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RESEARCH ARTICLE

Basic theorem as representation of heterogeneous concept lattices

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Abstract We propose a method for representing heterogeneous concept lattices as classical concept lattices. Particularly, we describe a transformation of heterogeneous formal context into a binary one, such that corresponding concept lattices will be isomorphic. We prove the correctness of this transformation by the basic theorem for heterogeneous as well as classical concept lattices.

Keywords basic theorem, heterogeneous concept lattice, representation

1 Introduction

Formal concept analysis (FCA) is a data-mining method used for identification of conceptual structures among data sets. It is also known as a theory of concept lattices based on the notion of formal context, which is represented by a binary relation between a set of objects and a set of attributes. In practice, there are natural examples of object-attribute models for which relationship between objects and attributes are represented by fuzzy relations. As a natural consequence there appeared various fuzzy generalizations of classical FCA. From many existing approaches we mention work of Bělohlávek [1,2], Krajči [3–6] or work on multi-adjoint concept lattices by Medina, Ojeda-Aciego and Ruiz-Calviño [7–9]. One can find comparison and survey of some existing approaches in [10]. As an efficient tool, formal concept analysis and its fuzzifications has been successfully applied to the domains such as decision systems, information retrieval, data mining and knowledge discovery. Hence the research in theoretical and practical applications of fuzzy FCA is very wide [11–24].

Recently, there were described two approaches with different types of complete lattices representing the structures of possible truth values on the side of objects as well as on the side of attributes. The first one, described in [25,26], has an input in the form of matrix consisting of Galois connections between particular objects and attributes. The second one, called heterogeneous concept lattices [27,28], works with different mutual relationships between the objects and the attributes. We note that both approaches are equivalent from some point of view, as it was proved in [27].

The main aim of this paper is to describe a representation of the heterogeneous concept lattices as the classical concept lattices. We propose a transformation of a heterogeneous formal context into a binary one, such that the corresponding concept lattices will be isomorphic. Also we give an explicit expression for the isomorphism between these two concept lattices. In order to prove the correctness of our transformation we will use the so-call basic theorem for concept lattices. The basic theorem for concept lattices represents one of the fundamental tool for a theoretical study of concept lattices and its validity allows to find a relatively simple proof of the

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proposed transformation.

The paper is organized as follows. In the next section we provide the basic notions concerning heterogeneous concept lattices and we recall the basic theorem for heterogeneous as well as classical concept lattices. Section 3 contains our main result. We propose the mentioned transformation of a heterogeneous formal context into a binary one. We prove its correctness and we find an explicit formula for the isomorphism between heterogeneous and corresponding classical concept lattice. A simple illustrative example of such representation is also described. At the end we provide an experiment concerning time complexity of the proposed transformation of the heterogeneous formal contexts into the binary ones.

2 Heterogeneous contexts and fuzzy concept lattices

In this section we briefly describe the so-called heterogeneous concept lattices, classical concept lattices and we recall the basic theorems for both types of concept lattices, [27–29].

2.1 Heterogeneous concept lattices

Let B and A be non-empty sets. Let $\mathcal{P} = ((P_{b,a}, \leqslant) : b \in B, a \in A)$ be a system of posets and R be a function with domain $B \times A$ such that $R(b,a) \in P_{b,a}$ for each $b \in B$ and $a \in A$. Let $C = ((C_b, \leqslant) : b \in B)$ and $\mathcal{D} = ((D_a) : a \in A)$ be systems of complete lattices. Further, let $\odot = ((\bullet_{b,a}) : b \in B, a \in A)$ be a system of operations such that $\bullet_{b,a}$ is from $C_b \times D_a$ to $P_{b,a}$ and it is isotone and left-continuous in both arguments, i.e.,

