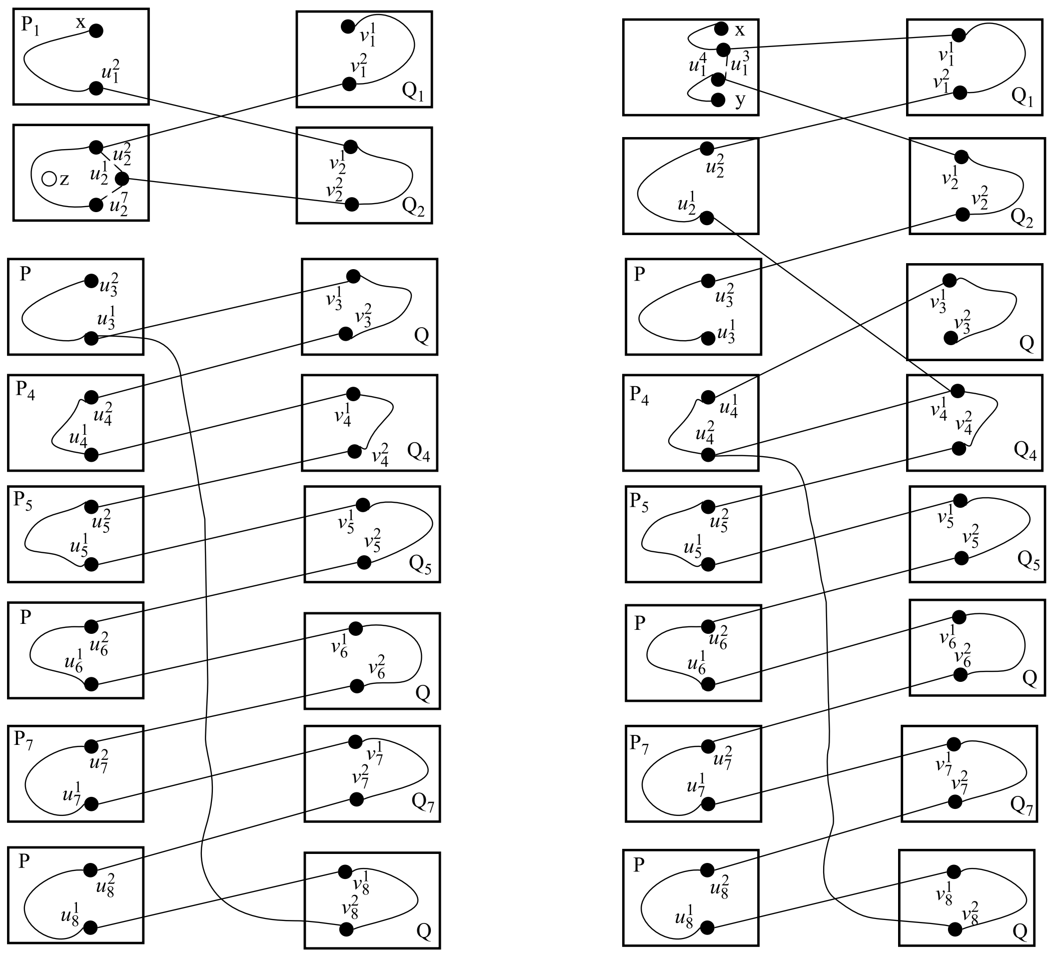
(a) (b)

Fig. 6 Illustrations for the proof of Cases 1.3 (a) and 1.4 (b) of Lemma 8

Case 2.1 x, y are in the same copy of $LTQ_3[L]$. Suppose that $x = u_1^1$, $y = u_1^8 \in V(LTQ_3[L_1])$, $z \in V(LTQ_3[R_3])$. By Lemma 2, there is a fault-free Hamiltonian cycle $HC[R_3]$ in $LTQ_3[R_3]$ and a fault-free Hamiltonian path $HP[L_1]$ in $LTQ_3[L_1]$. There exists a vertex v in $HC[R_3]$ such that the neighbour of v is not in $LTQ_3[L_1]$. Select an edge (v, v_3^1) (or (v, v_3^2)) in $HC[R_3]$ such that the neighbour of v_3^1 (v_3^2) is not in $LTQ_3[L_1]$. Select an edge (u_1^3, u_1^4) in $HP[L_1]$ such that the neighbours of u_1^3 and u_1^4 are not in $LTQ_3[R_3]$. Suppose that the neighbours of u_1^3, u_1^3, v, and v_3^1 are v_1^1, v_2^1, u_3^1, and u_4^1, respectively. Select $u_2^2, u_3^2, v_4^1, \ldots$, and v_8^2 such that v_1^2, v_2^2, $u_2^1, \ldots$, and u_4^2 are the neighbours of u_2^2, u_3^2, v_4^1, $\ldots$, and v_8^2, respectively. By Lemma 1, there is a Hamiltonian path in each copy LTQ_3 except for $LTQ_3[R_3]$. Thus, a desired fault-free Hamiltonian path between u and v can be constructed by linking the Hamiltonian paths, $\langle x, u_1^3\rangle$- path, $\langle v_3^1, v\rangle$-path, and $\langle u_1^4, y\rangle$-path with the edges E_1 in Fig. 7(b). That is, $P(x, u_1^3) + \langle u_1^3, v_1^1\rangle + Q_1 + \langle v_1^2, u_2^2\rangle + P_2 + \ldots, +Q_8 + \langle v_8^2, u_4^2\rangle + P_4 + \langle u_4^1, v_3^1\rangle + HC[R_3]$ - $\langle v, v_3^1\rangle$+ $P(v, y)$.

