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A Sliding Mode Observer for Uncertain Nonlinear Systems Based on Multiple Models Approach

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Abstract: This paper presents a method of state estimation for uncertain nonlinear systems described by multiple models approach. The uncertainties, supposed as norm bounded type, are caused by some parameters′ variations of the nonlinear system. Linear matrix inequalities (LMIs) have been established in order to ensure the stability conditions of the multiple observer which lead to determine the estimation gains. A sliding mode gain has been added in order to compensate the uncertainties. Numerical simulations through a state space model of a real process have been realized to show the robustness of the synthesized observer.

Keywords: Uncertain nonlinear system, norm bounded uncertainty, multiple models approach, multiple observer, sliding mode observer.

1 Introduction

The knowledge of state variables of nonlinear systems is necessary in control systems field. Although, in most real systems, this knowledge is partial because some variables of the state can not be measured and some sensors are expensive. As a solution, the researchers turn to the observer which is a dynamical system able to estimate the unmeasured state variables. It is noticed that the design of an observer is preceded by the step of modeling. Indeed, the modeling operation involves the construction of mathematical model describing the dynamic behavior of the real system. So, many works are focusing on this issue in order to reach the correct representation of nonlinear systems.

The multiple models is one of many tools of systems modeling. In the literature, methods of obtaining multiple models structure are studied. In fact, the work[1] treats the method based on the direct identification of model parameters (number of local models, the structure of weighting functions and data partitioning). In [2], the researchers present the method based on the linearization of an existing nonlinear model around the operating points. Other researchers use a method based on a transformation of the nonlinear system[3, 4]. As a definition, the multiple models approach is an interpolation of many submodels qualified as linear through activation functions. For each submodel, a local observer is synthesized. The interpolation of the local observers lead to the obtention of multiple observers.

State estimation based on multiple models structure is widely studied in the literature. Works as [5−7] are interested in determining the estimation gains using Linear matrix inequalities (LMIs). Later, many researchers have been focusing on the design of multiple observers in order to satisfy desired performances. In fact, the authors in [8−11] show designs of robust multiple observers for unknown input/uncertain systems based on LMI tools. Other works as [12–15], treat the design of robust sliding mode multiple observers in order to overcome the effect of unknown inputs, uncertainties and disturbances.

Most of works as [16−18], which deal with multiple observer design for uncertain systems using sliding mode techniques, assume that the uncertainties are bounded. The disadvantage of such assumption is the way of choosing the uncertainties′ upper bounds. In order to overcome this limitation, our contribution comes to enhance and improve some related works like [19] which assume that the uncertainties are norm bounded.

The objective of our study is to design a sliding mode observer for nonlinear system with time varying uncertainties supposed as norm bounded. To reach this aim, the multiple models approach is adopted in modeling such class of systems which leads to deal easily with the synthesis of the proposed observer by profiting from the linear tools like those used in [20−25].

This paper is organized as follows. In Section 2, we give the problem statement. The structure of the observer is presented in Section 3. The proof of the estimator′s convergence is demonstrated in Section 4. In Section 5, we present the simulation results and interpretations. A conclusion ends the paper.

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2 Problem statement

Consider a nonlinear system described by

$$\begin{cases} \dot{x} = f(x, u) \\ y = Cx \end{cases} \tag{1}$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ and $y \in \mathbf{R}^p$ are, respectively, the state vector, the input vector and the output vector of the system. $C = [I_p 0_{p\times(n-p)}]$ and verifies $CC^{\mathrm{T}} = I_p$. C^{T} is the transpose of matrix C and it presents also its right pseudoinverse.

The system (1) can be written under multiple model approach as follows:

$$\begin{cases} \dot{x} = \sum\limits_{i=1}^{M} \mu_i(z)(A_i x + B_i u) \\ y = Cx \end{cases} \tag{2}$$

where A_i and B_i are matrices with appropriate dimensions. $z = [z_1, z_2, \cdots, z_r]$ is considered as the vector of premise variables. The activation functions $\mu_i(z)$ verify the convex property:

$$\begin{cases} \sum\limits_{i=1}^{M} \mu_i(z) = 1 \\ 0 \le \mu_i(z) \le 1. \end{cases} \tag{3}$$

The aim of this paper is to develop a sliding mode observer for the reconstruction of unmeasured variables of uncertain nonlinear system described by the following multiple models:

$$\begin{cases} \dot{x} = \sum\limits_{i=1}^{M} \mu_i(z)((A_i + \Delta A_i)x + B_i u) \\ y = Cx \end{cases} \tag{4}$$

where ΔA_i represent the variation of parameters and verify the following hypothesis:

Hypothesis 1. We suppose that the uncertainties are norm bounded type:

$$\Delta A_i = M_i^a F_a N_i^a$$

where $F_a \in \mathbf{R}^{k\times l}$ respects the following constraint:

$$F_a^{\mathrm{T}} F_a \le I_{l\times l}.$$

It should be noted that F_a defines the structure of parameters′ uncertainties or variations.

3 Structure of the sliding mode observer

Consider the multiple observer presented in [5] which has the following form:

$$\begin{cases} \dot{\hat{x}} = \sum\limits_{i=1}^{M} \mu_i(z)(A_i \hat{x} + B_i u + L_i(y - \hat{y})) \\ \hat{y} = C\hat{x}. \end{cases} \tag{5}$$

The estimation gains $L_i \in \mathbf{R}^{n\times p}$ are determined by solving the following LMI′s :

$$(A_i - L_i C)^{\mathrm{T}} P + P(A_i - L_i C) < 0, \quad \text{for} \quad i = 1 \cdots M. \tag{6}$$

The structure of the proposed observer is an extended form of (5) and it is given by

$$\begin{cases} \dot{\hat{x}} = \sum\limits_{i=1}^{M} \mu_i(z)(A_i \hat{x} + B_i u + L_i(y - \hat{y}) + \alpha_i) \\ \hat{y} = C\hat{x} \end{cases} \tag{7}$$

where $\hat{x} \in \mathbf{R}^n$ is the estimate state vector, $L_i \in \mathbf{R}^{n\times p}$ are the estimation gains. α_i represent the sliding mode gains which play their role in compensating the uncertainties ΔA_i for each local model.

Theorem 1. The error estimation between the system (4) and (7) converges asymptotically to zero if there exist symmetric positive definite matrix $P \in \mathbf{R}^{n\times n}$ and matrices $W_i \in \mathbf{R}^{n\times p}$, and a positive scalar ε such that for $i = 1, \cdots, M$:

$$\begin{bmatrix} A_i^{\mathrm{T}} P + PA_i - C^{\mathrm{T}} W_i{}^{\mathrm{T}} + W_i C + 2\varepsilon^{-1} N_i^{a\,\mathrm{T}} N_i^a & PM_i^a \\ M_i^{a\,\mathrm{T}} P & -\varepsilon^{-1} I \end{bmatrix} < 0,$$

$$\varepsilon > 0 \tag{8}$$

where C and $M_i^a N_i^a$ are known matrices.

