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Bifurcation Control for a Duffing Oscillator with Delayed Velocity Feedback

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Abstract: In this paper, a Duffing oscillator model with delayed velocity feedback is considered. Applying the time delayed feedback control method and delayed differential equation theory, we establish some criteria which ensure the stability and the existence of Hopf bifurcation of the model. By choosing the delay as bifurcation parameter and analyzing the associated characteristic equation, the existence of bifurcation parameter point is determined. We found that if the time delay is chosen as a bifurcation parameter, Hopf bifurcation occurs when the time delay is changed through a series of critical values. Some numerical simulations show that the designed feedback controllers not only delay the onset of Hopf bifurcation, but also enlarge the stability region for the model.

Keywords: Hopf bifurcation, delay, stability, Duffing oscillator, time delayed feedback controller.

1 Introduction

It is well known that the dynamical behaviors of Duffing oscillator with delayed feedback have been one of the dominant themes in nonlinear dynamics due to their universal existence and importance. Over the past decade, there has been considerable interest in investigating the Duffing oscillator′s dynamics in various fields of mathematical and engineering fields. Many excellent and interesting results have been reported[1−19]. In 2009, Hu and Wang[17] introduced and discussed the following Duffing oscillator with delayed velocity feedback

$$\ddot{x}(t) + c\dot{x}(t) + \omega_0^2 x(t) + \mu x^3(t) = \nu\dot{x}(t-\tau) \tag{1}$$

where $x \in \mathbf{R}$, and the system parameters yield $c \geq 0, \omega_0 > 0, \mu > 0, c-\nu > 0$. Applying the singular perturbation methods, Hu and Wang[17] investigated the local Hopf bifurcation of (1). The singular perturbation methods (e.g., the method of multiple scales, the method of averaging, the energy analysis and the pseudo-oscillator analysis, etc.) have some advantages over the normal form theory and the center manifold theorem in studying the Hopf bifurcation. In many cases, the singular perturbation methods have some merits such as easier computation and higher accurate prediction on the local dynamics of time-delay near a Hopf bifurcation point[17]. It is highlighted as very important to make the delayed Duffing oscillator stable or improve the stability of delayed Duffing oscillator by applying feedback controller. In this paper, we are concerned with two problems: one is how to design the controller to improve the stability of original system (1) when the time delay in original system (1) is equal to the time delay in controller and the parameters of the original system are given, another is how to design the controller to enlarge the stability region of original system (1) when the time delay in original system (1) is not equal to the time delay in controller and the parameters of the original system are given. Based on the analysis above, we will add the following delayed feedback controller to system (1)

$$u(t) = k[x(t) - x(t-\sigma)] \tag{2}$$

where k is a feedback control parameter. σ is time delay. Then system (1) takes the form

$$\begin{aligned}\ddot{x}(t) + c\dot{x}(t) + \omega_0^2 x(t) + \mu x^3(t) = \\ \nu\dot{x}(t-\tau) + k[x(t) - x(t-\sigma)]\end{aligned} \tag{3}$$

where $c \geq 0, \omega_0 > 0, \mu > 0, c-\nu > 0$.

In this paper, we will devote our attention to investigating the stability and the existence of Hopf bifurcation of system (3). That is to say, we shall take the delay τ as the bifurcation parameter and show that when τ passes through a certain critical value, the equilibrium loses its stability and Hopf bifurcation will take place.

This paper is organized as follows. In Section 2, the stability of the equilibrium and the existence of Hopf bifurcation at the equilibrium are studied. In Section 3, numerical simulations are carried out to illustrate the validity of the designed feedback controllers.

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2 Stability and local Hopf bifurcation

In this section, by analyzing the characteristic equation of the linearized system of system (3) at the equilibrium, we investigate the stability of the equilibrium and the existence of the local Hopf bifurcations occurring at the equilibrium.

Let $y(t) = \dot{x}(t)$, then system (3) has the following equivalent form:

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = (k - \omega_0^2)x(t) - cy(t) + \nu y(t-\tau) - \\ \qquad \mu x^3(t) - kx(t-\sigma). \end{cases} \tag{4}$$

Obviously, (4) has a unique equilibrium $E(0,0)$. Then we obtain the linearized system of (4)

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = (k - \omega_0^2)x(t) - cy(t) + \nu y(t-\tau) - \\ \qquad kx(t-\sigma) \end{cases} \tag{5}$$

whose characteristic equation is

$$\lambda^2 + c\lambda + \omega_0^2 - k - \nu\lambda e^{-\lambda\tau} + ke^{-\lambda\sigma} = 0. \tag{6}$$

To study the stability of the equilibrium $E(0,0)$ of (4) and Hopf bifurcation, it is sufficient to investigate the distribution of roots of the transcendental (6). The following Lemma that is stated in [20] is useful in studying the location of roots of the transcendental (6).

Lemma 1[20]**.** For the transcendental equation

$$\begin{aligned} &P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m}) = \\ &\lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} + \\ &\left[p_1^{(1)}\lambda^{n-1} + \cdots + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right] e^{-\lambda\tau_1} + \cdots + \\ &\left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right] e^{-\lambda\tau_m} = 0 \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \cdots, \tau_m)$ vary, the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau_1}, \cdots, e^{-\lambda\tau_m})$ in the open right half plane can change, and only a zero appears on the imaginary axis or a zero crosses the imaginary axis.

Now we consider two cases.

Case 1. $\tau = \sigma$. In this case, (6) takes the form

$$\lambda^2 + c\lambda + \omega_0^2 - k - (\nu\lambda - k)e^{-\lambda\tau} = 0. \tag{7}$$

Lemma 2. When $\tau = 0$, the equilibrium $E(0,0)$ of system (3) is asymptotically stable.

Proof. For $\tau = 0$, the characteristic (7) becomes

$$\lambda^2 + (c - \nu)\lambda + \omega_0^2 = 0. \tag{8}$$

Since $c - \nu > 0, \omega_0^2 > 0$, then both roots of (8) are negative. Thus (3) is asymptotically stable.