1) $c_1 \leqslant c_2$ implies $c_1 \bullet_{b,a} d \leqslant c_2 \bullet_{b,a} d$ for all $c_1, c_2 \in C_b$ and $d \in D_a$,
2) $d_1 \leqslant d_2$ implies $c \bullet_{b,a} d_1 \leqslant c \bullet_{b,a} d_2$ for all $c \in C_b$ and $d_1, d_2 \in D_a$,
3) if $c \bullet_{b,a} d \leqslant p$ for some $d \in D_a$, $p \in P_{b,a}$ and for all $c \in X \subseteq C_b$, then $(\bigvee X) \bullet_{b,a} d \leqslant p$,
4) if $c \bullet_{b,a} d \leqslant p$ for some $c \in C_b$, $p \in P_{b,a}$ and for all $d \in Y \subseteq D_a$, then $c \bullet_{b,a} (\bigvee Y) \leqslant p$.

Then the tuple $\langle B, A, \mathcal{P}, R, C, \mathcal{D}, \odot \rangle$ is called heterogeneous formal context.

We recall the notion of direct product of lattices. If $(L_i : i \in I)$ is a family of lattices the direct product $\prod_{i \in I} L_i$ is defined as the set of all functions

$$f : I \to \bigcup_{i \in I} L_i,$$

such that $f(i) \in L_i$ for all $i \in I$ with the "componentwise" order, i.e., $f \leqslant g$ if $f(i) \leqslant g(i)$ for all $i \in I$. If $L_i = L$ for all $i \in I$ we get a direct power L^I. In this case the direct power L^I represents the structure of L-fuzzy sets, hence direct product of lattices can be seen as a generalization of the notion of L-fuzzy sets. The direct product of lattices forms complete lattice if and only if all members of the family are complete lattices. The straightforward computations show that the lattice operations in the direct product $\prod_{i \in I} L_i$ of complete lattices are calculated componentwise, i.e., for any subset $\{f_j : j \in J\} \subseteq \prod_{i \in I} L_i$ we obtain

$$\Big(\bigvee_{j \in J} f_j\Big)(i) = \bigvee_{j \in J} f_j(i) \quad \text{and} \quad \Big(\bigwedge_{j \in J} f_j\Big)(i) = \bigwedge_{j \in J} f_j(i),$$

where these equalities hold for each index $i \in I$.

Let $\langle B, A, \mathcal{P}, R, C, \mathcal{D}, \odot \rangle$ be heterogeneous formal context. There is a pair of mappings $\nearrow : \prod_{b \in B} C_b \to \prod_{a \in A} D_a$ and $\swarrow : \prod_{a \in A} D_a \to \prod_{b \in B} C_b$ defined by

$$\nearrow(f)(a) = \bigvee \{d \in D_a : (\forall b \in B) f(b) \bullet_{b,a} d \leqslant R(b,a)\},$$

$$\swarrow(g)(b) = \bigvee \{c \in C_b : (\forall a \in A) c \bullet_{b,a} g(a) \leqslant R(b,a)\}.$$

Proposition 1 The mappings $\nearrow$ and $\swarrow$ form a Galois connection.

This fact allows to use concept lattice construction as in the case of classical concept lattices. By a (heterogeneous fuzzy) concept we will understand a pair $\langle f, g \rangle$ from $\prod_{b \in B} C_b \times \prod_{a \in A} D_a$ such that $\nearrow (f) = g$ and $\swarrow (g) = f$. If $\langle f_1, g_1 \rangle$ and $\langle f_2, g_2 \rangle$ are two concepts we define an ordering $\langle f_1, g_1 \rangle \leqslant \langle f_2, g_2 \rangle$ if and only if $f_1 \leqslant f_2$ (or equivalently $g_1 \geqslant g_2$). The poset of all such concepts ordered by relation $\leqslant$ is called a heterogeneous concept lattice and denoted by $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$.

The following theorem, so-called the basic theorem on heterogeneous concept lattices, characterizes heterogeneous concept lattices.