Case 2.2 x, y are in different copy of $LTQ_3[L]$. Suppose that $x = u_1^1 \in V(LTQ_3[L_1])$, $y = u_2^1 \in V(LTQ_3[L_2])$, $z \in V(LTQ_3[R_2])$, and u_1^2 is the neighbour of v_1^2. By Lemma 2, there is a fault-free Hamiltonian cycle $\langle v_2^1, v_2^2, \ldots, v_2^7, v_2^1\rangle$ in $LTQ_3[R_2]$. Since LTQ_3 is Hamiltonian connected, we can choose a Hamiltonian path $\langle y, u_2^2, \ldots, u_2^8\rangle$ in $LTQ_3[L_2]$ such that the neighbours of u_2^3 and u_2^4 are not in $LTQ_3[R_2]$. Suppose that the neighbours of u_2^3, u_2^4 and u_2^8 are v_3^2, v_4^2 and v_5^2, respectively. Find an edge (v_2^3, v_2^4) in $HC[R_2]$ such that the neighbours of v_2^3 and v_2^4 are not in $LTQ_3[L_1]$ and $LTQ_3[L_2]$. Suppose that the neighbours of v_2^3 and v_2^4 are u_3^2 and u_4^2, respectively. Select v_1^1, v_3^1, $v_4^1, \ldots, v_8^1$, and u_5^2 such that $u_3^1, u_5^1, u_4^1, \ldots, u_8^1$, and v_8^2 are the neighbours of $v_1^1, v_3^1, v_4^1, \ldots, v_8^1$, and u_5^2, respectively. By Lemma 1, there is a Hamiltonian path in each copy LTQ_3 except for $LTQ_3[R_2]$. Thus, a desired fault-free Hamiltonian path between u and v can be constructed by linking the Hamiltonian paths, $\langle u_2^4, u_2^8\rangle$-path, $\langle v_2^3, v_2^4\rangle$-path, and $P(u_2^3, y)$ with the edges E_1 in Fig. 8(a). That is, $P(x, u_1^2) + \langle u_1^2, v_1^2\rangle + Q_1 + \langle v_1^1, u_3^1\rangle + P_3 + P(u_3^2, v_2^4) + P(v_2^4, u_2^8) + \langle u_2^8, u_5^2\rangle +, \ldots, +Q_8 + P(v_8^2, y)$.

Case 2.3 x is in $LTQ_3[L]$, y and z are in different copy of $LTQ_3[R]$. Suppose that $x = u_1^1 \in V(LTQ_3[L_1])$, $y \in V(LTQ_3[R_1])$, and $z \in V(LTQ_3[R_2])$. By Lemma 2, there exists a fault-free Hamiltonian cycle $\langle v_2^1, v_2^2, \ldots, v_2^7, v_2^1\rangle$ in $LTQ_3[R_2]$. Find an edge (v_2^1, v_2^2) ((v_2^2, v_2^7)) in $HP[R_2]$ such that the neighbour of v_2^1 and $v_2^2(v_2^7)$ are not in $LTQ_3[L_1]$. Without loss of generality, suppose that the neighbour of v_2^1 and v_2^2 are u_2^1 and u_3^1, respectively. Select $v_1^1 \in \{V(LTQ_3[R_1]) - y\}$ and $u_1^2 \in \{V(LTQ_3[L_1]) - x\}$ such that the neighbours of v_1^1 are not in $LTQ_3[L_1]$, $LTQ_3[L_2]$, and $LTQ_3[L_3]$ and the neighbours of u_1^2 are not in $LTQ_3[R_1]$ and $LTQ_3[R_2]$. Suppose that the neigh-



(a) (b)

Fig. 7 Illustrations for the proof of Cases 1.5 (a) and 2.1 (b) of Lemma 8

bours of v_1^1 and u_1^2 are u_4^1 and v_3^1 in $LTQ_3[R_3]$, respectively. Select v_3^2, u_4^2, $\ldots$, v_8^2 such that u_3^2, u_2^2, $\ldots$, u_4^2 are the neighbours of $u_3^2, v_4^2, \ldots, v_8^2$, respectively. Since LTQ_3 is Hamiltonian connected, a desired fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths and $\langle v_2^1, v_2^2\rangle$-path with the edges E_1 in Fig. 8(b). That is, $P(x, u_1^2) + \langle u_1^2, v_3^1\rangle + Q_3 + \langle v_3^2, u_3^2\rangle + P_3 + P(u_3^1, u_2^1) + P_2 + \langle u_2^2, v_4^2\rangle +, \ldots, +Q_8 + P(v_8^2, y)$.

Case 2.4 x is in $LTQ_3[L]$, y is in $LTQ_3[R]$. And y, z are in the same copy of $LTQ_3[R]$. Suppose that $x = u_1^1 \in V(LTQ_3[L_1])$, $y = v_1^1$, $z \in V(LTQ_3[R_1])$. Select a vertex u_1^2 in $V(LTQ_3[L_1]) - \{x\}$ such that the neighbours of u_1^2 are not in $LTQ_3[R_1]$. By Lemma 2, there exists a fault-free Hamiltonian cycle $\langle y, v_1^2, \ldots, v_1^7, y\rangle$ in $LTQ_3[R_1]$. Then, the neighbours of $v_1^2(v_1^7)$ are not in $LTQ_3[L_1]$. Suppose that the neighbours of v_1^2 and u_1^2 are u_2^2 and v_2^2, respectively. Select $u_2^1 v_2^1, \ldots, u_8^1$, and v_8^2 such that $v_8^1, u_3^1, \ldots, v_7^1$, and u_8^2 are the neighbours of $u_2^1 v_2^1, \ldots, u_8^1$, and v_8^2, respectively. Since LTQ_3 is Hamiltonian connected, a desired fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths and $\langle v_1^2, y\rangle$-path with the edges E_1 in Fig. 9(a). That is, $P(x, u_1^2) + \langle u_1^2, v_2^2\rangle + Q_2 +, \ldots, +Q_8 + P(v_8^1, y)$.

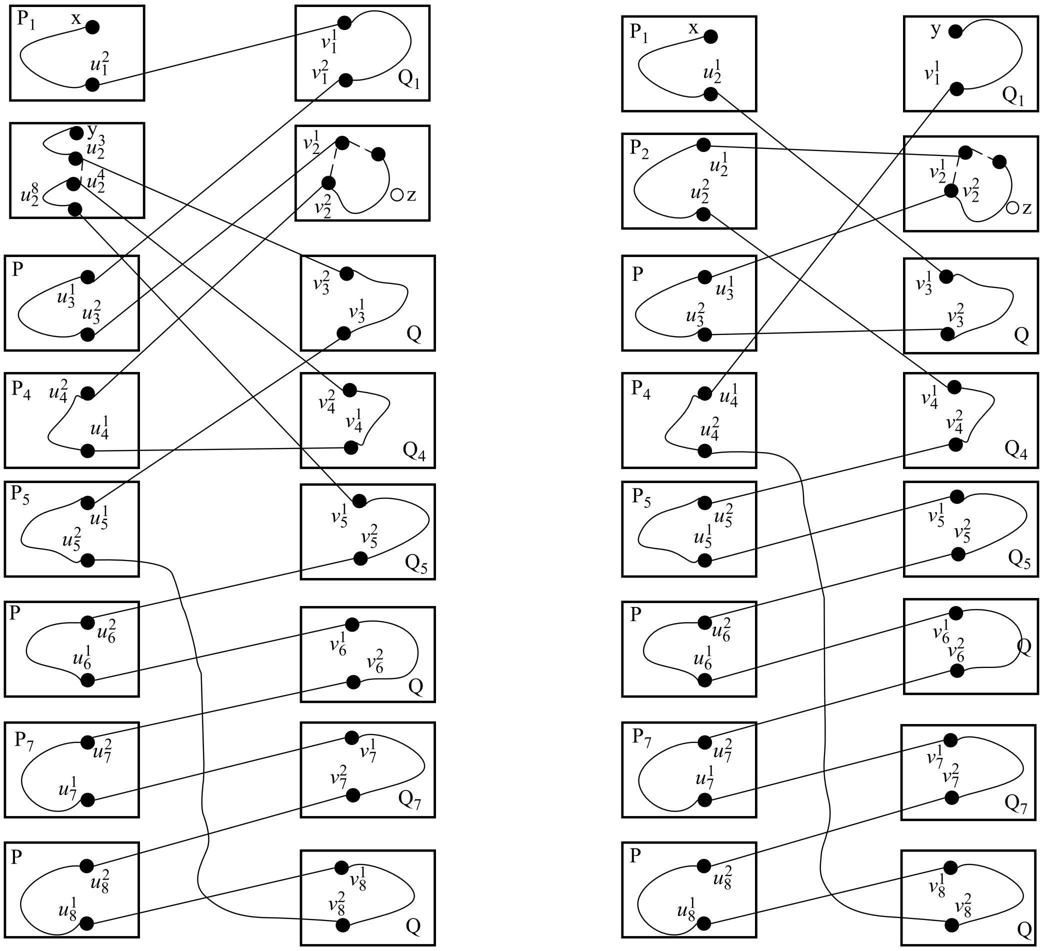
Case 2.5 x, y, and f are in the same copy of $LTQ_3[R]$. The case is the same as Case 1.1.