For $\omega > 0$, $i\omega$ is a root of (7) if and only if

$$-\omega^2 + c\omega i + \omega_0^2 - k - (\nu\omega i - k)e^{-\omega\tau i} = 0. \tag{9}$$

Separating the real and imaginary parts, we get

$$\begin{cases} k\cos\omega\tau - \nu\omega\sin\omega\tau = \omega^2 - \omega_0^2 + k \\ \nu\omega\cos\omega\tau + k\sin\omega\tau = c\omega \end{cases} \tag{10}$$

which leads to the following fourth order polynomial equation:

$$\omega^4 + (2k - 2\omega_0^2 - \nu^2 + c^2)\omega^2 + \omega_0^4 - 2k\omega_0^2 = 0. \tag{11}$$

It is easy to see that if either

$$2k - 2\omega_0^2 - \nu^2 + c^2 > 0 \tag{12}$$

or

$$\Delta = (2k - 2\omega_0^2 - \nu^2 + c^2)^2 - 4(\omega_0^4 - 2k\omega_0^2) < 0 \tag{13}$$

then (11) has no positive root. Assume that

$$\begin{cases} 2k - 2\omega_0^2 - \nu^2 + c^2 < 0 \\ \Delta = (2k - 2\omega_0^2 - \nu^2 + c^2)^2 - 4(\omega_0^4 - 2k\omega_0^2) > 0 \end{cases} \tag{14}$$

then (11) has two positive roots

$$\begin{cases} \omega_\pm = \dfrac{\sqrt{2}}{2}\Bigg[-(2k - 2\omega_0^2 - \nu^2 + c^2) \pm \\ \sqrt{(2k - 2\omega_0^2 - \nu^2 + c^2)^2 - 4(\omega_0^4 - 2k\omega_0^2)}\Bigg]^{\frac{1}{2}}. \end{cases} \tag{15}$$

Without loss of generality, we assume that (11) has two positive roots $\omega_\pm$. Then, from (10), we can obtain

$$\tau_j^\pm = \frac{1}{\omega_\pm}\arccos\left[\frac{k(\omega^2 - \omega_0^2 + k) + c\nu\omega^2}{k^2 + \nu^2\omega^2}\right] + \frac{2j\pi}{\omega_\pm} \tag{16}$$

at which (7) has a pair of purely imaginary roots $\pm i\omega_\pm$, where $j = 0, 1, \cdots$.

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of (7) satisfying $\alpha(\tau_j^\pm) = 0, \omega(\tau_j^\pm) = \omega_\pm$. Due to functional differential equation theory, for $\tau_j^\pm$, there exists $\varepsilon > 0$ such that $\lambda(\tau)$ is continuously differentiable in τ for $|\tau - \tau_j^\pm| < \varepsilon$. Then the following transversality condition holds. □

Lemma 3. If (14) is satisfied, then

$$\frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}\Big|_{\tau=\tau_j^+} > 0, \ \frac{\mathrm{dRe}\lambda(\tau)}{\mathrm{d}\tau}\Big|_{\tau=\tau_j^-} < 0. \tag{17}$$

Proof. Differentiating the (7) with respect to τ leads to

$$\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1} = -\frac{2\lambda + c - \nu e^{-\lambda\tau}}{\lambda(\nu\lambda - k)e^{-\lambda\tau}} - \frac{\tau}{\lambda} = -\frac{(2\lambda + c)e^{\lambda\tau} - \nu}{\lambda(\nu\lambda - k)} - \frac{\tau}{\lambda}.$$

Then,

$$\begin{aligned} &\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_j^\pm} = \\ &-\mathrm{Re}\left[\frac{(2\omega_\pm i + c)e^{\omega_\pm i\tau_j^\pm} - \nu}{\omega_\pm i(\nu\omega_\pm i - k)}\right] - \mathrm{Re}\left[\frac{\tau}{\omega_\pm i}\right] = \\ &-\mathrm{Re}\left[\frac{(2\omega_\pm i + c)(\cos\omega_\pm\tau_j^\pm + i\sin\omega_\pm\tau_j^\pm) - \nu}{\omega_\pm i(\nu\omega_\pm i - k)}\right] = \\ &\frac{\varrho}{(\nu\omega_\pm^2)^2 + (k\omega_\pm)^2} \end{aligned} \tag{18}$$

where $\varrho = (c^2\omega_\pm^2 + 2k\omega_\pm^2)\cos\omega_\pm\tau_j^\pm + (ck\omega_\pm - 2c\omega_\pm^3) \times \sin\omega_\pm\tau_j^\pm - c\nu\omega_\pm^2$. By (10), we get

$$\begin{cases} \cos\omega_\pm\tau_j^\pm = \dfrac{k(\omega_\pm^2 - \omega_0^2 + k) + c\nu\omega_\pm^2}{k^2 + \nu^2\omega_\pm^2} \\ \sin\omega_\pm\tau_j^\pm = \dfrac{ck\omega_\pm - (\omega_\pm^2 - \omega_0^2 + k)\nu\omega_\pm}{k^2 + \nu^2\omega_\pm^2}. \end{cases} \tag{19}$$

In view of (18) and (19), we have

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_j^\pm} = \frac{\omega_\pm^2}{s^2\omega_\pm^4 + k^2\omega_\pm^2} \times$$

$$\left\{\sqrt{(2k - 2\omega_0^2 - \nu^2 + c^2)^2 - 4(\omega_0^4 - 2k\omega_0^2)}\right\}. \tag{20}$$

Thus

$$\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_j^+} > 0,\ \mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau_j^-} < 0. \tag{21}$$

□

From Lemmas 1–3, we can obtain the following result about the distribution of the characteristic roots of (7).

Lemma 4. Let $\omega_\pm, \tau_j^\pm (j = 0, 1, 2, \cdots)$ be defined by (15) and (16), respectively.

1) If (12) or (13) holds, then all the roots (7) have negative real parts for all $\tau \geq 0$;

2) If (14) holds and $\tau = \tau_j^+ (\tau = \tau_j^-$, respectively), then (7) has a pair of imaginary roots $\pm\omega_+\mathrm{i}$ ($\pm\omega_-\mathrm{i}$, respectively). Furthermore, if there is a positive integer k such that

$$\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup .... \cup (\tau_{k-1}^-, \tau_k^+)$$

then all roots of (7) have negative real parts; when $\tau = \tau_j^+ j = 0, 1, 2, \cdots, k$ ($\tau = \tau_j^- j = 0, 1, 2, \cdots, k-1$, respectively), then all roots of (7) have negative real parts except $\pm\omega_+\mathrm{i}$ ($\pm\omega_-\mathrm{i}$, respectively), and when

$$\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup .... \cup (\tau_{k-1}^+, \tau_{k-1}^-)$$

and $\tau > \tau_k^+$, then (4) has at least one root with positive real part.