And the following conditions are fulfilled

$$\begin{cases} \text{If } r \ne 0, \text{then } \alpha_i = \varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} P^{-1} C^{\mathrm{T}} \dfrac{s}{\| s \|^2} \\ \text{If } r = 0, \text{then } \alpha_i = 0 \end{cases} \tag{9}$$

where s is the sliding mode surface. The gains of the observer are derived from $L_i = P^{-1} W_i$.

4 Stability analysis of the observer

In the proof of theorem, the following lemma is used:

Lemma 1. For every two matrices X and Y with appropriate dimensions, the following property holds:

$$X^{\mathrm{T}} Y + Y^{\mathrm{T}} X \le \beta X^{\mathrm{T}} X + \beta^{-1} Y^{\mathrm{T}} Y, \quad \beta > 0 \tag{10}$$

Lemma 2[26]**.** For a symmetric matrix M partitioned into blocks:

$$M = \begin{bmatrix} Z & S \\ S^{\mathrm{T}} & R \end{bmatrix} \tag{11}$$

where both Z and R are symmetric and square. Assume that $R > 0$, then the following properties are equivalent:

$$M > 0$$
$$Z - SR^{-1} S^{\mathrm{T}} > 0.$$

In order to prove the stability of the observer, let us define the state and output errors:

$$e = x - \hat{x} \tag{12}$$

$$r = y - \hat{y}. \tag{13}$$

The dynamics of estimation error is given by

$$\dot{e} = \dot{x} - \dot{\hat{x}} \tag{14}$$

$$\dot{e} = \sum_{i=1}^{M} \mu_i(z)((A_i - L_iC)e + \Delta A_i x - \alpha_i) \tag{15}$$

defining the sliding surface

$$s = r = Ce = C(x - \hat{x}). \tag{16}$$

Considering the following candidate Lyapunov function:

$$V = s^{\mathrm{T}} Q s \tag{17}$$

where Q is a symmetric positive definite matrix and $\mathrm{rank}(Q) = p$. Using (15) and (16), its time derivative is given by

$$\begin{aligned}\dot{V} &= \dot{s}^{\mathrm{T}} Q s + s Q \dot{s}\\ \dot{V} &= \sum_{i=1}^{M} \mu_i(z)(e^{\mathrm{T}}(A_i - L_iC)^{\mathrm{T}} C^{\mathrm{T}} Q C e +\\ &x^{\mathrm{T}} \Delta A_i{}^{\mathrm{T}} C^{\mathrm{T}} Q C e - \alpha_i^{\mathrm{T}} C^{\mathrm{T}} Q C e +\\ &e^{\mathrm{T}} C^{\mathrm{T}} Q C (A_i - L_iC)e +\\ &e^{\mathrm{T}} C^{\mathrm{T}} Q C \Delta A_i x - e^{\mathrm{T}} C^{\mathrm{T}} Q C \alpha_i).\end{aligned} \tag{18}$$

Supposing that $P = C^{\mathrm{T}} Q C$, one can obtain:

$$\begin{aligned}\dot{V} = &\sum_{i=1}^{M} \mu_i(z)(e^{\mathrm{T}}((A_i - L_iC)^{\mathrm{T}} P + P(A_i - L_iC))e +\\ &x^{\mathrm{T}} \Delta A_i{}^{\mathrm{T}} P e + e^{\mathrm{T}} P \Delta A_i x - 2e^{\mathrm{T}} P \alpha_i).\end{aligned} \tag{19}$$

Applying the Lemmas 1 and 2, one can obtain:

$$\begin{aligned}&x^{\mathrm{T}} \Delta A_i^{\mathrm{T}} P e + e^{\mathrm{T}} P \Delta A_i x = x^{\mathrm{T}} (M_i^a F_a N_i^a)^{\mathrm{T}} P e +\\ &e^{\mathrm{T}} P M_i^a F_a N_i^a x = e^{\mathrm{T}} P M_i^a F_a N_i^a x +\\ &x^{\mathrm{T}} N_i^{a\,\mathrm{T}} F_a^{\mathrm{T}} M_i^{a\,\mathrm{T}} P e \le \varepsilon e^{\mathrm{T}} P M_i^a M_i^{a\,\mathrm{T}} P e +\\ &\varepsilon^{-1} x^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a x.\end{aligned} \tag{20}$$

So, the derivative of the Lyapunov function can be obtained as follows:

$$\begin{aligned}\dot{V} = &\sum_{i=1}^{M} \mu_i(z)(e^{\mathrm{T}}((A_i - L_iC)^{\mathrm{T}} P + P(A_i - L_iC) +\\ &\varepsilon P M_i^a M_i^{a\,\mathrm{T}} P)e + \varepsilon^{-1} x^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a x - 2e^{\mathrm{T}} P \alpha_i).\end{aligned} \tag{21}$$

Replacing the state by its new expression:

$$x = \hat{x} + e \tag{22}$$

one can obtain the following equality:

$$\begin{aligned}&\varepsilon^{-1} x^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a x = \varepsilon^{-1} (\hat{x} + e)^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a (\hat{x} + e) =\\ &\varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} + \varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a e +\\ &\varepsilon^{-1} e^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} + \varepsilon^{-1} e^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a e.\end{aligned} \tag{23}$$

Now, using the Lemma 1 with $\beta = 1$, this leads to obtain:

$$\begin{aligned}&\varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} + \varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a e +\\ &\varepsilon^{-1} e^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} + \varepsilon^{-1} e^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a e \le\\ &2\varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} + 2\varepsilon^{-1} e^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a e.\end{aligned} \tag{24}$$

As a consequence, a new inequality can be obtained as

$$\begin{aligned}\dot{V} \le &\sum_{i=1}^{M} \mu_i(z)(e^{\mathrm{T}}((A_i - L_iC)^{\mathrm{T}} P + P(A_i - L_iC) +\\ &\varepsilon P M_i^a M_i^{a\,\mathrm{T}} P + 2\varepsilon^{-1} N_i^{a\,\mathrm{T}} N_i^a)e + 2\varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x}\\ &- 2e^{\mathrm{T}} P \alpha_i).\end{aligned} \tag{25}$$

According to the output error, two cases can be investigated:

Case 1. If $r \ne 0$, then

$$2\varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} - 2e^{\mathrm{T}} P \alpha_i = 0 \tag{26}$$

$$\alpha_i = \varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} P^{-1} C^{\mathrm{T}} \frac{s}{\parallel s \parallel^2}. \tag{27}$$