Case 3 $z \in E_1$. There exists 63 fault-free E_1 edges between $LTQ_3[L]$ and $LTQ_3[R]$. By the location of x and y, we have the following cases.

Case 3.1 x and y are in the same copy of LTQ_3. Suppose that $x = u_1^1$, $y = u_1^8 \in V(LTQ_3[L_1])$. There is a Hamiltonian path $HP[L_1]$ between x and y in $LTQ_3[L_1]$ by Lemma 1. Select an edge (u_1^i, u_1^{i+1}) $(1 \leqslant i \leqslant 6)$ in $HP[L_1]$ such that the edges who are composed by u_1^i and its neighbour which is in $LTQ_3[R]$ and u_1^{i+1} and its neighbour which is in $LTQ_3[R]$ are fault-free. Suppose that the neighbours of u_1^i and u_1^{i+1} are v_1^1 and v_2^1. Using the similar method to Case 1.3, it can be constructed a desired fault-free Hamiltonian path between x and y. That is, $P(x, u_1^i) + \langle u_1^i, v_1^1\rangle + Q_1 +, \ldots, +Q_8 + P(v_8^2, y)$ (refer to Fig. 9(b)).

Case 3.2 x and y are in different copy of LTQ_3. Without loss of generality, suppose that $x = u_1^1$, $y = v_1^8 \in V(LTQ3[R_8])$. We can select 16 fault-free edges E_1 between $LTQ_3[L_i]$ and $LTQ_3[R_j]$ $(1 \leqslant i \leqslant 8, 1 \leqslant j \leqslant 8)$. By Lemma 1, there exists a Hamiltonian path in each copy $LTQ3$. A desired fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths. The method of constructing is similar to Case 2.3. Thus, we omit it. □

Lemma 9 For $s \geqslant 4$, $LeTQ_{s,s}$ is $(s-2)$-Hamiltonian connected.



(a) (b)

Fig. 8 Illustrations for the proof of Cases 2.2 (a) and 2.3 (b) of Lemma 8

Proof By Lemma 3, $LeTQ_{s,s}$ can be seen as the disjoint union of 2^s copies of $LTQ_s[L]$ and 2^s copies of $LTQ_s[R]$. Hence, we can denote $LTQ_s[L_1]$, $LTQ_s[L_2]$, ..., and $LTQ_s[L_{2^s}]$ as 2^s copies of LTQ_s that contain the edges E_2, and $LTQ_s[1]$, $LTQ_s[2]$, ..., and $LTQ_s\{R_{2^s}\}]$ as 2^s copies of LTQ_s that contain the edges E_3. Let F be a faulty set of $LeTQ_{s,s}$ with $F_l = F \cap LTQ_s[L]$, $F_r = F \cap LTQ_s[R]$, and $F_1 = F \cap E_1$. Among them, let $f_l = |F_l|$, $f_r = |F_r|$, and $f_1 = |E_1|$. By the location of faults, we have the following cases.

Case 1 All the faults are located in E_1 or the same copy of LTQ_s. The proof is the same as $LeTQ_{3,3}$ in Lemma 13.

Case 2 The faults are scattered in $LTQ_s[L]$, $LTQ_s[R]$, and E_1. Suppose that f_l is the greatest of f_l, f_r and f_l. Since at least two of f_l, f_r and f_l are greater than zero, then $f_l \leqslant s-3$, $f_r \leqslant s-3$ and $f_r + f_1 \leqslant s-3 (s \geqslant 4)$. By the location of x and y, we have the following cases.

Case 2.1 x and y are in the same copy of LTQ_s. Suppose that $x = u_1^1$ and $y = u_1^{2^s}$ are in $LTQ_s[L_1]$. By Lemma 2, there is a fault-free Hamiltonian path $HP(x, y)$ in $LTQ_s[L_1]$. Since $2^s - 1 - 2 - (s-3) > 2(s-3)$, there is an edge $(u_1^k, u_1^k + 1)$ $(1 \leqslant k \leqslant 2^s - 1)$ such that the neighbours of u_1^k and $u_1^k + 1$ are in $LTQ_s[R_i]$ and $LTQ_s[R_j]$ $(1 \leqslant i \leqslant 2^s, 1 \leqslant j \leqslant 2^s$, and $i, j)$, respectively. And both $LTQ_s[R_i]$ and $LTQ_s[R_j]$ are fault-free. Since LTQ_s is $(s-3)$- Hamiltonian connected, by the Case 1.3 of Lemma 13, a desired fault-free Hamiltonian path between x and y is constructed by linking the Hamiltonian paths, $\langle x, u_1^k\rangle$-path, and $\langle u_1^{k+1}, y\rangle$-path with the edges E_1 (refer to Fig. 6(a)).

Case 2.2 x and y are in different copy of LTQ_s. We have the following cases.