Spectral properties of (7) immediately lead to the properties of the zero solutions of (4), equivalently, the properties of the equilibrium $E(0, 0)$ for system (3).

Theorem 1. Let $\omega_\pm, \tau_j^\pm (j = 0, 1, 2, \cdots)$ be defined by (15) and (16), respectively. For (3), we have

1) If (11) or (13) holds, then the equilibrium $E(0, 0)$ is asymptotically stable for all $\tau \geq 0$;

2) If (14) holds, then there is a positive integer k such that the equilibrium $E(0, 0)$ switches k times from stability to instability to stability; that is, the equilibrium $E(0, 0)$ is asymptotically stable when

$$\tau \in [0, \tau_0^+) \cup (\tau_0^-, \tau_1^+) \cup \cdots \cup (\tau_{k-1}^-, \tau_k^+)$$

and unstable when

$$\tau \in (\tau_0^+, \tau_0^-) \cup (\tau_1^+, \tau_1^-) \cup \cdots \cup (\tau_{k-1}^+, \tau_{k-1}^-)$$

and $\tau > \tau_k^+$. Equation (3) undergoes a Hopf bifurcation near $E(0, 0)$.

Case 2. $\tau \neq \sigma$. In this case, the characteristic equation of system (3) takes the form

$$\lambda^2 + c\lambda + \omega_0^2 - k - \nu\lambda\mathrm{e}^{-\lambda\tau} + k\mathrm{e}^{-\lambda\sigma} = 0. \tag{22}$$

Lemma 5[21]**.** Consider the following equation

$$(z^2 + pz + q)\mathrm{e}^z + r = 0 \tag{23}$$

then all the roots of (23) have negative real parts it and only if $r \geq 0$ and $\frac{r\sin a_n}{pa_n} < 1$ or $-q < r < 0$ and $\frac{r\sin a_n}{pa_n} < 1, a_\kappa(\kappa \geq 0)$ is the root of the equation $\cot a = \frac{a^2-q}{ap}, a_\kappa \in (\kappa\pi, \kappa\pi + 1)$, where the positive integer n is defined as follows:

1) If $r \geq 0$ and $p^2 \geq 2q$, then $n = 1$;

2) If $r \geq 0$ and $p^2 < 2q$, then n is odd κ such that a_κ is closest to $\sqrt{q - \frac{p^2}{2}}$;

3) If $r < 0$ and $p^2 \geq 2q$, then $n = 2$;

4) If $r < 0$ and $p^2 < 2q$, then n is even such that a_κ is closest to $\sqrt{q - \frac{p^2}{2}}$.

For $\sigma \neq 0$, we consider the two cases: $\tau = 0$ and $\tau > 0$.

Case 3. When $\sigma \neq 0, \tau = 0$. We have the following result.

Theorem 2. For (3) with $\sigma \neq 0$. If the following conditions are satisfied, $k \geq 0$ and $\frac{k\sin a_n}{c-\nu} < 1$ or $k < 0$ and $\frac{k\sin a_n}{c-\nu} < 1$, a_κ is the root of the equation $\cot a = \frac{a^2-\omega_0^2+k\sigma}{a(c-\nu)\sigma}$, where the positive integer n is defined as follows:

1) If $k \geq 0$ and $(c-\nu)^2\sigma^2 \geq 2(\omega_0^2 - k\sigma^2)$, then $n = 1$;

2) If $k \geq 0$ and $(c-\nu)^2\sigma^2 < 2(\omega_0^2 - k\sigma^2)$, then n is odd κ such that a_κ is closest to $\sqrt{\omega_0^2 - k\sigma^2 - \frac{(c-\nu)^2\sigma^2}{2}}$;

3) If $k < 0$ and $(c-\nu)^2\sigma^2 \geq 2(\omega_0^2 - k\sigma^2)$, then $n = 2$;

4) If $k < 0$ and $(c-\nu)^2\sigma^2 < 2(\omega_0^2 - k\sigma^2)$, then n is even such that a_κ is closest to $\sqrt{\omega_0^2 - k\sigma^2 - \frac{(c-\nu)^2\sigma^2}{2}}$ are satisfied, then the equilibrium $E_0(0, 0)$ for system (3) is asymptotically stable.

Proof. Let $z = \lambda\sigma$, then (22) takes the form

$$[z^2 + (c-\nu)\sigma z + (\omega_0^2 - k)\sigma^2]\mathrm{e}^z + k\sigma^2 = 0. \tag{24}$$

Corresponding to (23), we have $p = (c-\nu)\sigma, q = \omega_0^2 - k\sigma^2, r = k\sigma^2$. By Lemma 5, we can conclude that if the conditions in Theorem 2 are fulfilled, then all the roots of (22) have negative real parts. Thus the equilibrium $E(0, 0)$ for (3) is asymptotically stable. □

Case 4. When $\sigma \neq 0, \tau > 0$. We have the following result.

Theorem 3. If $0 < k < \frac{2}{3}\omega_0^2$ holds, then

1) holds when

$$\left\{\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau^*, \lambda=\mathrm{i}\omega^*}\right\} < 0$$

then for all $\tau \geq 0$ and $\tau \neq \tau^*$, the equilibrium $E(0, 0)$ of system (3) is asymptotically stable.

2) When

$$\left\{\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]^{-1}_{\tau=\tau^*, \lambda=\mathrm{i}\omega^*}\right\} > 0$$

then for all $\tau \in [0, \tau^*)$, the equilibrium $E(0,0)$ of system (3) is asymptotically stable and for all $\tau > \tau^*$, the equilibrium $E(0,0)$ of system (3) is unstable and $\tau = \tau^*$ is the bifurcation value.