So, the inequality becomes:

$$\begin{aligned}\dot{V} \le &\sum_{i=1}^{M} \mu_i(z)(e^{\mathrm{T}}((A_i - L_iC)^{\mathrm{T}} P + P(A_i - L_iC) +\\ &\varepsilon P M_i^a M_i^{a\,\mathrm{T}} P + 2\varepsilon^{-1} N_i^{a\,\mathrm{T}} N_i^a)e).\end{aligned} \tag{28}$$

Case 2. If $r = 0$, then $\alpha_i = 0$,

$$\begin{aligned}\dot{V} \le &\sum_{i=1}^{M} \mu_i(z)(e^{\mathrm{T}}((A_i - L_iC)^{\mathrm{T}} P + P(A_i - L_iC) +\\ &\varepsilon P M_i^a M_i^{a\,\mathrm{T}} P + 2\varepsilon^{-1} N_i^{a\,\mathrm{T}} N_i^a)e).\end{aligned} \tag{29}$$

As a conclusion, the system converges asymptotically to zero if and only if:

$$\begin{aligned}&(A_i - L_iC)^{\mathrm{T}} P + P(A_i - L_iC) +\\ &\varepsilon P M_i^a M_i^{a\,\mathrm{T}} P + 2\varepsilon^{-1} N_i^{a\,\mathrm{T}} N_i^a < 0.\end{aligned} \tag{30}$$

Supposing $W_i = PL_i$, we obtain

$$\begin{aligned}&A_i^{\mathrm{T}} P + P A_i^{\mathrm{T}} - C^{\mathrm{T}} W_i^{\mathrm{T}} - W_i^{\mathrm{T}} C +\\ &\varepsilon P M_i^a M_i^{a\,\mathrm{T}} P + 2\varepsilon^{-1} N_i^{a\,\mathrm{T}} N_i^a < 0.\end{aligned} \tag{31}$$

By using Schur Complement, cited in Lemma 2, the inequality can be written in the LMI form:

$$\begin{bmatrix} A_i^{\mathrm{T}} P + P A_i - C^{\mathrm{T}} W_i{}^{\mathrm{T}} + W_i C + 2\varepsilon^{-1} N_i^{a\,\mathrm{T}} N_i^a & P M_i^a \\ M_i^{a\,\mathrm{T}} P & -\varepsilon^{-1} I \end{bmatrix} < 0. \tag{32}$$

After determining P and W_i, the values of estimation gains are given by

$$L_i = P^{-1} W_i.$$

The sliding mode gains take the following expressions:

$$\begin{cases} \text{If } r \ne 0, \text{then } \alpha_i = \varepsilon^{-1} \hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} P^{-1} C^{\mathrm{T}} \dfrac{s}{\parallel s \parallel^2} \\ \text{If } r = 0, \text{then } \alpha_i = 0. \end{cases} \tag{33}$$

By using $CC^{\mathrm{T}} = I_p$, the value of Q can be deduced from P as follows:

$$\begin{aligned}P &= C^{\mathrm{T}} Q C\\ C P C^{\mathrm{T}} &= C C^{\mathrm{T}} Q C C^{\mathrm{T}}\\ Q &= C P C^{\mathrm{T}}.\end{aligned}$$

In conclusion, starting with $V = s^{\mathrm{T}} Q s > 0$, where $Q^{\mathrm{T}} = Q > 0$, we obtain $\dot{V} < 0$. So, the estimation error converges asymptotically to zero.

5 Numerical example

5.1 Description of denitrification process

In this section, the proposed design approach is applied to a model of denitrification process. This process is a bacterial culture of Pseudomonas denitrificas where biomass, X, begins consuming acetic acid, S_3, and the nitrate, S_1, and rejects nitrites, S_2. Then it continues to use acetic acid and nitrite product.

The process is described by the following model[27]:

$$\begin{cases} \dot{S}_1 = -y_{11}\mu_1 X + D(S_{1in} - S_1) \\ \dot{S}_2 = (y_{12}\mu_1 - y_{22}\mu_2)X + D(S_{2in} - S_2) \\ \dot{S}_3 = -(y_{13}\mu_1 - y_{23}\mu_2)X + D(S_{3in} - S_3) \\ \dot{X} = (\mu_1 + \mu_2)X - k_d X - DX \end{cases} \tag{34}$$

where S_1, S_2, S_3 and X are respectively the concentrations of the respective species. S_{1in}, S_{2in} S_{3in} are the respective supply of S_1, S_2 and S_3 concentrations. k_d is the mortality rate of the microorganisms. D is the dilution rate, and the y_{ij} denote yield coefficients and finally μ_1 and μ_2 are respectively the specific growth rates of the biomass on the acetic acid and nitrite and have the following expressions:

$$\begin{aligned} \mu_1 &= \mu_{1\max} \frac{S_3}{(S_3 + k_{S_3})} \frac{S_1}{(S_1 + k_{S_1})} \\ \mu_2 &= \mu_{2\max} \frac{S_3}{(S_3 + k_{S_3})} \frac{S_2}{(S_2 + k_{S_2})}. \end{aligned}$$

Let′s define the state vector x, the input vector u and the output vector y of the system: $x = [S_1 \quad S_2 \quad S_3 \quad X]^{\mathrm{T}}$, $u = [S_{1in} \quad S_{2in} \quad S_{3in}]^{\mathrm{T}}$ and $y = [S_1 \quad S_2 \quad S_3]^{\mathrm{T}}$.

5.2 Multiple model form

The multiple model is adopted as an approach in order to design the observer for state estimation. We employ the procedure presented in [3, 4]. Considering the process, we define the following nonlinearities as the premise variables:

$$z_1 = D \tag{35}$$

$$z_2 = \frac{S_3}{(S_3 + k_{S_3})} \frac{S_1}{(S_1 + k_{S_1})} \tag{36}$$

$$z_3 = \frac{S_3}{(S_3 + k_{S_3})} \frac{S_2}{(S_2 + k_{S_2})}. \tag{37}$$

The nonlinear model can be written in the following quasi-linear parameter varying (LPV) form:

$$\dot{x} = A(z)x + B(z)u \tag{38}$$

with matrix $A(z)$ and $B(z)$ are expressed by using the premise variables:

$$A(z) = \begin{bmatrix} -z_1 & 0 & 0 & a_{14} \\ 0 & -z_1 & 0 & a_{24} \\ 0 & 0 & -z_1 & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \tag{39}$$

$$B(z) = \begin{bmatrix} -z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & -z_1 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

where

$$\begin{aligned} a_{14} &= -y_{11}\mu_{1\max} z_2 \\ a_{24} &= y_{12}\mu_{1\max} z_2 - y_{22}\mu_{2\max} z_3 \\ a_{34} &= -y_{13}\mu_{1\max} z_2 - y_{23}\mu_{2\max} z_3 \\ a_{44} &= -z_1 + \mu_{1\max} z_2 + \mu_{2\max} z_3 - k_d. \end{aligned} \tag{41}$$