Case 2.2.1 x and y are in different copy of $LTQ_s[L]$. Suppose that $x = u_1^1 \in V(LTQ_s[L_1])$ and $y = u_2^1 \in V(LTQ_s[L_2])$. Since $2^s - 2 > s - 3$, we can select a fault-free $LTQ_s[L_s]$ $(s \neq 1, 2)$. Suppose that $LTQ_s[L_3]$ is fault-free. By Lemma 6, there is a Hamiltonian path $HP(u_3^1, u_3^{2^s})$ in $LTQ_s[L_3]$. Since $2^s - 3 > 2(s-3)$, there is an edge (u_3^k, u_3^{k+1}) $(2 \leqslant k \leqslant 2^s - 1)$ such that the neighbours of u_3^k and u_3^{k+2} are in $LTQ_s[R_i]$ and $LTQ_s[R_j]$ $(1 \leqslant i \leqslant 2^s, 1 \leqslant j \leqslant 2^s$, and $i, j)$, respectively. And both $LTQ_s[R_i]$ and $LTQ_s[R_j]$ are fault-free. Select a vertex u_1^2 in $V(LTQ_s[L_1]) - F_l - \{x\}$ such that the fault-free neighbours of u_1^2 and u_3^k are in the same copy of $LTQ_s[R_i]$. Select a vertex u_2^2 in $V(LTQ_s[L_2]) - F_l - \{y\}$ such that the fault-free neighbours of u_2^2 and u_3^{k+1} are in the same copy of $LTQ_s[R_j]$. Without loss of generality, suppose that the neighbours of $i = 1$ and $j = 2$. Since LTQ_s is $(s-3)$-Hamiltonian connected, by the construction method of Case 1.4 of Lemma 13, a desired fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths, $\langle u_3^1, u_3^k\rangle$-path, and $\langle u_3^{k+1}, u_3^{2^s}\rangle$-path with the edges E_1 (refer to Fig. 6(b)).

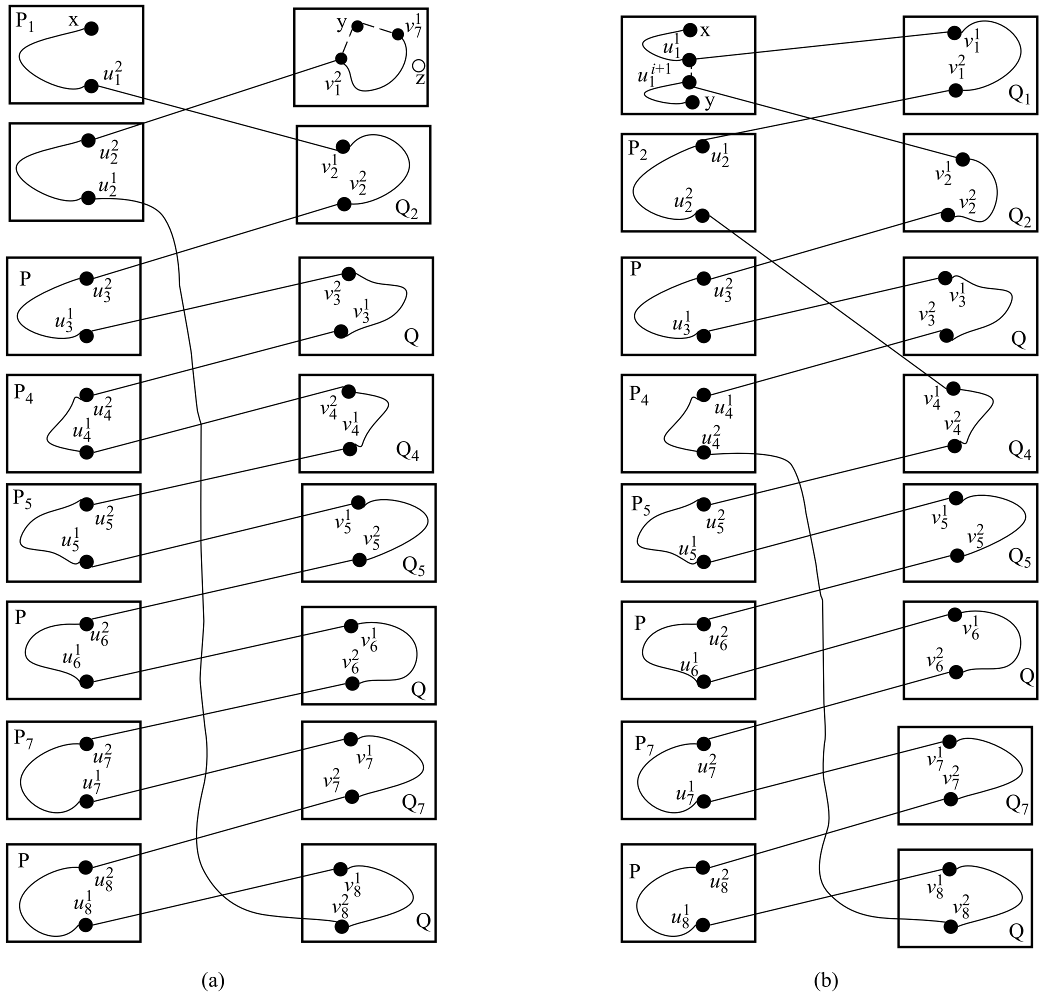


Fig. 9 Illustrations for the proof of Cases 2.4 (a) and 3.1 (b) of Lemma 8

Case 2.2.2 x and y are in different copy of $LTQ_s[R]$. The case is the same as Case 2.2.1.

Case 2.2.3 x is in $LTQ_s[L]$ and y is in $LTQ_s[R]$. Suppose that $x = u_1^1 \in V(LTQ_s[L_1])$ and $y = v_1^1 \in V(LTQ_s[R_1])$. Since $2^{s-2} - (s-3) > s-3$, it can be selected a fault-free vertex u_1^2 in $V(LTQ_s[L_1]) - \{x\}$ such that the neighbour of u_1^2 is in fault-free $LTQ_s[R_i]$ $(1 \leqslant i \leqslant 2^s)$. Since $2^{s-2} - (s-3) > s-3$, it can be selected a fault-free vertex v_1^2 in $V(LTQ_s[R_1]) - \{y\}$ such that the neighbours of v_1^2 is in fault-free $LTQ_s[L_j]$ $(1 \leqslant j \leqslant 2^s)$. Suppose that the neighbours of u_1^2 and v_1^2 are v_2^2 and u_2^2. Select $v_2^1, v_3^2, u_4^1, \ldots, v_{2^s}^1$ such that $u_3^1, u_3^2, v_3^1, \ldots, u_2^1$ are the neighbours of $v_2^1, v_3^2, u_4^1, \ldots, v_{2^s}^1$, respectively. By Lemma 2, there is a fault-free Hamiltonian paths in each copy LTQ_s. Thus, a required fault-free Hamiltonian path between x and y can be constructed by linking the Hamiltonian paths with the edges E_1 (refer to the construction method of Case 1.5 of Lemma 13 in Fig. 7(a)). □

Theorem 1 If $LeTQ_{s,k}$ is $(s-1)$-Hamiltonian and $(s-2)$-Hamiltonian connected for $s \geqslant 2$, $k \geqslant 3$ and $s \leqslant k$, then $LeTQ_{s,k+1}$ is $(s-1)$-Hamiltonian.