Proof. Let $\lambda_{1,2} = \pm \mathrm{i}\omega$ be a pair of purely imaginary roots. Substituting it into (22) and separating the real and imaginary parts, we get

$$\begin{cases} \nu\omega\cos\omega\tau = -\omega^2 + \omega_0^2 - k + k\cos\omega\sigma \\ \nu\omega\sin\omega\tau = c\omega - k\sin\omega\sigma. \end{cases} \tag{25}$$

Then we have

$$\omega^4 + [c^2 - 2(\omega_0^2 - k) - 2k\cos\omega\sigma - \nu^2]\omega^2 - 2ck\omega\sin\omega\sigma + k^2\cos^2\omega\sigma - 2(\omega_0^2 - k)k\cos\omega\sigma + k^2\sin^2\omega\sigma = 0.$$

Denote

$$f(\omega) = \omega^4 + [c^2 - 2(\omega_0^2 - k) - 2k\cos\omega\sigma - \nu^2]\omega^2 - 2ck\omega\sin\omega\sigma + k^2\cos^2\omega\sigma - 2(\omega_0^2 - k)k\cos\omega\sigma + k^2\sin^2\omega\sigma = 0.$$

If $0 < k < \frac{2}{3}\omega_0^2$ holds, then $f(0) < 0$. Since $\lim_{\omega\to+\infty} f(\omega) = +\infty$, then there exists a $\omega^* > 0$ such that $f(\omega^*) = 0$. Substituting $\omega = \omega^*$ into the second equation of (25), we get

$$\tau^j = \frac{1}{\omega^*}\left[\arccos\left(\frac{-\omega^{*2} + \omega_0^2 - k + k\cos\omega^*\sigma}{\nu\omega^*}\right) + 2j\pi\right] \tag{26}$$

where $j = 0, 1, 2, \cdots$. Denote $\tau^* = \min\{\tau^j\}$. Differentiating (22) with respect to τ, we get

$$\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = -\frac{2\lambda^2 + c - \nu\mathrm{e}^{-\lambda\tau} - k\mathrm{e}^{-\lambda\sigma}\sigma}{\nu\lambda^2\mathrm{e}^{-\lambda\tau}} - \frac{\tau}{\lambda}. \tag{27}$$

Then

$$\begin{aligned} &\left[\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right]^{-1}_{\tau=\tau^*,\lambda=\mathrm{i}\omega^*} = \\ &-\mathrm{Re}\left[\frac{-2\omega^{*2} + c - \nu\mathrm{e}^{-\mathrm{i}\omega^*\tau^*} - k\mathrm{e}^{-\mathrm{i}\omega^*\sigma}\sigma}{-\nu\omega^{*2}\mathrm{e}^{-\mathrm{i}\omega^*\tau^*}}\right] - \\ &\mathrm{Re}\left[\frac{\tau^*}{i\omega^*}\right] = \\ &\mathrm{Re}\left[\frac{-2\omega^{*2} + c - \nu\mathrm{e}^{-\mathrm{i}\omega^*\tau^*} - k\mathrm{e}^{-\mathrm{i}\omega^*\sigma}\sigma}{\nu\omega^{*2}\mathrm{e}^{-\mathrm{i}\omega^*\tau^*}}\right] = \\ &\mathrm{Re}\left[\frac{-2\omega^{*2} + c - \nu\mathrm{e}^{-\mathrm{i}\omega^*\tau^*} - k\mathrm{e}^{-\mathrm{i}\omega^*\sigma}\sigma}{\nu\omega^{*2}\mathrm{e}^{-\mathrm{i}\omega^*\tau^*}}\right] = \\ &\frac{(c - 2\omega^{*2})\cos\omega^*\tau^* - \nu - k\sigma}{\nu\omega^{*2}}. \end{aligned} \tag{28, 29}$$

If

$$\left[\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right]^{-1}_{\tau=\tau^*,\lambda=\mathrm{i}\omega^*} < 0$$

then the characteristic (22) only has one pair of purely imaginary roots and the other roots have negative real parts when $\tau = \tau^*$. Moreover, Reλ decreases with the increase of τ. If

$$\left[\mathrm{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\right]^{-1}_{\tau=\tau^*,\lambda=\mathrm{i}\omega^*} > 0$$

then the characteristic (22) will add one pair of roots with positive real parts when $\tau > \tau^*$. □

3 Numerical examples

In this section, we give numerical simulations to verify the correctness of our designed feedback controllers. As an example, we consider the system (3) with $c = 2, \omega_0 = 2, \mu = 1, \nu = 2$. That is

$$\ddot{x}(t) + 2\dot{x}(t) + 4x(t) + x^3(t) = 2\dot{x}(t-\tau) + k[x(t) - x(t-\sigma)]. \tag{30}$$

Similar to the process of transforming (3) into (4), system (29) can become the following form

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = (k-4)x(t) - 2y(t) - 3y(t-\tau) - 2x^3(t) - \\ \qquad kx(t-\sigma) \end{cases} \tag{31}$$

which has a unique equilibrium $E(0,0)$. Next, we consider five cases.

Case 5. Let $\tau = \sigma$ and the feedback control parameter $k = 0$. Then system (30) takes the form

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -4x(t) - 2y(t) - 3y(t-\tau) - 2x^3(t). \end{cases} \tag{32}$$

It is easy to check that the condition 2) of Theorem 2.1 is satisfied. By Matlab 7.0 software, we get $\tau^* \approx 0.612$. The zero equilibrium $E(0,0)$ is stable when $\tau = 0.61 < \tau^*$ which is illustrated by the computer simulations (see Fig. 1). When $\tau = 0.63$ which passes through the critical value $\tau^* \approx 0.612$, the zero equilibrium $E(0,0)$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcate from the zero equilibrium $E(0,0)$ (see Fig. 2).

Case 6. Let $\tau = \sigma$ and the feedback control parameter $k = -0.05$. Then system (30) takes the form

$$\begin{cases} \dot{x}(t) = y(t) \\ \dot{y}(t) = -4.05x(t) - 2y(t) - 3y(t-\tau) - 2x^3(t) - \\ \qquad 0.05x(t-\tau). \end{cases} \tag{33}$$

It is easy to check that the condition 2) of Theorem 1 is satisfied. By Matlab 7.0 software, we get $\tau^* \approx 0.65$. The zero equilibrium $E(0,0)$ is stable when $\tau = 0.63 < \tau^*$ which is illustrated by the computer simulations (see Fig. 3). When $\tau = 0.8$ which passes through the critical value $\tau^* \approx 0.65$, the zero equilibrium $E(0,0)$ loses its stability and a Hopf bifurcation occurs from the zero equilibrium $E(0,0)$ (see Fig. 4).