Each one of the premise variables can be expressed as

$$z_j = F_{j1} z_{j1} + F_{j2} z_{j2}, \ \text{for } j = 1, 2, 3 \tag{42}$$

where

$$\begin{aligned} z_{j1} &= \max\{z_j\} \\ z_{j2} &= \min\{z_j\} \\ F_{j1} &= \frac{z_j - z_{j2}}{z_{j1} - z_{j2}} \\ F_{j2} &= \frac{z_{j1} - z_j}{z_{j1} - z_{j2}}. \end{aligned} \tag{43}$$

The constant matrices A_i and B_i defining the 8 submodels are determined by using the matrices $A(z)$ and $B(z)$ and $z(j, i)$, $i = 1, 2$ and $j = 1, 2, 3$.

$$\begin{aligned} &A_1 = A(z_{11}, z_{21}, z_{31}), \ A_2 = A(z_{11}, z_{21}, z_{32}) \\ &A_3 = A(z_{11}, z_{22}, z_{31}), \ A_4 = A(z_{11}, z_{22}, z_{32}) \\ &A_5 = A(z_{12}, z_{21}, z_{31}), \ A_6 = A(z_{12}, z_{21}, z_{32}) \\ &A_7 = A(z_{12}, z_{22}, z_{31}), \ A_8 = A(z_{12}, z_{22}, z_{32}) \\ &B_1 = B(z_{11}), \ B_2 = B_3 = B_4 = B_1 \\ &B_5 = B(z_{12}), \ B_6 = B_7 = B_8 = B_5. \end{aligned} \tag{44}$$

The activation functions have the following expressions:

$$\begin{aligned} \mu_1(z) &= F_{11}F_{21}F_{31} \\ \mu_2(z) &= F_{11}F_{21}F_{32} \\ \mu_3(z) &= F_{11}F_{22}F_{31} \\ \mu_4(z) &= F_{11}F_{22}F_{32} \\ \mu_5(z) &= F_{12}F_{21}F_{31} \\ \mu_6(z) &= F_{12}F_{21}F_{32} \\ \mu_7(z) &= F_{12}F_{22}F_{31} \\ \mu_8(z) &= F_{12}F_{22}F_{32}. \end{aligned} \tag{45}$$

Finally the nonlinear model can be written in multiple model form:

$$\begin{cases} \dot{x} = \displaystyle\sum_{i=1}^{8} \mu_i(z)(A_i x + B_i u) \\ y = Cx. \end{cases} \tag{46}$$

The multiple model used for denitrification process is modified in order to take account of the uncertainties affecting the parameters $\mu_{1\max}$ and $\mu_{2\max}$. These parameters appear in the coefficients a_{14}, a_{24}, a_{34}, a_{44}. So, the structure of ΔA_i will have the following form:

$$\Delta A_i = \begin{bmatrix} 0 & 0 & 0 & \Delta a_{14i} \\ 0 & 0 & 0 & \Delta a_{24i} \\ 0 & 0 & 0 & \Delta a_{34i} \\ 0 & 0 & 0 & \Delta a_{44i} \end{bmatrix} \tag{47}$$

$$\begin{aligned} \Delta a_{14i} &= -y_{11}\Delta\mu_{1max} z_{2i} \\ \Delta a_{24i} &= y_{12}\Delta\mu_{1max} z_{2i} - y_{22}\Delta\mu_{2max} z_{3i} \\ \Delta a_{34i} &= -y_{13}\Delta\mu_{1max} z_{2i} - y_{23}\Delta\mu_{2max} z_{3i} \\ \Delta a_{44i} &= \Delta\mu_{1max} z_{2i} + \Delta\mu_{2max} z_{3i}. \end{aligned} \tag{48}$$

As we have supposed that the uncertainties are of norm bounded type: $\Delta A_i = M_i^a F_a N_i^a$ where

$$M_i^a = \begin{bmatrix} -y_{11} & 0 \\ y_{12} z_{2i} & -y_{22} z_{3i} \\ -y_{13} z_{2i} & -y_{23} z_{3i} \\ z_{2i} & z_{3i} \end{bmatrix}, \ F_a = \begin{bmatrix} \Delta\mu_{1\max} \\ \Delta\mu_{2\max} \end{bmatrix}$$

$$N_i^a = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}. \tag{49}$$

The nonlinear model of the denitrification process and the proposed observer are simulated under the initial conditions and parameters values given, respectively, by Tables 1 and 2.

Table 1 Initial conditions

Variables	Values
$S_1(0)$	0.6 g/L
$S_2(0)$	0 g/L
$S_3(0)$	2.77 g/L
$X(0)$	0.15 g/L
$\hat{S}_1(0)$	0.6 g/L
$\hat{S}_2(0)$	0 g/L
$\hat{S}_3(0)$	2.77 g/L
$\hat{X}(0)$	0.2 g/L

Table 2 Parameters values

Parameters	Values
y_{11}	6.2
y_{12}	3.3
y_{22}	1.2
y_{13}	1.1
y_{23}	1.6
$\mu_{1\max 0}$	0.17 h^{-1}
$\mu_{2\max 0}$	0.085 h^{-1}
k_{S_1}	0.05 g/L
k_{S_2}	0.07 g/L
k_{S_3}	0.1 g/L
k_d	0.025 h^{-1}

The variations of some parameters are shown in Fig. 1.

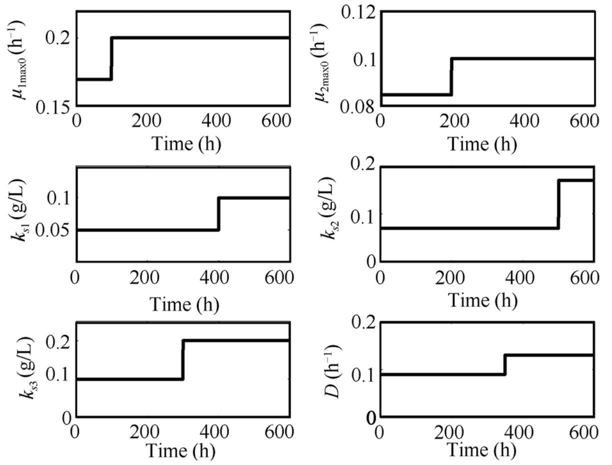


Fig. 1 Parameters variations

The uncertainties $\Delta\mu_{1\max}$ and $\Delta\mu_{2\max}$ are given as follow:

$$\begin{cases} \Delta\mu_{1\max} = 0.05 \times \sin(\dfrac{2\pi t}{50}) \\ \Delta\mu_{2\max} = 0.02 \times \sin(\dfrac{2\pi t}{50}). \end{cases} \tag{50}$$

The evolution of the known inputs of the denitrification process are shown by Fig. 2.