Proof Let E_c be the set of crossing edges and $E_c = \{(u_0, u_1) | (u_0, u_1) \in E_3,\ u_0 \in LeTQ^0_{s,k+1} \text{ and } u_1 \in LeTQ^1_{s,k+1}\}$. Let F be a faulty set of $LeTQ_{s,k+1}$ with $F_l = F \cap LeTQ^0_{s,k+1}$, $F_r = F \cap LeTQ^1_{s,k+1}$, and $F_1 = F \cap E_1$. And let $f_l = |F_l|$, $f_r = |F_r|$, and $f_c = |E_c|$. By the location of faults, we have the following cases.

Case 1 All the faults are located in the same copy of $LeTQ^i_{s,k+1} (i \in 0, 1)$. Suppose that all of the faults are in $LeTQ^0_{s,k+1}$ and $|F_l| = s-1$. Since $LeTQ_{s,k}$ is $(s-1)$-Hamiltonian and $LeTQ_{s,k+1}$ at least have $2^{s+k} - (s-1) \geqslant 2$ fault-free E_c edges, there exist a fault-free edge $(u_0, v_0) \in E_3$ in $LeTQ^0_{s,k+1}$ such that there is a Hamiltonian path $HP(x_0, y_0)$ between x_0 and y_0. Let x_1 and y_1 be the neighbours of x_0 and y_0 in $LeTQ^1_{s,k+1}$. Since $LeTQ^1_{s,k+1}$ is $(s-2)$-Hamiltonian connected, there is a Hamiltonian path $HP(u_1, v_1)$ between u_1 and v_1. Thus, $\langle u_0, HP(u_0, v_0), v_0, v_1, HP(u_1, v_1), u_1, u_0\rangle$ is a fault-free Hamiltonian cycle.

Case 2 All of the faults are located in E_c. Since $LeTQ_{s,k+1}$ has at least $2^{s+k} - (s-1) \geqslant 2$ fault-free crossing edges where $s \geqslant 2$ and $k \geqslant 3$. We always can choose two fault-free crossing edges (u_0, u_1) and (v_0, v_1). Because both $LeTQ^0_{s,k+1}$ and $LeTQ^1_{s,k+1}$ are Hamiltonian connected, there exists a Hamiltonian path $HP(u_0, v_0)$ in $LeTQ^0_{s,k+1}$ and Hamiltonian path $HP(u_1, v_1)$ in $LeTQ^1_{s,k+1}$. Thus, $\langle x_0, HP(x_0, y_0), y_0, y_1, HP(x_1, y_1), x_1, x_0\rangle$ is a fault-free Hamiltonian cycle in $LeTQ_{s,k+1}$.

Case 3 The faults are scattered in $LeTQ^0_{s,k+1}$, E_c and

Algorithm 1 Fault-tolerant hamiltonian cycle (a, b)

Input: $LeTQ_{s,t}$ and an edge $e = (a, b)$; Faulty elements F.
Output: A Hamiltonian cycle in $LeTQ_{s,t}$-F.

if $(s, t) \in \{(2, 2)\}$ **then**
return A Hamiltonian cycle in $LeTQ_{2,2}$-F;
end if
if ($s \geqslant 2$ and $t \geqslant 2$ **then**
return $LeTQHP_{s,t,a,b}$+(b, a);
end if
if ($s = 1$ and $t \geqslant 1$ **then**
return $HC(1, t, e)$;
end if
if $(1, t) \in (1, 1), (1, 2), (1, 3)$ **then**
return A Hamiltonian cycle in $LeTQ_{1,t}$-F;
else
  $C^* = HC(1, t-1, e)$;
  $C_0 = C^*i$;
  $C_1 = C^*i(1-i)$;
  Select $(x_0, y_0) \in E_3)$ on C_0 and $(x_1, y_1) \in E_3)$ on C_1, such that (x_0, y_0) and (x_1, y_1) are all belong to the crossing edges of E_3;
return $(C_0 - (x_0, y_0)) + (C_1 - (x_1, y_1))$;
end if

$LeTQ^1_{s,k+1}$. Then $3 \leqslant s \leqslant k$. Suppose that f_l is the greatest of f_l, f_r and f_c. Since at least two of f_l, f_r and f_c greater than zero, then $f_r \leqslant f_l \leqslant s-2$ and $f_r + f_c \leqslant s-2$. Because $2^{s+k} - (s-2) \geqslant 2$, it can be found two fault-free crossing edges (u_0, u_1) and (v_0, v_1). Since both $LeTQ^0_{s,k+1}$ and $LeTQ^1_{s,k+1}$ are $(s-2)$-Hamiltonian connected, there is a Hamiltonian path $HP(u_0, v_0)$ in $LeTQ^0_{s,k+1}$ and a Hamiltonian path $HP(u_1, v_1)$ in $LeTQ^1_{s,k+1}$. Thus, $\langle u_0, HP(u_0, v_0), v_0, v_1, HP(u_1, v_1), u_1, u_0\rangle$ is a faultfree Hamiltonian cycle in $LeTQ_{s,k+1}$.

The theorem is thus proved. □

Theorem 2 For $s \geqslant 2$, $t \geqslant 3$, and $s \leqslant t$, $LeTQ_{s,t}$ is $(s-1)$-Hamiltonian.

Proof We prove this by induction on t. It is clearly holds for $LeTQ_{s,s}$ $(s \geqslant 3)$ by Lemma 6 and Lemma 7. Suppose that $LeTQ_{s,k}$ $(3 \leqslant t = k)$ is $(s-1)$-Hamiltonian and $(s-2)$-Hamiltonian connected. By Theorem 1, the conclusion holds for $t = k+1$. Since $LeTQ_{2,3}$ is 1-Hamiltonian and Hamiltonian connected by Lemma 5, we can easily obtain that $LeTQ_{2,t}$ is 1-Hamiltonian by Theorem 1. Therefore, $LeTQ_{s,t}$ is $(s-1)$-Hamiltonian. □

Lemma 10 If $LeTQ_{s,k}$ is $(s-1)$-Hamiltonian and $(s-2)$-Hamiltonian connected for $3 \leqslant s \leqslant k$, then $LeTQ_{s,k+1}$ is $(s-2)$-Hamiltonian connected.

Proof Let E_c be a class of crossing edges and $E_c = \{(u_0, u_1)|(u_0, u_1) \in E_3,\ u_0 \in LeTQ^0_{s,k+1}$ and $u_1 \in LeTQ^1_{s,k+1}\}$. Let F be a faulty set of $LeTQ_{s,k+1}$ with $F_l = F \cap LeTQ^0_{s,k+1}$, $F_r = F \cap LeTQ^1_{s,k+1}$, and $F_1 = F \cap E_1$. And let $f_l = |F_l|$, $f_r = |F_r|$, and $f_c = |E_c|$. By the location of faults, we have the following cases.