Theorem 1 1) The heterogeneous concept lattice $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$ is a complete lattice in which

$$\bigwedge_{i \in I} \langle f_i, g_i \rangle = \Big\langle \bigwedge_{i \in I} f_i, \nearrow \Big(\swarrow \big(\bigvee_{i \in I} g_i \big) \Big) \Big\rangle,$$

and

$$\bigvee_{i \in I} \langle f_i, g_i \rangle = \Big\langle \swarrow \Big(\nearrow \big(\bigvee_{i \in I} f_i \big) \Big), \bigwedge_{i \in I} g_i \Big\rangle.$$

2) A complete lattice L is isomorphic to $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$ if and only if there are two mappings $\beta : \bigcup_{b \in B} (\{b\} \times C_b) \to L$ and $\alpha : \bigcup_{a \in A} (\{a\} \times D_a) \to L$ such that:

i) β does not decrease in the second argument (for the fixed first one);

ii) α does not increase in the second argument (for the fixed first one);
iii) $\mathrm{Rng}(\beta)$ is supremum-dense in L;
iv) $\mathrm{Rng}(\alpha)$ is infimum-dense in L;
v) For every $b \in B$, $a \in A$ and $c \in C_b$, $d \in D_a$

$$\beta(b, c) \leqslant \alpha(a, d) \quad \text{if and only if} \quad c \bullet_{b,a} d \leqslant R(b, a).$$

2.2 Classical (binary) concept lattices

Now we briefly recall the basic notions of FCA [29].

Let $\langle G, M, I\rangle$ be a formal context, i.e., $G, M \neq \emptyset$ and $I \subseteq G \times M$. There is a pair of mappings ${}^{\uparrow}\colon \mathbf{P}(G) \to \mathbf{P}(M)$ and ${}^{\downarrow}\colon \mathbf{P}(M) \to \mathbf{P}(G)$, which forms a Galois connection between the power sets of G and M respectively.

$$X^{\uparrow} = \{y \in M : \langle x, y\rangle \in I, \forall x \in X\},$$

$$Y^{\downarrow} = \{x \in G : \langle x, y\rangle \in I, \forall y \in Y\}.$$

The corresponding concept lattice is denoted by $\underline{\mathfrak{B}}(G, M, I)$. The following theorem, the basic theorem on concept lattices, represents the well-known characterization of concept lattices [29].

Theorem 2 1) The concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:

$$\bigwedge_{t\in T} \langle A_t, B_t\rangle = \Big\langle \bigcap_{t\in T} A_t, (\bigcup_{t\in T} B_t)^{\downarrow\uparrow}\Big\rangle,$$

$$\bigvee_{t\in T} \langle A_t, B_t\rangle = \Big\langle (\bigcup_{t\in T} A_t)^{\uparrow\downarrow}, \bigcap_{t\in T} B_t\Big\rangle.$$

2) A complete lattice L is isomorphic to $\underline{\mathfrak{B}}(G, M, I)$ if and only if there are mappings $\bar{\gamma}\colon G \to L$ and $\bar{\mu}\colon M \to L$ such that $\bar{\gamma}(G)$ is supremum-dense in L, $\bar{\mu}(M)$ is infimum-dense in L and $\langle g, m\rangle \in I$ is equivalent to $\bar{\gamma}(g) \leqslant \bar{\mu}(m)$ for all $g \in G$ and all $m \in M$.

3 Basic theorem as a representation of heterogeneous concept lattices

We say that a heterogeneous concept lattice $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$ is representable as a (classical) concept lattice, if there exists a formal context $\langle G, M, I\rangle$ such that the concept lattices $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$ and $\underline{\mathfrak{B}}(G, M, I)$ are isomorphic. In order to find such representation we will use the both basic theorems presented in the previous section. First, we transform a heterogeneous formal context into a binary one such that corresponding concept lattices will be isomorphic.