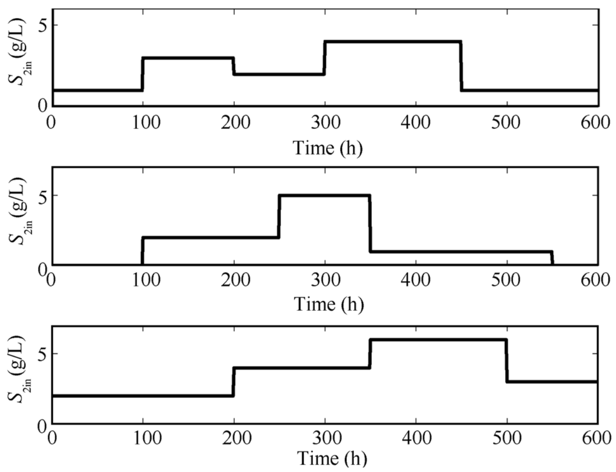


Fig. 2 Evolution of the known inputs

5.3 Luenberger observer

The resolution of the inequalities in (6) leads to obtain the matrices P and L_i:

$$P = \begin{bmatrix} 1.3769 & 0 & 0 & 0.6245 \\ 0 & 1.3769 & 0 & -0.0206 \\ 0 & 0 & 1.3769 & 0.5265 \\ 0.6245 & -0.0206 & 0.5265 & 2.0289 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} -0.7252 & -34.3741 & 10.8032 \\ 28.3965 & 0.5580 & 4.3824 \\ -9.3101 & -9.5083 & 1.7835 \\ 2.3929 & 13.2588 & -3.7223 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 1.326\,2 & -78.905\,8 & -7.289\,1 \\ 65.398\,9 & 0.803\,7 & 8.535\,9 \\ 7.050\,2 & -19.863\,2 & -0.528\,4 \\ -2.130\,4 & 29.711\,4 & 2.323\,1 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} 4.445\,2 & 21.629\,0 & -29.755\,7 \\ -18.100\,0 & 0.239\,4 & -1.869\,0 \\ 28.713\,0 & 5.105\,9 & -3.423\,6 \\ -9.007\,6 & -8.077\,9 & 9.894\,1 \end{bmatrix}$$

$$L_4 = \begin{bmatrix} 2.091\,1 & 37.702\,5 & -11.052\,2 \\ -32.073\,7 & 0.172\,8 & 1.410\,3 \\ 10.878\,0 & 3.437\,8 & -1.017\,6 \\ -3.816\,8 & -12.536\,9 & 3.602\,3 \end{bmatrix}$$

$$L_5 = \begin{bmatrix} 2.425\,9 & 23.495\,0 & -11.340\,5 \\ -20.391\,0 & 0.325\,6 & 2.862\,8 \\ 11.390\,1 & -0.136\,3 & -0.947\,2 \\ -4.400\,7 & -6.989\,0 & 3.601\,4 \end{bmatrix}$$

$$L_6 = \begin{bmatrix} 0.702\,3 & -30.556\,1 & -0.200\,3 \\ 25.716\,9 & 0.589\,1 & -0.302\,2 \\ 0.510\,8 & -3.773\,3 & 0.498 \\ -0.600\,3 & 10.650\,2 & -0.178\,5 \end{bmatrix}$$

$$L_7 = \begin{bmatrix} 0.341\,3 & 0.708\,8 & 1.254\,0 \\ -0.439\,4 & 0.421\,2 & -0.842\,2 \\ -1.024\,7 & 1.056\,8 & 0.686\,4 \\ 0.195\,5 & -0.587\,5 & -0.670\,4 \end{bmatrix}$$

$$L_8 = \begin{bmatrix} 0.496\,0 & -0.137\,0 & 0.101\,2 \\ 0.133\,9 & 0.430\,0 & -0.008\,8 \\ 0.047\,3 & 0.005\,0 & 0.508\,3 \\ -0.145\,4 & 0.002\,0 & -0.204\,7 \end{bmatrix}.$$

The system and the observer are simulated with the presence of the uncertainties described by (50). Figs. 3−6 show the Luenberger estimator tracking the real state. But, it is mentioned that the accuracy is bad because the observer presents a non robust behavior. Moreover, it should be highlighted that the non robustness appears very clear in

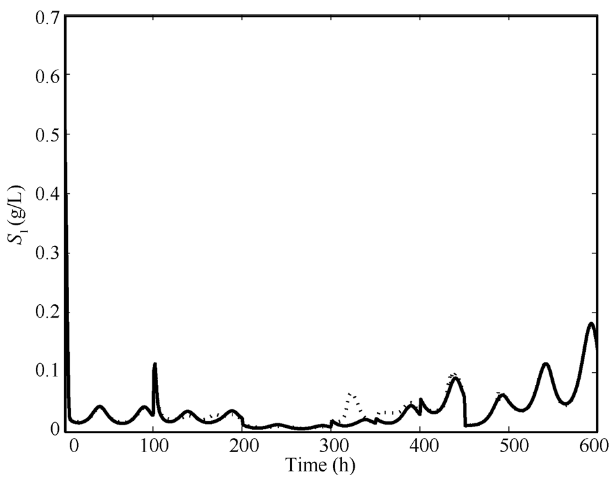


Fig. 3 Evolution of S_1: Real value (solid) and estimated value (dotted)

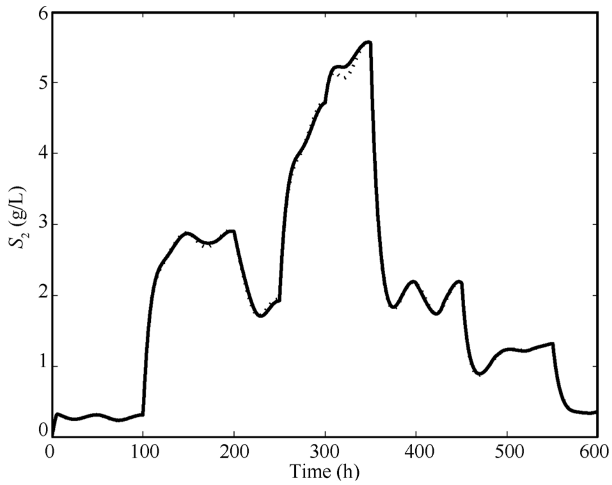


Fig. 4 Evolution of S_2: Real value (solid) and estimated value (dotted)