Case 7. Let $\tau=\sigma$ and the feedback control parameter $k=-0.099$. Then system (30) takes the form

$$\begin{cases} \dot{x}(t)=y(t) \\ \dot{y}(t)=-4.099x(t)-2y(t)-3y(t-\tau)-2x^3(t)- \\ \qquad 0.099x(t-\tau). \end{cases} \tag{34}$$

It is easy to check that the condition 2) of Theorem 1 is fulfilled. By Matlab 7.0 software, we get $\tau^*\approx 0.67$. The zero equilibrium $E(0,0)$ is stable when $\tau=0.63<\tau^*$ which is illustrated by the computer simulations (see Fig. 5). When $\tau=0.9$ passes through the critical value $\tau^*\approx 0.67$, the zero equilibrium $E(0,0)$ loses its stability and a Hopf bifurcation occurs from the zero equilibrium $E(0,0)$ (see Fig. 6).

Case 8. Let $\tau\neq\sigma$ and the feedback control parameter $k=0.5$. Then system (30) takes the form

$$\begin{cases} \dot{x}(t)=y(t) \\ \dot{y}(t)=-3.5x(t)-2y(t)-3y(t-\tau)-2x^3(t)+ \\ \qquad 0.5x(t-\sigma). \end{cases} \tag{35}$$

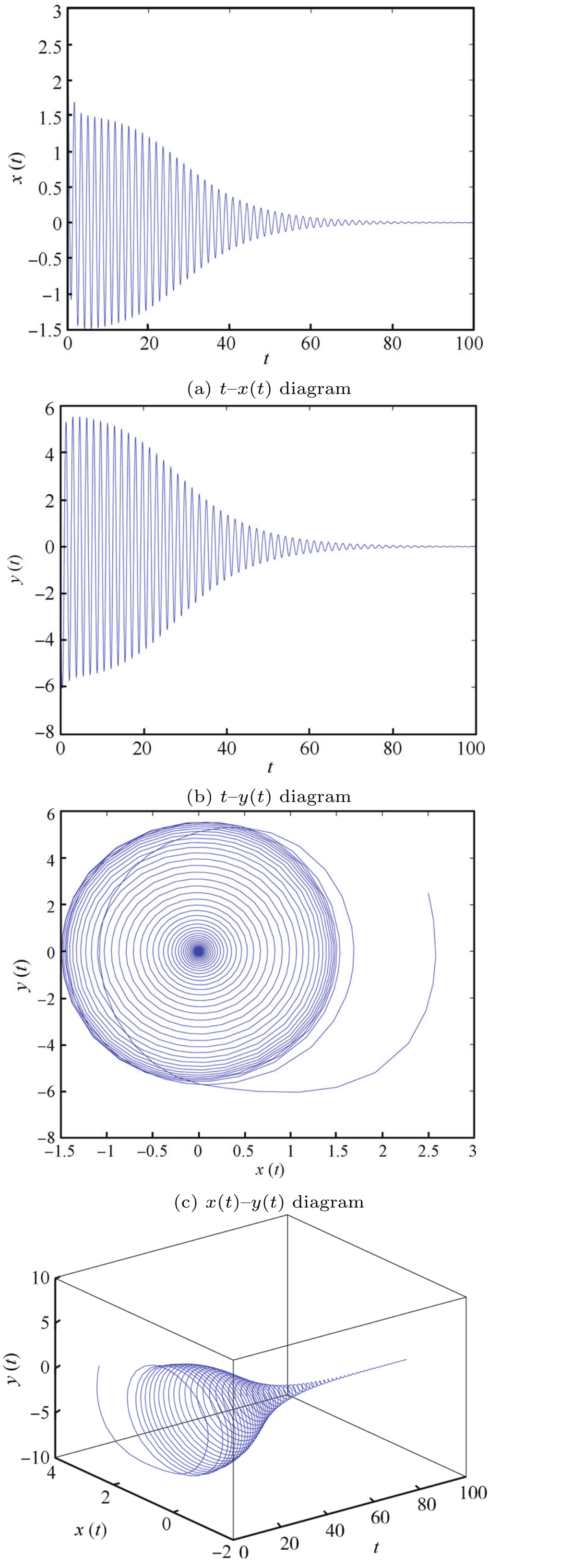
Fix $\sigma=0.7$. It is easy to check that the condition 2) of Theorem 3 is fulfilled. By Matlab 7.0 software, we get $\tau^*\approx 0.619$. The zero equilibrium $E(0,0)$ is stable when $\tau=0.6<\tau^*$ which is depicted in Fig. 7. When $\tau=1.2$ passes through the critical value $\tau^*\approx 0.619$, the zero equilibrium $E(0,0)$ loses its stability and a Hopf bifurcation occurs from the zero equilibrium $E(0,0)$ (see Fig. 8).

Case 9. Let $\tau\neq\sigma$ and the feedback control parameter $k=0.8$. Then system (30) takes the form

$$\begin{cases} \dot{x}(t)=y(t) \\ \dot{y}(t)=-3.2x(t)-2y(t)-3y(t-\tau)-2x^3(t)+ \\ \qquad 0.8x(t-\sigma). \end{cases} \tag{36}$$

Fix $\sigma=0.7$. It is easy to check that the condition 2) of Theorem 3 is satisfied. By Matlab 7.0 software, we get $\tau^*\approx 0.63$. The zero equilibrium $E(0,0)$ is stable when $\tau=0.62<\tau^*$ which is depicted in Fig. 9. When $\tau=1.3$ passes through the critical value $\tau^*\approx 0.63$, the zero equilibrium $E(0,0)$ loses its stability and a Hopf bifurcation occurs from the zero equilibrium $E(0,0)$ (see Fig. 10).