Let $\langle B, A, \mathcal{P}, R, C, \mathcal{D}, \odot\rangle$ be a heterogeneous formal context. We put

$$G = \bigcup_{b\in B} (\{b\} \times C_b) \quad \text{and} \quad M = \bigcup_{a\in A} (\{a\} \times D_a)$$

and define a binary relation $I \subseteq G \times M$ as

$$\langle\, \langle b, c\rangle , \langle a, d\rangle \,\rangle \in I \quad \text{iff} \quad c \bullet_{b,a} d \leqslant R(b, a), \tag{1}$$

for all $b \in B$, $a \in A$ and for all $c \in C_b$ and $d \in D_a$. In this case the set of objects consists of the elements of the form $\langle b, c\rangle$ for some $b \in B$ and $c \in C_b$. Similarly, the set of attributes consists of the ordered pairs $\langle a, d\rangle$ for some $a \in A$ and $d \in D_a$.

Theorem 3 The concept lattices $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$ and $\underline{\mathfrak{B}}(G, M, I)$ are isomorphic.

Proof Denote by L the heterogeneous concept lattice $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$. Since L is isomorphic to itself, there is a pair of mappings $\beta\colon \bigcup_{b\in B}(\{b\} \times C_b) \to L$ and $\alpha\colon \bigcup_{a\in A}(\{a\} \times D_a) \to L$ satisfying conditions (i)–(v) of Theorem 1.

Now we define $\bar{\gamma}\colon G \to L$ and $\bar{\mu}\colon M \to L$ by $\bar{\gamma}(b, c) = \beta(b, c)$ for all $b \in B$ and $c \in C_b$; and $\bar{\mu}(a, d) = \alpha(a, d)$ for all $a \in A$ and $d \in D_a$. According to the conditions iii) and iv) of Theorem 1 sets $\bar{\gamma}(G)$ and $\bar{\mu}(M)$ are supremum-dense in L and infimum-dense in L respectively. The condition v) and rule 1) give the following equivalent assertions

$$\langle\, \langle b, c\rangle , \langle a, d\rangle \,\rangle \in I \quad \text{iff} \quad c \bullet_{b,a} d \leqslant R(b, a),$$
$$\text{iff} \quad \bar{\gamma}(b, c) \leqslant \bar{\mu}(a, d),$$

for all $b \in B$, $a \in A$ and for all $c \in C_b$ and $d \in D_a$.

This yields that the assumptions of the Basic Theorem for concept lattices (Theorem 2) are fulfilled, hence the concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is isomorphic to the lattice L. □

The proof of the basic theorem for concept lattices allows to find such isomorphism. First we recall an important fact. Let $\underline{\mathfrak{B}}(G, M, I)$ be a concept lattice and L be a complete lattice such that $\bar{\gamma}(G)$ is supremum-dense in L, $\bar{\mu}(M)$ is infimum-dense in L and $\langle g, m\rangle \in I$ is equivalent to $\bar{\gamma}(g) \leqslant \bar{\mu}(m)$ for all $g \in G$ and all $m \in M$. Then the mapping $\varphi\colon \underline{\mathfrak{B}}(G, M, I) \to L$ given by

$$\varphi(X, Y) = \bigvee\{\bar{\gamma}(g) : g \in X\}, \tag{2}$$

is an order isomorphism between $\underline{\mathfrak{B}}(G, M, I)$ and L (see [29] page 21 for details).