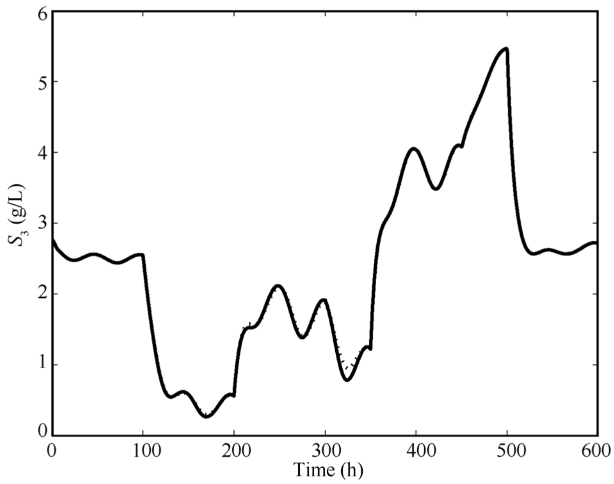


Fig. 5 Evolution of S_3: Real value (solid) and estimated value (dotted)

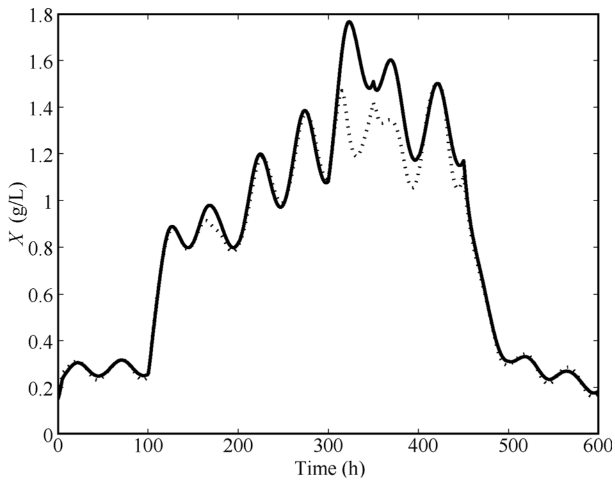


Fig. 6 Evolution of X: Real value (solid) and estimated value (dotted)

Fig. 5 because it corresponds to the unmeasured variable.

5.4 Sliding mode observer

Solving the inequalities described in (8) leads to obtain the numeric values of the matrices P, Q and L_i and the scalar ε:

$$P = \begin{bmatrix} 1.0874 & 1.4427 & 0.1906 & 2.0136 \\ 1.4427 & 2.0491 & -0.0427 & 2.2236 \\ 0.1906 & -0.0427 & 3.1531 & 4.7511 \\ 2.0136 & 2.2236 & 4.7511 & 10.3426 \end{bmatrix}$$

$$\varepsilon = 3.0410$$

$$L_1 = 10^3 \begin{bmatrix} -0.7252 & -34.3741 & 10.8032 \\ 0.0079 & 0.0663 & 0.7673 \\ 0.0379 & -0.3873 & -0.0280 \\ -0.0251 & 0.2629 & 0.0719 \end{bmatrix}$$

$$L_2 = 10^3 \begin{bmatrix} -0.6291 & -0.9309 & 1.0621 \\ 0.3696 & 0.5799 & -0.6273 \\ -0.0896 & -0.0285 & 0.1177 \\ 0.0841 & 0.0696 & -0.1261 \end{bmatrix}$$

$$L_3 = \begin{bmatrix} -95.9986 & -449.0547 & -373.7676 \\ 158.0132 & 278.7043 & 152.0034 \\ 124.3318 & -15.9638 & -114.2906 \\ -72.4568 & 34.7669 & 92.4422 \end{bmatrix}$$

$$L_4 = 10^3 \begin{bmatrix} -0.5643 & -1.4401 & -0.5381 \\ 0.4208 & 0.6655 & 0.3393 \\ 0.0422 & -0.3853 & -0.0333 \\ -0.0001 & 0.3142 & 0.0470 \end{bmatrix}$$

$$L_5 = \begin{bmatrix} 245.1670 & 330.3931 & -109.9807 \\ -113.6146 & -120.0244 & 21.6150 \\ 73.8630 & 134.9207 & -57.7524 \\ -57.2677 & -100.5330 & 43.2144 \end{bmatrix}$$

$$L_6 = \begin{bmatrix} -111.9137 & -431.2422 & -250.2063 \\ 101.7201 & 196.6560 & 154.9055 \\ 31.8156 & -120.2406 & -17.7648 \\ -14.7305 & 96.8756 & 23.4912 \end{bmatrix}$$

$$L_7 = \begin{bmatrix} -22.8856 & -20.7668 & 334.8536 \\ 11.2114 & 59.7660 & -205.3840 \\ -8.3108 & 67.1396 & 29.9431 \\ 5.8307 & -39.6865 & -34.8679 \end{bmatrix}$$

$$L_8 = \begin{bmatrix} -38.8176 & -223.4972 & -94.3856 \\ 42.9070 & 109.0936 & 152.0034 \\ 20.1841 & -52.3846 & -114.2906 \\ -10.9715 & 44.0847 & 7.1643 \end{bmatrix}.$$

Each sliding mode gain α_i is modified as follows:

$$\begin{cases} \alpha_i = \varepsilon^{-1}\hat{x}^{\mathrm{T}} N_i^{a\,\mathrm{T}} N_i^a \hat{x} P^{-1} C^{\mathrm{T}} \dfrac{s}{\parallel s \parallel^2 + \delta}, & \text{if } \ s \neq 0 \\ \alpha_i = 0, \quad \text{if } \ s = 0. \end{cases} \tag{51}$$

The parameter δ is a small scalar and it is used in order to smooth out the discontinuity.

The simulation of the system and the sliding mode observer took place with the same conditions of the observer of Section 5.3. Figs. 7−10 show the evolution of the four

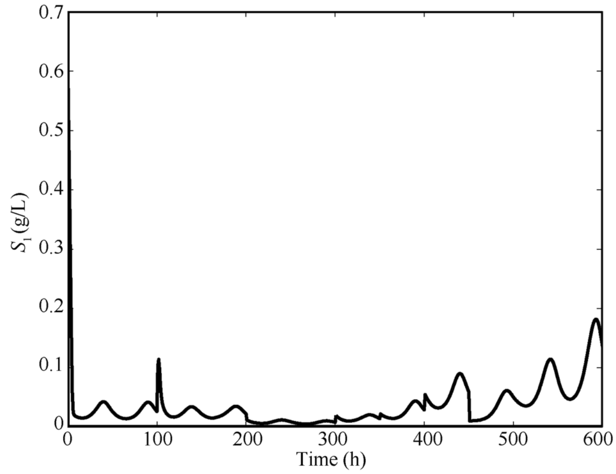


Fig. 7 Evolution of S_1: Real value (solid) and estimated value (dotted)