Case 1 All faults are located in the same copy of $LeTQ^i_{s,k+1}(i \in \{0, 1\})$. Suppose that all of the faults are in $LeTQ^0_{s,k+1}$ and $f_l = s-2$. There are three subcases.

Case 1.1 x and y are in different $LeTQ^i_{s,k+1}(i \in \{0, 1\})$. Suppose that $x \in V(LeTQ^0_{s,k+1})$ and $y \in V(LeTQ^1_{s,k+1})$. Select a fault-free vertex $u \in V(LTQ_t)$ in $V(LeTQ^0_{s,k+1}) - \{x\}$ such that its neighbour u' in $LeTQ^1_{s,k+1}$ is different from y. Since $LeTQ_{s,k}$ is $(s-2)$-Hamiltonian connected, there is a Hamiltonian path $HP(x, u)$ in $LeTQ^0_{s,k+1}$ and a Hamiltonian path $HP(u', y)$ in $LeTQ^1_{s,k+1}$. Thus, $\langle x, HP(x, u), u, u', HP(u', y), y\rangle$ is a fault-free Hamiltonian path between x and y in $LeTQ_{s,k}$.

Case 1.2 Both x and y are in $LeTQ^0_{s,k+1}$. Since $LeTQ_{s,k}$ is $(s-2)$-Hamiltonian connected, there is a fault-free Hamiltonian path $HP(x, y)$ in $LeTQ^0_{s,k+1}$. Since $2^{s+k-1} - 3 - (s-2) > 1$, there exists an edge $(u_0, v_0) \in E_3$ in $HP(x, y)$ such that their neighbours u_1 and v_1 are in $LeTQ^1_{s,k+1}$. By the condition of the lemma, there is a Hamiltonian path $HP(u_1, v_1)$ between u_1 and v_1 in $LeTQ^1_{s,k+1}$. Thus, $\langle x, u_0\rangle + \langle u_0, u_1\rangle + HP(u_1, v_1) + \langle v_1, v_0\rangle + \langle v_0, y\rangle$ is a fault-free Hamiltonian path between x and y in $LeTQ_{s,k+1}$.

Case 1.3 Both x and y are in $LeTQ^1_{s,k+1}$. By the condition of the lemma, there is a Hamiltonian path $HP(x, y)$ between x and y in $LeTQ^1_{s,k+1}$. Since $2^{s+k-1} - 3 > 2(s-2)$, there is an edge $(u_1, v_1) \in E_3$ in $LeTQ^1_{s,k+1}$ such that the neighbours u_0 and v_0 are fault-free in $LeTQ^0_{s,k+1}$. Since $LeTQ_{s,k}$ is $(s-2)$-Hamiltonian connected, there is a fault-free Hamiltonian path $HP(u_0, v_0)$ in $LeTQ^0_{s,k+1}$. Thus, $\langle x, u_1\rangle + \langle u_1, u_0\rangle + HP(u_0, v_0) + \langle v_0, v_1\rangle + \langle v_1, y\rangle$ is a fault-free Hamiltonian path between x and y in $LeTQ_{s,k+1}$.

Case 2 All of the faults are located in E_c. There are two subcases.

Case 2.1 x and y are in different $LeTQ^i_{s,k+1}(i \in \{0, 1\})$. Suppose that $x \in V(LeTQ^0_{s,k+1})$ and $y \in V(LeTQ^1_{s,k+1})$. Since $2^{s+k} - (s-2) \geqslant 3$, we can select a fault-free vertex $u \in V(LTQ_t)$ in $V(LeTQ^0_{s,k+1}) - \{x\}$ such that its neighbour u' in $LeTQ^1_{s,k+1}$ is different from y. Since $LeTQ_{s,k}$ is Hamiltonian connected, there is a Hamiltonian path $HP(x, u)$ in $LeTQ^0_{s,k+1}$ and a Hamiltonian path $HP(u', y)$ in $LeTQ^1_{s,k+1}$. Thus, $\langle x, HP(x, u),\ u, u',\ HP(u', y), y\rangle$ is a fault-free Hamiltonian path between x and y in $LeTQ_{s,k+1}$.

Case 2.2 Both x and y are in $LeTQ^i_{s,k+1}(i \in 0, 1)$. Suppose that $x, y \in V(LeTQ^0_{s,k+1})$. There exists a Hamiltonian path $HP(x, y)$ between x and y in $LeTQ^0_{s,k+1}$. Since $2^{s+k-1} - 2(s-2) \geqslant 2$, we always can choose an edge (u_0, v_0) in $HP(x, y)$ such that the two crossing edges (u_0, u_1) and (v_0, v_1) are fault-free. Since $LeTQ^1_{s,k+1}$ is Hamiltonian connected, there is a fault-free Hamiltonian path $HP(u_1, v_1)$ between u_1 and v_1 in $LeTQ^1_{s,k+1}$. Thus, $\langle x, u_0\rangle, \langle u_0, u_1\rangle, HP(u_1, v_1), \langle v_1, v_0\rangle, \langle v_0, y\rangle$ is a fault-free Hamiltonian path between x and y in $LeTQ(s, k+1)$.

Case 3 The faults are scattered in $LeTQ^0_{s,k+1}$, E_c and $LeTQ^1_{s,k+1}$. Without loss of generality, suppose that f_l is the greatest of f_l, f_r and f_c. Then $f_l \leqslant s-3$ and $f_r + f_c \leqslant s-3$. We have the following cases.

Case 3.1 x and y are in different $LeTQ^i_{s,k+1}(i \in \{0, 1\})$. Suppose that $x \in V(LeTQ^0_{s,k+1})$ and $y \in V(LeTQ^1_{s,k+1})$. Select a fault-free vertex $u \in V(LTQ_t)$ in $V(LeTQ^0_{s,k+1}) - \{x\}$ such that the neighbour $u'(u' \neq y)$ in $LeTQ^1_{s,k+1}$ and the edge (u, u') are both fault-free. Since $LeTQ_{s,k}$ is $(s-2)$-Hamiltonian connected, there is a Hamiltonian path $HP(x, u)$ in $LeTQ^0_{s,k+1}$ and a Hamiltonian path $HP(u', y)$ in $LeTQ^1_{s,k+1}$. Thus,

$\langle x, HP(x,u), u, u', HP(u',y), y\rangle$ is a fault-free Hamiltonian path between x and y in $LeTQ_{s,k+1}$.