As in the previous proof, denote by L the heterogeneous concept lattice $\mathsf{HCL}(B, A, \mathcal{P}, R, C, \mathcal{D}, \odot, \nearrow, \swarrow, \leqslant)$. We define the following singleton functions $T_b^c \in \prod_{b\in B} C_b$ for each

$b \in B$ and $c \in C_a$, and $S^d_a \in \prod_{a\in A} D_a$ for each $a \in A$ and $d \in D_a$:

$$T^c_b(x) = \begin{cases} c, & \text{if } x = b;\\ 0_{C_b}, & \text{otherwise,}\end{cases}$$

$$S^d_a(x) = \begin{cases} d, & \text{if } x = a;\\ 0_{D_a}, & \text{otherwise.}\end{cases}$$

From the basic properties of Galois connections it can be easily shown ([28]) that the mappings $\beta\colon \bigcup_{b\in B}(\{b\}\times C_b) \to L$ and $\alpha\colon \bigcup_{a\in A}(\{a\}\times D_a) \to L$ defined by

$$\beta(b, c) = \langle \swarrow (\nearrow (T^c_b)), \nearrow (T^c_b)\rangle,$$

$$\alpha(a, d) = \langle \swarrow (S^d_a), \nearrow (\swarrow (S^d_a))\rangle,$$

fulfill the conditions i)–v) of Theorem 1.

Theorem 4 The mapping $\varphi\colon \mathfrak{B}(G, M, I) \to L$ given by $\varphi(X, Y) = \langle f, g\rangle$, where $f(b) = \bigvee\{c \in C_b : \langle b, c\rangle \in X\}$ for all $b \in B$ and $g(a) = \bigvee\{d \in D_a : \langle a, d\rangle \in Y\}$ for all $a \in A$, is an order isomorphism.

Proof According to the definition of the formal context (G, M, I) the mappings $\bar{\gamma}\colon G \to L$ and $\bar{\mu}\colon M \to L$ defined by

$$\bar{\gamma}(b, c) = \beta(b, c) = \langle \swarrow (\nearrow (T^c_b)), \nearrow (T^c_b)\rangle,$$

$$\bar{\mu}(a, d) = \alpha(a, d) = \langle \swarrow (S^d_a), \nearrow (\swarrow (S^d_a))\rangle,$$

fulfill the conditions of the basic theorem for concept lattices. Due to Eq.(2) the mapping $\varphi\colon \mathfrak{B}(G, M, I) \to L$ defined by

$$\varphi(X, Y) = \bigvee \{\bar{\gamma}(b, c) : \langle b, c\rangle \in X\}$$

is an order isomorphism.

From this and using the expression for supremum in L we obtain

$$\begin{aligned}\varphi(X, Y) &= \bigvee \{\bar{\gamma}(b, c) : \langle b, c\rangle \in X\}\\ &= \bigvee_{\langle b,c\rangle\in X} \langle \swarrow (\nearrow (T^c_b)), \nearrow (T^c_b)\rangle\\ &= \Big\langle \swarrow \Big(\nearrow (\bigvee_{\langle b,c\rangle\in X} T^c_b)\Big), \nearrow (\bigvee_{\langle b,c\rangle\in X} T^c_b)\Big\rangle.\end{aligned}$$

First, we show that

$$\nearrow \Big(\bigvee\{T^c_b : \langle b, c\rangle \in X\Big)(a) = \bigvee \{d \in D_a : \langle a, d\rangle \in Y\}$$

is valid for all $a \in A$.

Obviously for all $b \in B$ we have

$$\Big(\bigvee\{T^c_b : \langle b, c\rangle \in X\Big)(b) = \bigvee \{c \in C_b : \langle b, c\rangle \in X\}.$$

Let $a \in A$ be any fixed element. Properties 1) and 3) of the operations $\bullet_{b,a}$ imply the following two equivalent assertions

$$(\forall b \in B) \quad \Big(\bigvee_{\langle b,c\rangle\in X} c\Big) \bullet_{b,a} d \leqslant R(b, a),$$

$$(\forall b \in B) \quad c \bullet_{b,a} d \leqslant R(b, a), \forall c : \langle b, c\rangle \in X.$$