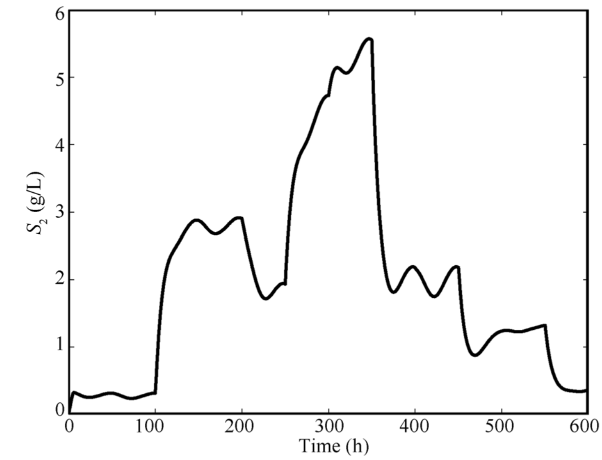


Fig. 8 Evolution of S_2: Real value (solid) and estimated value (dotted)

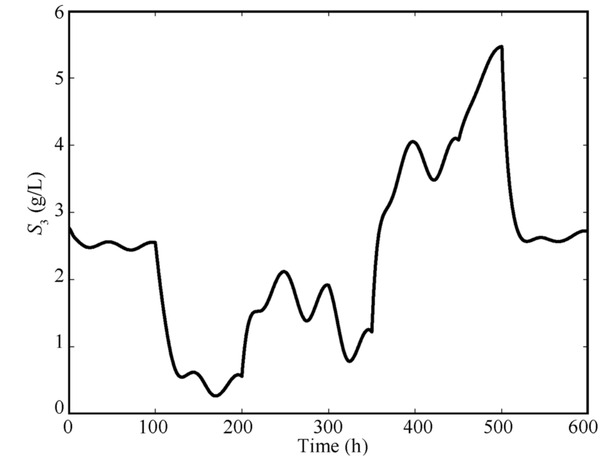


Fig. 9 Evolution of S_3: Real value (solid) and estimated value (dotted)

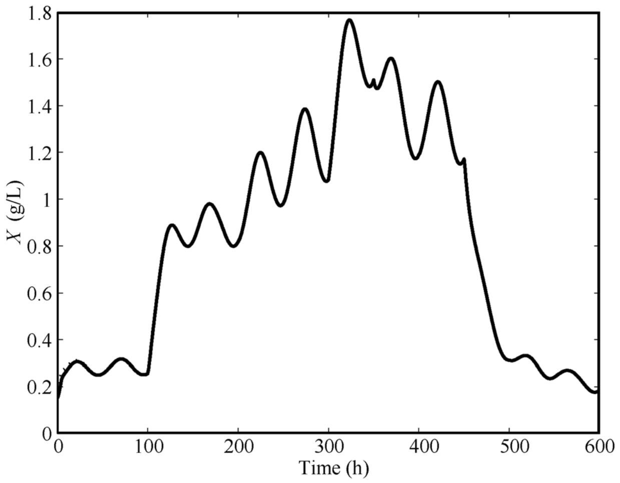


Fig. 10 Evolution of X: Real value (solid) and estimated value (dotted)

state variables of denitrification process. It is clear that the estimated values converge to the real ones with an improved accuracy which lead to say that the proposed observer presents a robust behavior via parametric variations in comparison with the observer of Section 5.3.

In Fig. 11, additional curves of estimation errors are presented in order to support the previous figures of state variables in showing the performance of the proposed observer. It is noticed that e_1, e_2, e_3 and e_4 are the estimation errors of S_1, S_2, S_3 and X, respectively.

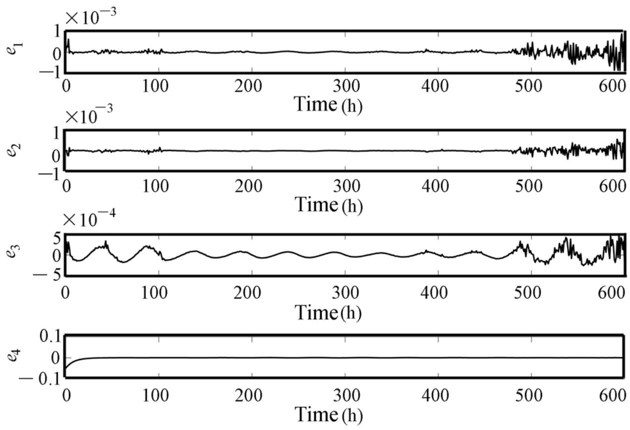


Fig. 11 Evolution of estimation errors

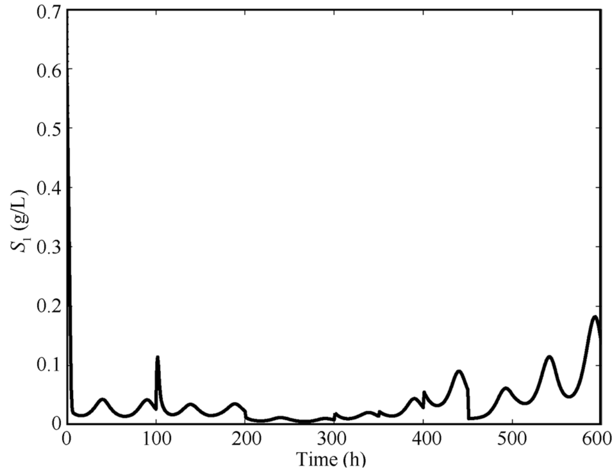
Another simulation is performed with different initial conditions for the observer:

$$\hat{S}_1(0) = 0.7\,\text{g/L}, \quad \hat{S}_2(0) = 0.01\,\text{g/L},$$
$$\hat{S}_3(0) = 2.9\,\text{g/L}, \quad \hat{X}(0) = 0.2\,\text{g/L}.$$

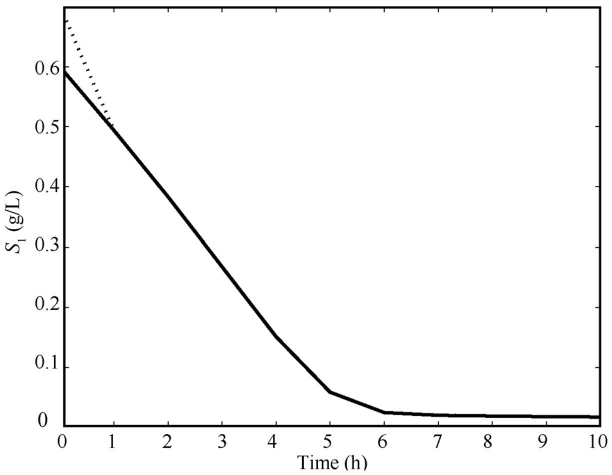
The results are presented in Figs. 12–15 which show the evolution of the state variables, for both the system and the observer, and provide a zoom captured near the initial conditions in order to check the behavior of the estimator in the transient phase.