Remark 1. From Cases 5–7, we know that when feedback control parameter $k=0$ (i.e., without control), the system (31) is asymptotically stable when $\tau\in[0,0.612]$ when feedback control parameter $k=0.05$, the system (32) is asymptotically stable when $\tau\in[0,0.65]$ and when feedback control parameter $k=0.099$, the system (33) is asymptotically stable when $\tau\in[0,0.67]$. Thus we can conclude that a suitable feedback controller can enlarge the stable region of the original system with appropriate choice of feedback control parameter k.



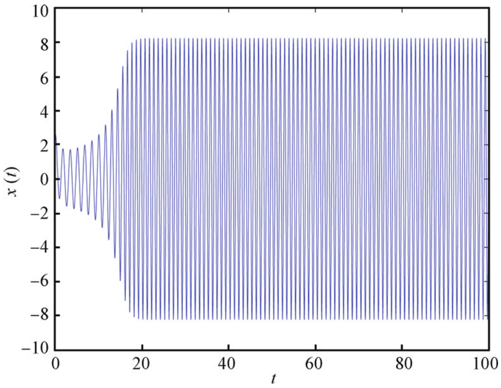
(a) t–$x(t)$ diagram

(b) t–$y(t)$ diagram

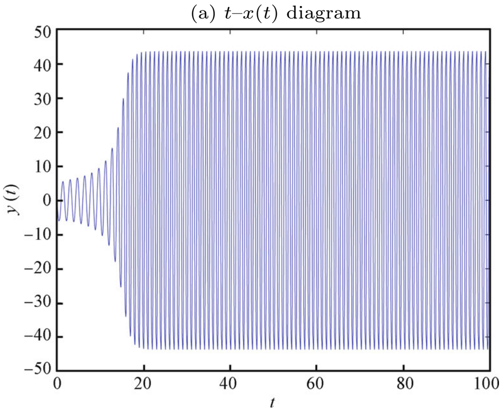
(c) $x(t)$–$y(t)$ diagram

(d) t–$x(t)$–$y(t)$ diagram

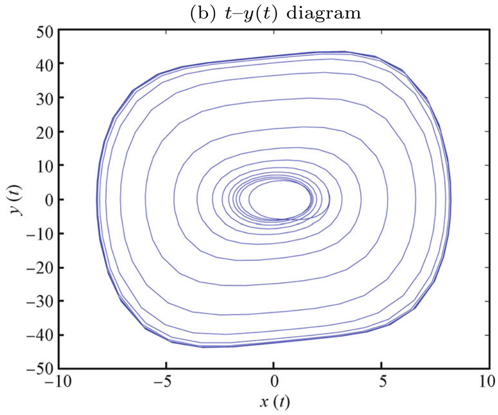
Fig. 1 Response of state variables and phase portrait of system (3) with $\tau=0.61<\tau^*\approx 0.612$. The equilibrium is asymptotically stable.



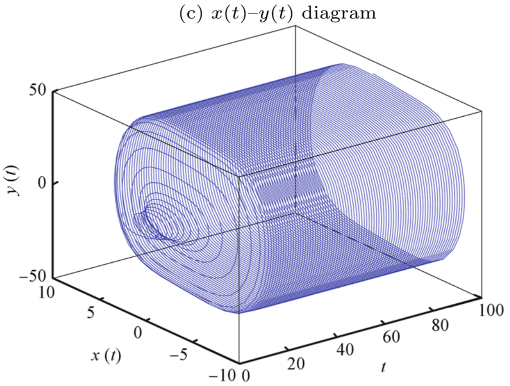
(a) t–$x(t)$ diagram



(b) t–$y(t)$ diagram

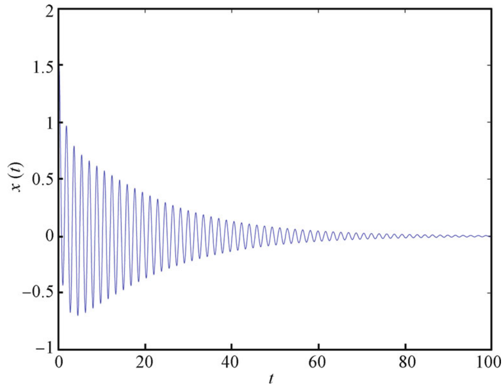


(c) $x(t)$–$y(t)$ diagram

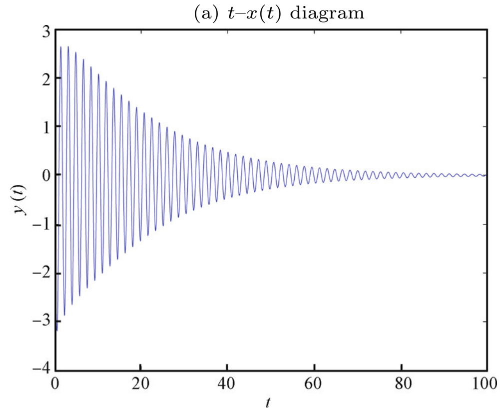


(d) t–$x(t)$–$y(t)$ diagram

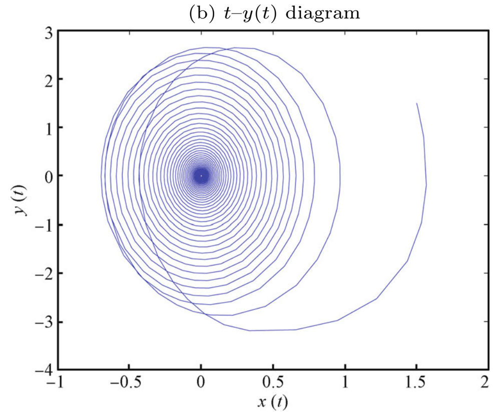
Fig. 2 Response of state variables and phase portrait of system (31) with $\tau = 0.63 > \tau_0 \approx 0.612$. Hopf bifurcation occurs from the equilibrium.