Due to the condition i) and fact that $\langle\langle b, c\rangle, \langle a, d\rangle\rangle \in I$ for all $\langle b, c\rangle \in X$ if and only if $\langle a, d\rangle \in X^{\uparrow} = Y$, we obtain equality between these sets

$$\begin{aligned}&\{d \in D_a : (\forall b \in B)(\bigvee_{\langle b,c\rangle\in X} c) \bullet_{b,a} d \leqslant R(b, a)\}\\ &= \{d \in D_a : \langle a, d\rangle \in Y\}.\end{aligned}$$

Hence, from the definition of the mapping $\nearrow$ we obtain for all $a \in A$

$$\nearrow (\bigvee\{T^c_b : \langle b, c\rangle \in X)(a) = \bigvee \{d \in D_a : \langle a, d\rangle \in Y\}.$$

Due to this fact, the equality

$$\swarrow \Big(\nearrow (\bigvee_{\langle b,c\rangle\in X} T^c_b)\Big)(b) = \{c \in C_b : \langle b, c\rangle \in X\},$$

for all $b \in B$ can be proved in the same way. □

Example 1 For a positive integer $n \geqslant 2$ denote by L_n an n-element chain $L_n = \{\frac{i}{n-1} : 0 \leqslant i \leqslant n-1\} \subseteq [0, 1]$. Given positive integers $n, m, k \geqslant 2$ let us consider the mapping $\mathbf{P}^k_{n,m}\colon L_n \times L_m \to L_k$ defined for each $c \in L_n$ and $d \in L_m$ as

$$\mathbf{P}^k_{n,m} = \frac{\lceil (k-1) \cdot c \cdot d\rceil}{k-1},$$

where $\lceil\ \rceil$ is the ceiling function and $\cdot$ denotes the usual product of rational numbers. Any $\mathbf{P}^k_{n,m}$ satisfies the conditions 1)–4) of the product operations in the definition of heterogeneous formal context.

Let $B = \{b_1, b_2\}$ be a set of elements, $A = \{a_1, a_2\}$ be a set of attributes. Further we put $C_{b_1} = L_3$, $C_{b_2} = L_2$ and $D_{a_1} = L_2$, $D_{a_2} = L_3$. To define the system of operations $\odot$ we set $\bullet_{b_1,a_1} = \mathbf{P}^3_{3,2}$, $\bullet_{b_1,a_2} = \mathbf{P}^2_{3,3}$, $\bullet_{b_2,a_1} = \mathbf{P}^4_{2,2}$ and $\bullet_{b_2,a_2} = \mathbf{P}^3_{2,3}$. Let us note, that the system $\mathcal{P}$ of posets is given by the corresponding operations $\bullet_{b,a}$, e.g., $P_{b_1,a_1} = (L_3, \leqslant)$. Finally, the incidence relation R is given in Table 1.

Table 1 Incidence relation R

Object/Attribute	a_1	a_2
b_1	1/2	0
b_2	2/3	1/2

Now we transform the heterogeneous formal context $\langle B, A, \mathcal{P}, R, C, \mathcal{D}, \odot\rangle$ into the binary one. Hence we put $G = \{\langle b_1, 0\rangle, \langle b_1, 1/2\rangle, \langle b_1, 1\rangle, \langle b_2, 0\rangle, \langle b_2, 1\rangle\}$, $M = \{\langle a_1, 0\rangle, \langle a_1, 1\rangle, \langle a_2, 0\rangle, \langle a_2, 1/2\rangle, \langle a_2, 1\rangle\}$ and using rule 1) we obtain the incidence relation $I \subseteq G \times M$ depicted in Table 2. Let us note that the symbol $\times$ on a position in the table indicates that the corresponding object is in the relation I with the corresponding attribute.