Case 3.2 Both x and y are in $LeTQ^i_{s,k+1}$ ($i \in \{0,1\}$). Suppose that $x, y \in V(LeTQ^0_{s,k+1})$. Since $LeTQ_{s,k}$ is $(s-2)$-Hamiltonian connected, there is a fault-free Hamiltonian path $HP(x,y)$ in $LeTQ^0_{s,k+1}$. Because $2^{s+k-1} - 3 - (s-3) > 2(s-3)$, there is an edge (u_0, v_0) in $HP(x,y)$ such that u_0's and v_0's neighbours u_1 and v_1 in $LeTQ^1_{s,k+1}$, (u_0,u_1), and (v_0,v_1) are all fault-free. By the condition of the lemma, there is a Hamiltonian path $HP(u_1,v_1)$ between u_1 and v_1 in $LeTQ^1_{s,k+1}$. Thus, $\langle x,u_0\rangle+\langle u_0,u_1\rangle+ HP(u_1,v_1)+\langle v_1,v_0\rangle+\langle v_0,y\rangle$ is a fault-free Hamiltonian path between x and y in $LeTQ_{s,k+1}$. □

Theorem 3 For $s \geqslant 2$, $t \geqslant 3$, and $s \leqslant t$, $LeTQ_{s,t}$ is $(s-2)$-Hamiltonian connected.

Proof We prove this by induction on t. It is clearly holds for $LeTQ_{s,s}$ ($s \geqslant 3$) by Lemmas 8 and 9. Suppose that $LeTQ_{s,k}$ ($3 \leqslant t = k$) is $(s-1)$-Hamiltonian and $(s-2)$- Hamiltonian connected. By Lemma 10, the conclusion holds for $t = k+1$. We easily obtain that $LeTQ_{2,t}$ is Hamiltonian connected by Lemma 4. Thus $LeTQ_{s,t}$ is $(s-2)$-Hamiltonian connected. □

Theorem 4 For $s \geqslant 2$, $t \geqslant 3$, and $s \leqslant t$, let a and b be two different vertices in $LeTQ_{s,t}$. There exists an ($N\log N$) time algorithm which can construct a Hamiltonian path and Hamiltonian cycle between a and b in $LeTQ_{s,t}$, where N is the number of vertices of $LeTQ_{s,t}$.

Proof In a graph $LeTQ_{s,t}$ with fault elements, given a source vertex $a = (a_{n-1}, a_{n-2}, \ldots\ a_0)$ and a target vertex $b = (b_{n-1}, b_{n-2}, \ldots\ b_0)$. Our algorithm needs to output a fault-free Hamilton path from a to b. We first select a as the starting vertex and record the vertices in the path with the linear table P. Add vertices a and b to P, and then find any vertex a_1 that is adjacent to a but does not join path P. Next, vertex a_1 is added to P to further find any vertex a_2 adjacent to a_1 but not added to path P. Until a vertex v is reached, all its fault-free adjacent vertices have been added to P. In this case, if P contains all fault-free vertices and vertex b is adjacent to vertex b, then the construction is successful. P is a Hamiltonian path from a to b. Otherwise, other fault-free adjacent vertices of vertex a are selected to perform the above process. Until the construction is successful or all the fault-free adjacent vertices of vertex b are searched, the return fails.

We use $T(n)$ to represent the time complexity of Algorithm 1 (Algorithm 2), for $n = s+t+1$. In the algorithm description, iP^*0 (or jP^*1) means adding 1 bit i (or j) to each vertex on the path P^*0 (or P^*1), where P^*0 and P^*1 represent the Hamiltonian path on $LeTQ_{s-1,t}$. Therefore, statement 14 of Algorithms takes $O(N)$. It takes $2T(n_1)$ to find the Hamiltonian Path P^*0 (Hamiltonian cycle) and Hamiltonian Path P^*1 (Hamiltonian cycle). It is easy to verify that when $s = 2$ and $t = 3$, $T(6) = O(1)$.

From the above discussion, the following recursive equation can be obtained:

$$T(n) = 2(T(n-1)) + O(2^n), (n \geqslant 6).$$

Therefore, $T(n) = O(N\log N)$. □

4 Simulations and experiments

In this section, we will verify the effectiveness of the algorithm Hamiltonian Cycle through simulation experiments. Our experimental platform consists of three CPUs with Intel (R) Xeon (R) E5420/8 core/2.50GHz and 32GB memory. The operating system is Ubuntu 16.04 Linux. Based on the algorithm Hamiltonian Cycle, we also write the corresponding C language program, and generate an executable program through the GCC compiler. The simulation experiment shows how to constructs Hamiltonian cycles on $LeTQ_{s,t}$ network.

In a faulty $LeTQ_{s,t}$, we first select a fault free vertex s as the starting vertex, and record the vertices in the cycle with vertex set C. Add a vertex s to C, and then find any vertex s^1 adjacent to s but not joined in cycle C. Next, add vertex s^1 to C, and further find any vertex s^2 that is adjacent to s^1 but does not join cycle C. Until a vertex t is reached, all its fault free adjacency vertices have been added to C. In this case, if C contains all the fault free vertices and vertex t is the adjacency vertex of vertex s, then the construction is successful, and C is the Hamilto-

Algorithm 2 Fault-tolerant hamiltonian path (a, b, A, P)

Input: Starting node a, ending node b; Available node set A; Node set P.
Output: A Hamiltonian path P in $LeTQ_{s,t}$-F or return failure;

```
1: if A = ∅ then
2:     if a is the neighbour of b then;
3: return (true, P);
4:     else
5: return (false, P);
6:     end if
7: else
8:     while there exists a neighbor a′ of a such that a′ ∈ A do;
9:         A = A − {a′};
10:        P = P ∪ {a′};
11:        (v, P∗)=conHC(a′, b, A, P)
12:        if A = ∅ then
13:            if w is a neighbor of t then
14:                return (true , P);
15:            else
16:                return (false , P);
17:            end if
18:        else
19:            while there exists a neighbor w′ of w such that w′ ∈ A do
20:                A = A − {w′};
21:                P = P ∪ {w′};
22:                (b, P′)=conPath(w′, t, A, P)
23:                if b==true then
24:                    return (true , P′);
25:                else
26:                    A = A ∪ {w′};
27:                    P = P − {w′};
28:                end if
29:            end while
30:        end if
31:        if v==true then
32: return (true,P∗);
33:        else
34:            A = A ∪ {a′};
35:            P = P − {a′};
36:        end if
37:    end while
38: end if
39: return (false,P);
```

nian cycle. Otherwise, select other fault free adjacent contacts of vertex s to perform the above process. Until the construction is successful or all the fault free adjacency vertices of vertex s are found, return failure.