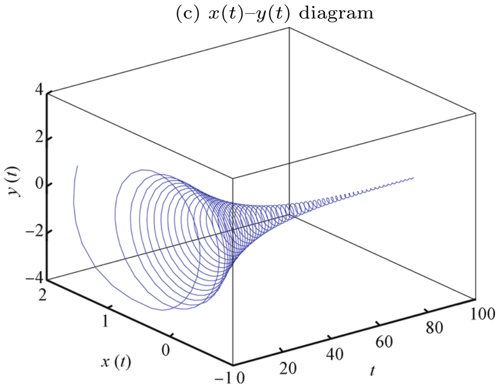
(a) t–$x(t)$ diagram



(b) t–$y(t)$ diagram

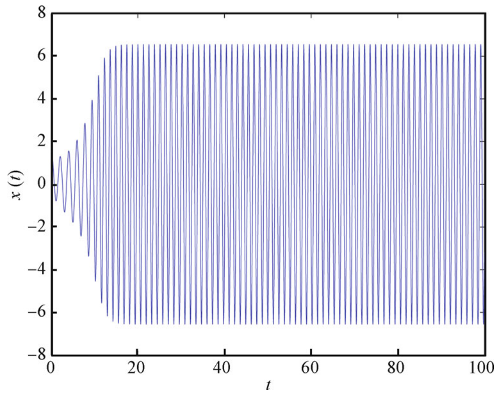


(c) $x(t)$–$y(t)$ diagram

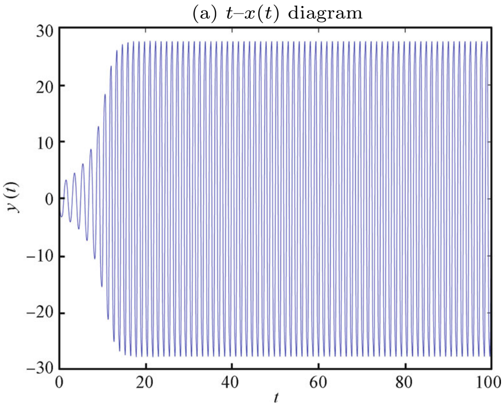


(d) t–$x(t)$–$y(t)$ diagram

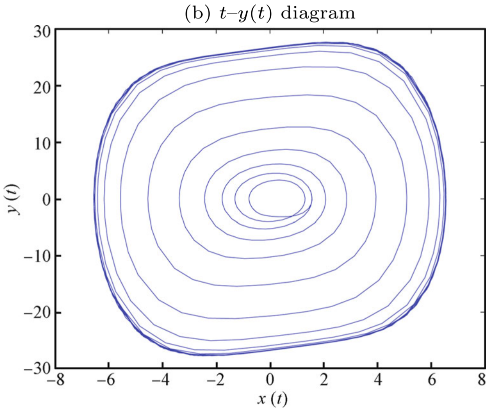
Fig. 3 Response of state variables and phase portrait of system (32) with $\tau = 0.63 < \tau^* \approx 0.65$. The equilibrium is asymptotically stable.



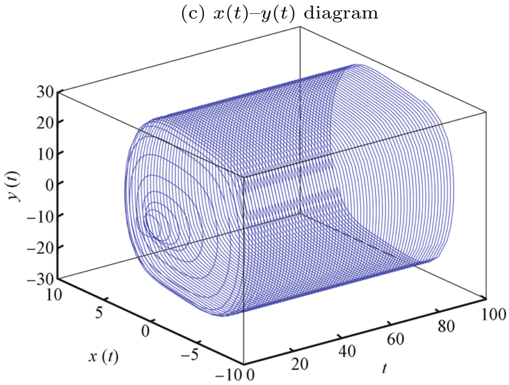
(a) t–$x(t)$ diagram



(b) t–$y(t)$ diagram

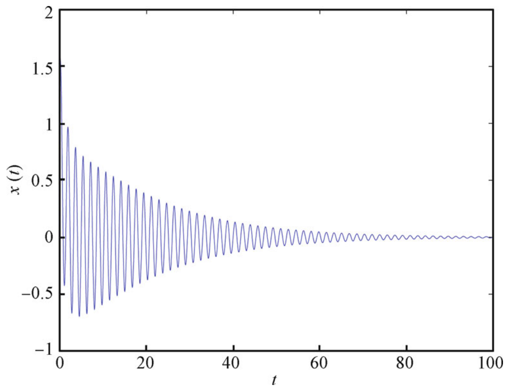


(c) $x(t)$–$y(t)$ diagram

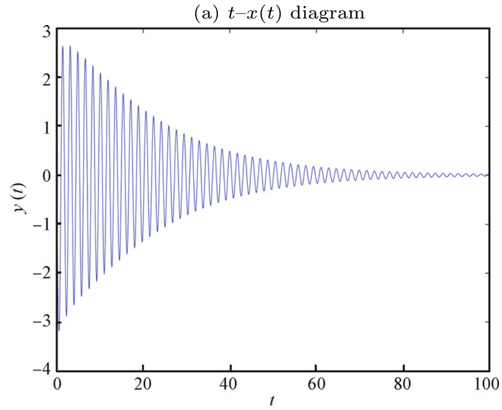


(d) t–$x(t)$–$y(t)$ diagram

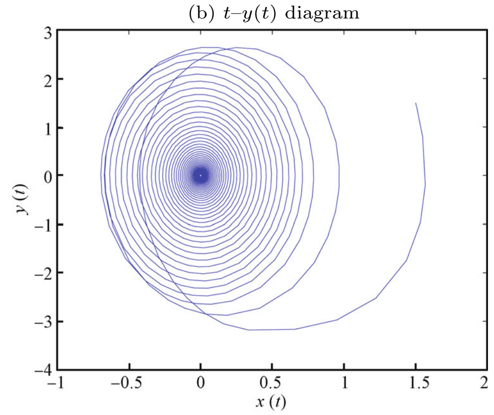
Fig. 4 Response of state variables and phase portrait of system (32) with $\tau = 0.8 > \tau^* \approx 0.612$. Hopf bifurcation occurs from the equilibrium.