Table 2 Incidence relation I of the transformed binary context

Object/Attribute	$\langle a_1, 0\rangle$	$\langle a_1, 1\rangle$	$\langle a_2, 0\rangle$	$\langle a_2, 1/2\rangle$	$\langle a_2, 1\rangle$
$\langle b_1, 0\rangle$	$\times$	$\times$	$\times$	$\times$	$\times$
$\langle b_1, 1/2\rangle$	$\times$	$\times$	$\times$		
$\langle b_1, 1\rangle$	$\times$		$\times$		
$\langle b_2, 0\rangle$	$\times$	$\times$	$\times$	$\times$	$\times$
$\langle b_2, 1\rangle$	$\times$		$\times$	$\times$	

From this formal context we obtain concept lattice $\underline{\mathfrak{B}}(G, M, I)$ (Fig. 1). Consequently, using the formula from Theorem 4 we obtain the isomorphic heterogeneous concept lattice (Fig. 2).

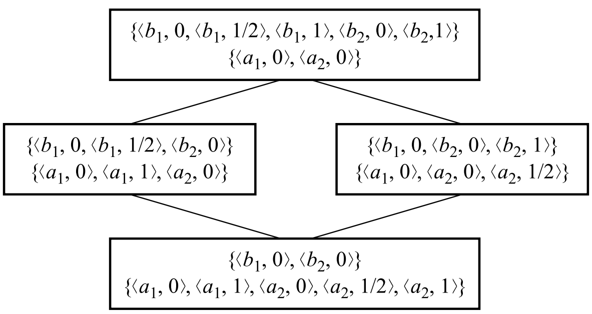

Fig. 1 Classical concept latice corresponding to Table 2

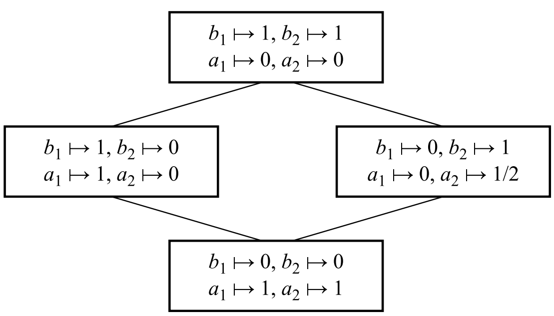

Fig. 2 Heterogeneous concept lattice corresponding to Table 1

Let $\langle B, A, \mathcal{P}, R, C, \mathcal{D}, \odot\rangle$ be a heterogeneous formal context. We will assume that the object set B, the attribute set A as well as all the posets in $\mathcal{P}$, C and $\mathcal{D}$ are finite. Denote by n and m the number of objects and attributes respectively. Further let $c = \max\{|C_b| : b \in B\}$ and $d = \max\{|D_a| : a \in A\}$, i.e., c denotes the maximal cardinality of some lattice in C and d denotes the maximal cardinality of some lattice in $\mathcal{D}$. Our aim is to provide an estimation for time complexity of transformation of a heterogeneous formal context into a binary one.

Let $\langle G, M, I\rangle$ be the transformed binary context. Then $G = \bigcup_{b \in B}(\{b\} \times C_b)$, $M = \bigcup_{a \in A}(\{a\} \times D_a)$ and we obtain $|G| \leqslant n \cdot c$ and $|M| \leqslant m \cdot d$. Further, we assume that for all $b \in B$, $a \in A$ and for any given values $c_1 \in C_b$, $d_1 \in D_a$ the decision whether $c_1 \bullet_{b.a} d_1 \leqslant R(b, a)$ can be done in a constant time $t \in \mathbb{R}$. Since the incidence relation $I \subseteq G \times M$ is given by the condition

$$\big\langle \langle b, c_1\rangle, \langle a, d_1\rangle \big\rangle \in I \quad \text{iff} \quad c_1 \bullet_{b,a} d_1 \leqslant R(b, a),$$

we obtain $T \leqslant t \cdot n \cdot c \cdot m \cdot d$ for time complexity T of this transformation.

From a practical point of view it is interesting to experimentally compare computational times of transformation with classical binary algorithm and direct application of an algorithm for heterogeneous fuzzy concept lattices. For this purpose we use the algorithm for binary concept lattices [30] and an algorithm for a special type of heterogeneous concept lattices, the so-called GOSCL algorithm [31].