6 Conclusions

A robust sliding mode observer based on multiple models is developed for a nonlinear uncertain system. The design of observer gain is based on LMI′s tools that guarantee the asymptotic convergence of estimation error. A sliding mode

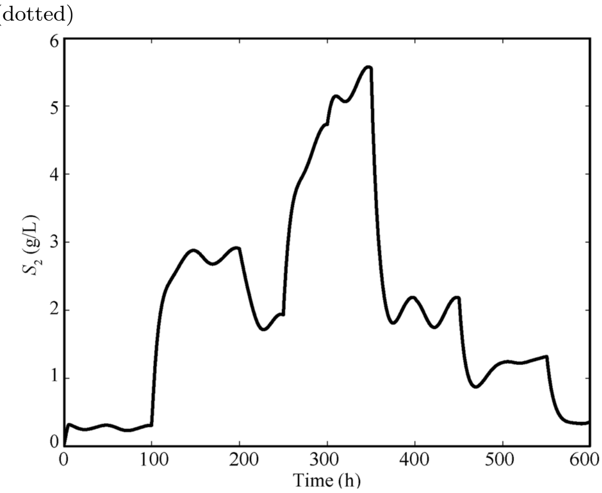


(a) Evolution of S_1

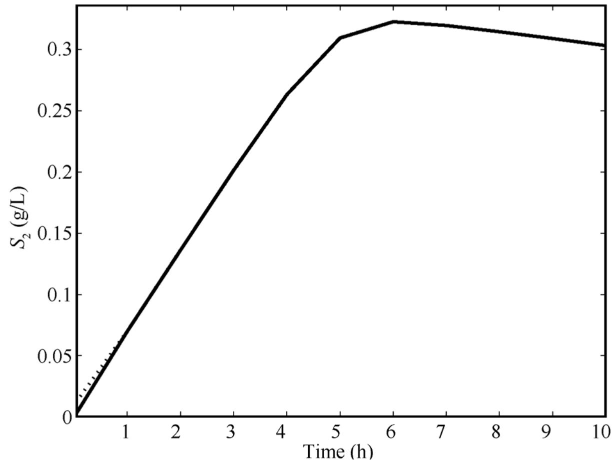


(b) View near initial conditions of S_1

Fig. 12 Curves of S_1: Real value (solid) and estimated value (dotted)

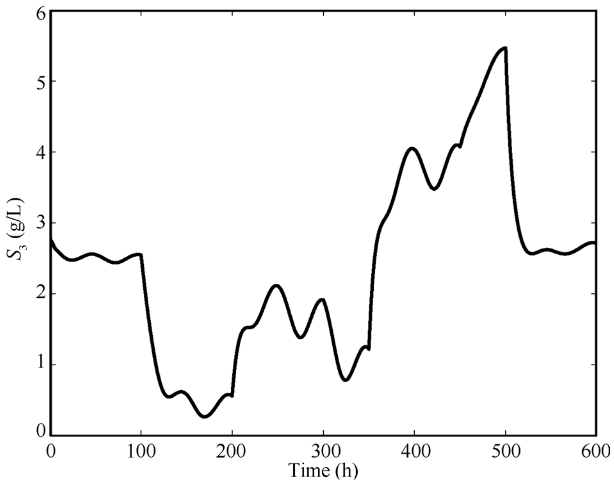


(a) Evolution of S_2

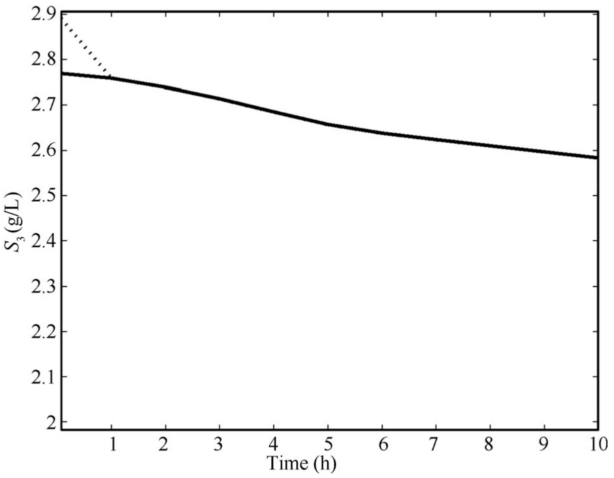


(b) View near initial conditions of S_2

Fig. 13 Curves of S_2: Real value (solid) and estimated value (dotted)



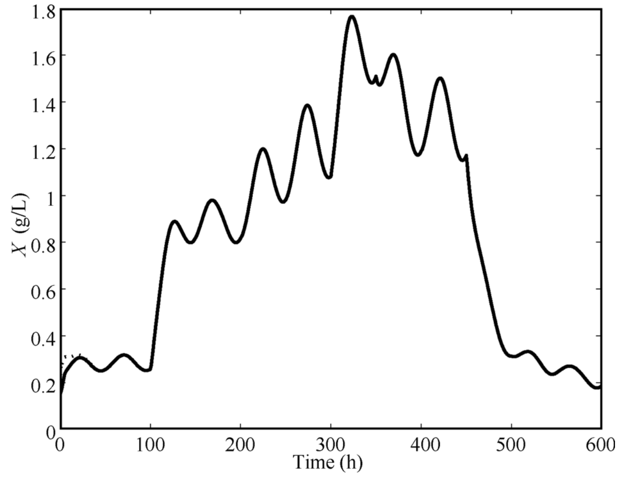
(a) Evolution of S_3



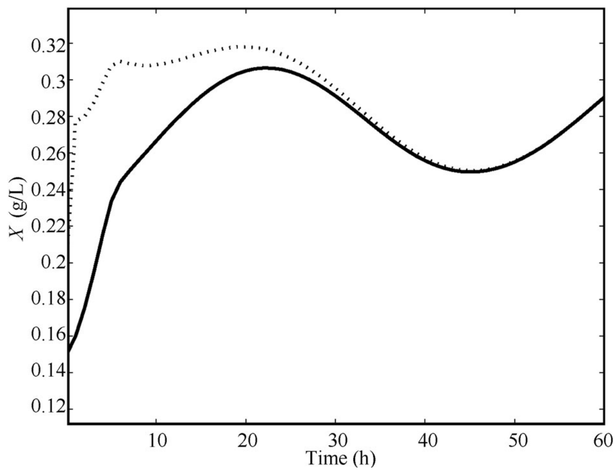
(b) View near initial conditions of S_3

Fig. 14 Curves of S_3: Real value (solid) and estimated value (dotted)

term is added in order to ensure the robustness against the uncertainties. A classic Luenberger multiples observer and the proposed multiple observer are applied to estimate the state of a denitrification process under uncertainties that affected the maximum of specific growth rates. The simulation results demonstrate the effectiveness of the proposed multiple observer.



(a) Evolution of X



(b) View near initial conditions of X

Fig. 15 Curves of X: Real value (solid) and estimated value (dotted)

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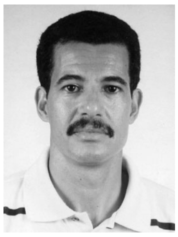

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