In the experiment, we first simulate $LeTQ_{1,1}$, $LeTQ_{1,2}$, $LeTQ_{1,3}$, $LeTQ_{1,4}$, $LeTQ_{1,5}$, $LeTQ_{2,2}$, $LeTQ_{2,3}$ networks according to the definition of locally exchanged twisted cube networks. Then we run the corresponding programs of the algorithm Hamiltonian Cycle to construct corresponding Hamiltonian cycles on $LeTQ_{1,1}$, $LeTQ_{1,2}$, $LeTQ_{1,3}$, $LeTQ_{1,4}$, $LeTQ_{1,5}$, $LeTQ_{2,2}$, $LeTQ_{2,3}$ networks (See Fig. 10). The experimental results further verify the validity of the algorithm Hamiltonian cycle.

We compare the time consumption of N from 1-Dimension to 10-Dimension by using Algorithms 1 and 2, for $N = s+t+1$. The results are illustrated in Fig. 11. It shows that the time consumption for constructing a Hamiltonian path is approximately equal when N = 3, 5 or N = 6, 8. The trend of time consumption of constructing a Hamiltonian path is similar to that of constructing a Hamiltonian cycle. It can be explained by the proof of Theorems 2 and 3. For $s \geqslant 2$ and $t \geqslant 3$, a Hamiltonian path is constructed by calling the function HC in Algorithm 2. Compared with the construction of Hamiltonian cycles, the time consumption of constructing Hamiltonian paths is slightly higher in the same dimension. The experimental results show that the algorithms have good performance and simulation results indicate that both the time complexity of Algorithms 1 and 2 meet $O(N\log N)$.

5 Conclusions

We studied the tolerant Hamiltonian properties of a faulty locally exchanged twisted cube, $LeTQ_{s,t}-(f_v+f_e)$, with f_v faulty vertices and f_e faulty edges. We showed that an $LeTQ_{s,t}$ can

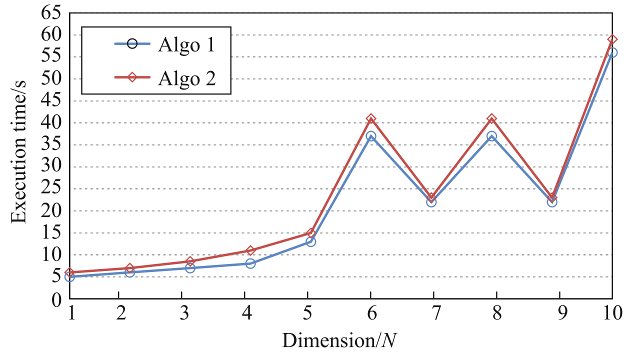


Fig. 11 Time consumptions of Algorithms 1 and 2

tolerate a set F of up to $s-1$ faulty vertices and edges when embedding a Hamiltonian cycle provided that $s \geqslant 2$, $t \geqslant 3$, and $s \leqslant t$. We have also showed another result that there is a Hamiltonian path between any two distinct fault-free vertices in a faulty $LeTQ_{s,t}$ with up to $s-2$ faulty vertices and edges provided that $s \geqslant 2$, $t \geqslant 3$, and $s \leqslant t$. The results are optimal that the fault-tolerant Hamiltonicity of $LeTQ_{s,t}$ is at most $s-1$, and the fault-tolerant Hamiltonian connectivity is at most $s-2$. This paper reveals the fact that faulty $LeTQ_{s,t}$ nearly remains the fault-tolerant Hamiltonicity although it has about one half edges of LTQ_n. Although the architecture of locally exchanged twisted cube has not been really applied in practice, it brings opportunities for future parallel systems.

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$LeTQ_{s,t}$	Hamiltonian cycle
$s=t=1$	000 100 101 111 110 010 011 001 000
$s=1, t=2$	0000 0001 1001 1101 1100 0100 0101 0111 0110 1110 1111 1011 1010 0010 0011 0001 0000
$s=1, t=3$	00000 10000 11000 01000 01001 01101 01111 01011 01010 00010 10010 11010 11011 11001 11101 11111 11110 01110 00110 10110 10111 10011 10001 10101 10100 11100 01100 00100 00101 00111 00011 00001 00000
$s=t=2$	00000 10000 11000 01000 01001 01101 01111 01011 01010 00010 10010 11010 11011 11001 11101 11111 11110 01110 00110 10110 10111 10011 10001 10101 10100 11100 01100 00100 00101 00111 00011 00001 00000
$s=1, t=4$	000000 100000 110000 010000 010001 011001 011101 010101 010111 011011 011111 010011 010010 110010 100010 000010 000011 001111 001011 000111 000101 001101 001100 011100 111100 101100 101101 100101 100100 000100 010100 110100 110101 111101 111001 110001 110011 111111 111110 011110 001110 101110 101111 101011 101010 001010 011010 111010 111011 110111 110110 010110 000110 100110 100111 100011 100001 101001 101000 111000 011000 001000 001001 000001 000000
$s=2, t=3$	000000 100000 110000 010000 010001 011001 011101 010101 010111 011011 011111 010011 010010 110010 100010 000010 000011 001111 001011 000111 000101 001101 001100 011100 111100 101100 101101 100101 100100 000100 010100 110100 110101 111101 111001 110001 110011 111111 111110 011110 001110 101110 101111 101011 101010 001010 011010 111010 111011 110111 110110 010110 000110 100110 100111 100011 100001 101001 101000 111000 011000 001000 001001 000001 000000
$s=1, t=5$	0111100 0111101 0011101 0011100 1011100 1011101 1011111 1011110 0011110 0011111 0011011 0011010 1011010 1011011 1011001 1011000 0011000 0011001 0010001 0010000 1010000 1010001 1010011 1010010 0010010 0010011 0010111 0010110 1010110 1010111 1010101 1010100 0010100 0010101 0000101 0000100 1000100 1000101 1000111 1000110 0000110 0000111 0000011 0000010 1000010 1000011 1000001 1000000 0000000 0000001 0001001 0001000 1001000 1001001 1001011 1001010 0001010 0001011 0001111 0001110 1001110 1001111 1001101 1001100 0001100 0001101 0101101 0101100 1101100 1101101 1101111 1101110 0101110 0101111 0101011 0101010 1101010 1101011 1101001 1101000 0101000 0101001 0100001 0100000 1100000 1100001 1100011 1100010 0100010 0100011 0100111 0100110 1100110 1100111 1100101 1100100 0100100 0100101 0110101 0110100 1110100 1110101 1110111 1110110 0110110 0110111 0110011 0110010 1110010 1110011 1110001 1110000 0110000 0110001 0111001 0111000 1111000 1111001 1111011 1111010 0111010 0111011 0111111 0111110 1111110 1111111 1111101 1111100 0111100

Fig. 10 Hamiltonian cycles in $LeTQ_{s,t}$

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