In this special case of heterogeneous concept lattices the system C consists of two-element lattices and the system $\mathcal{D}$ corresponding to the attribute set contains arbitrary complete lattices.

In these experiments all attributes had randomly assigned an i-element chain from the set $\{L_i\}_{i=2}^{10}$ as a truth value structure. For all objects $b \in B$ and $a \in A$ the product $\bullet_{b,a}$ was defined as $\mathbf{P}^2_{2,i}$ where i denotes an index such that truth value for the attribute a was L_i. Also an incidence relation R was generated randomly with the uniform distribution.

In Table 3 are averaged computation times of GOSCL algorithm for different number of objects n and number of attributes $m = 3, 5, 7, 10$. The most experimental run for different input context setup was repeated 100 times (those with duration greater than 1 000 seconds were repeated 20 times) and computation times were averaged. Experiments were performed on the machine with Intel i5/750 (2.66 GHz) processor.

Table 3 Computation times (in seconds) for GOSCL

n	m			
	3	5	7	10
10	0.002 8	0.009 3	0.032	0.135 3
20	0.009 3	0.027	0.084 9	0.538 9
30	0.019 2	0.167 9	1.547 9	4.780 1
40	0.050 8	0.432	5.245 1	23.56
50	0.103 9	1.603 2	14.678	63.907
60	0.161 1	3.395 5	35.045	192.29
70	0.310 2	5.245	63.418	451.49
80	0.385 7	10.034 9	128.45	1 892
90	0.526 4	17.225	199.05	4 651
100	0.824 2	32.257 1	318.87	7 923

The results for computation times of transformation and classical binary algorithm are in Table 4.

Table 4 Computation times (in seconds) for tr. + bin.

n	m			
	3	5	7	10
10	0.013 4	0.031 8	0.123	2.329 3
20	0.023 8	0.117 6	1.584 9	14.316
30	0.071 3	0.474 1	7.487 1	105.71
40	0.103 6	1.249	25.141	523.56
50	0.746 2	2.930 7	68.215	963.46
60	1.938 2	8.352 1	97.404	1 622
70	3.230 4	13.138	145.28	3 059
80	7.365 1	24.398	208.51	4 532
90	15.028	39.274	291.75	6 023
100	22.242	60.083	378.93	7 821

In general, a computational time is influenced by number of concepts which should be created. Thus it depends on parameters n and m respectively. The greater value of m, the larger is the lattice $\prod_{a\in A} D_a$. Hence, the likelihood for creating a concept is greater than for a smaller value of m. Consequently, more concepts are created and the computational time is greater.

In the case when the number of attributes is bounded as well as the maximal cardinality of complete lattices c and d, the time complexity of the proposed transformation is bounded by a linear function $T \leqslant K \cdot n$ according to the parameter n (number of objects). This situation appeared in our experiments. As one can see, for the small values of n the effect of the transformation is significant. However, with the greater values n, the ratio of computation times becomes smaller. Hence the transformation effect is not so significant as for relatively small n. Although presented results also depend on the implementation aspects of both algorithms, this fact can indicate that GOSCL algorithm is not so effective as combination of transformation with the next application of relatively fast binary algorithm. However, in order to achieve the greater statistical credibility, this hypothesis should be verified on the larger samples.

4 Conclusions

We described a representation of the heterogeneous concept lattices, introduced in [28], as the classical concept lattices. This representation is based on the so-called basic theorems for concept lattices (for the classical as well as for the heterogeneous case). We also provided an expression describing isomorphism between the classical and heterogeneous concept lattices. Such representation allows to use various algorithms designed for classical contexts also for heterogeneous ones. Hence, the efficiency of any new algorithm developed for the heterogeneous concept lattices can be compared with those classical ones. Also we think that some well developed notions from FCA could be transformed into heterogeneous approach via this representation.

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