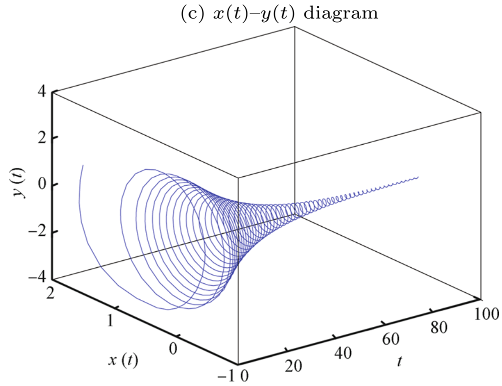
(a) t–$x(t)$ diagram



(b) t–$y(t)$ diagram

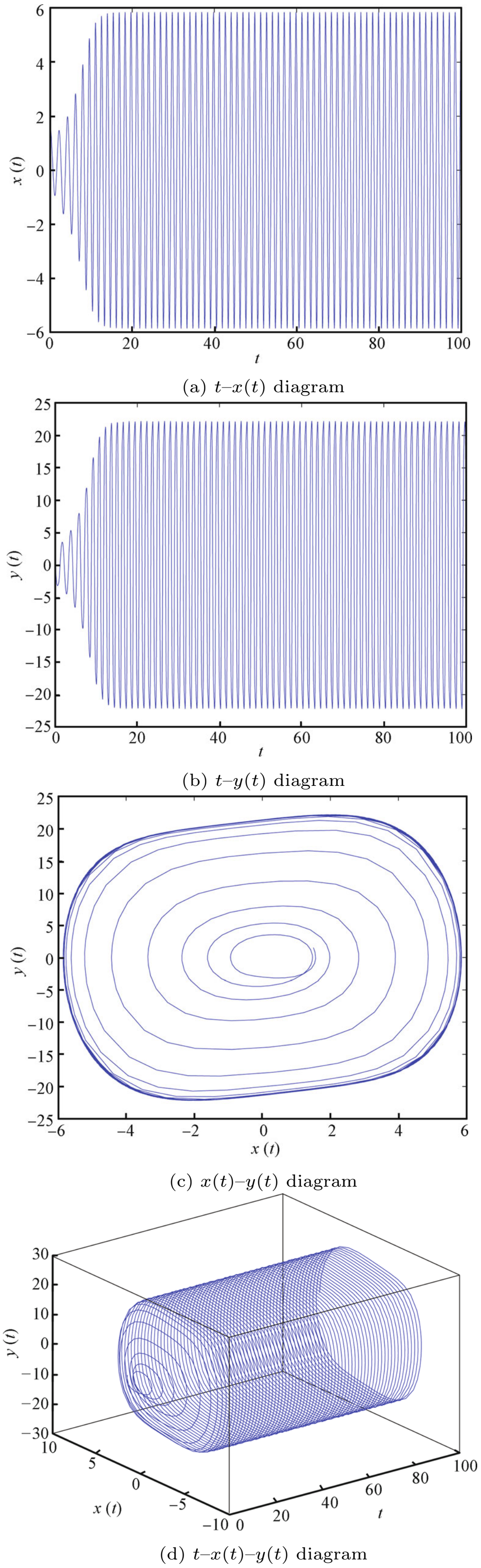


(c) $x(t)$–$y(t)$ diagram



(d) t–$x(t)$–$y(t)$ diagram

Fig. 5 Response of state variables and phase portrait of system (33) with $\tau = 0.63 < \tau^* \approx 0.67$. The equilibrium is asymptotically stable.



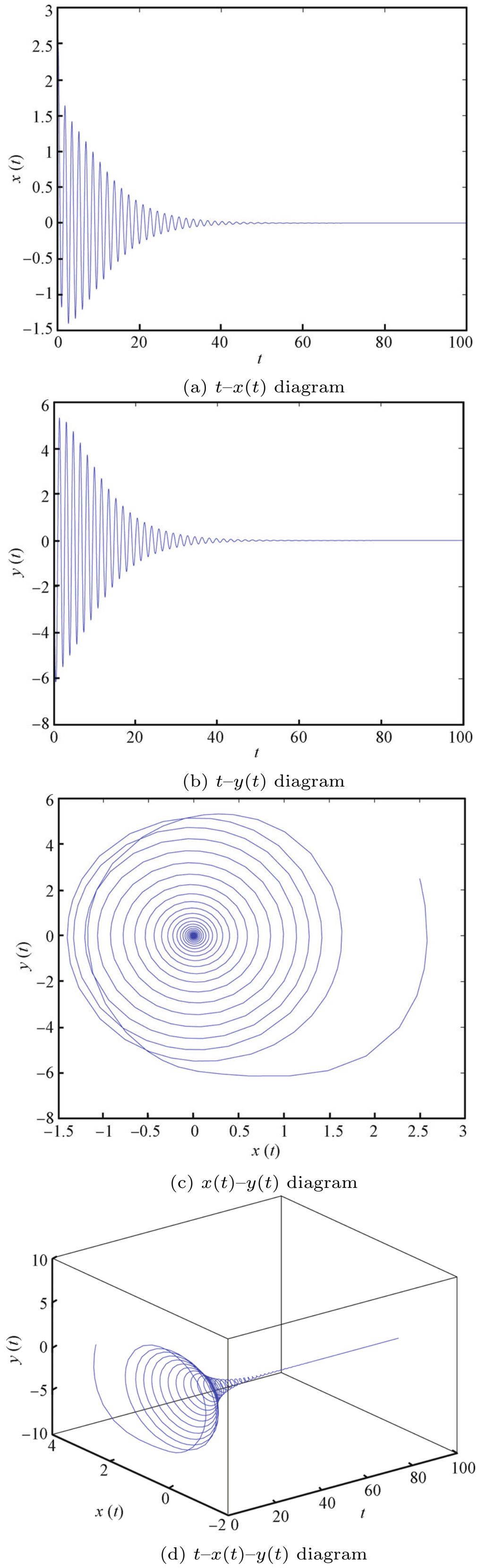
(a) t–$x(t)$ diagram

(b) t–$y(t)$ diagram

(c) $x(t)$–$y(t)$ diagram

(d) t–$x(t)$–$y(t)$ diagram

Fig. 6 Response of state variables and phase portrait of system (33) with $\tau = 0.9 > \tau^* \approx 0.67$. Hopf bifurcation occurs from the equilibrium.



(a) t–$x(t)$ diagram

(b) t–$y(t)$ diagram

(c) $x(t)$–$y(t)$ diagram

(d) t–$x(t)$–$y(t)$ diagram

Fig. 7 Response of state variables and phase portrait of system (34) with $\tau = 0.6 < \tau^* \approx 0.619$. The equilibrium is asymptotically stable.

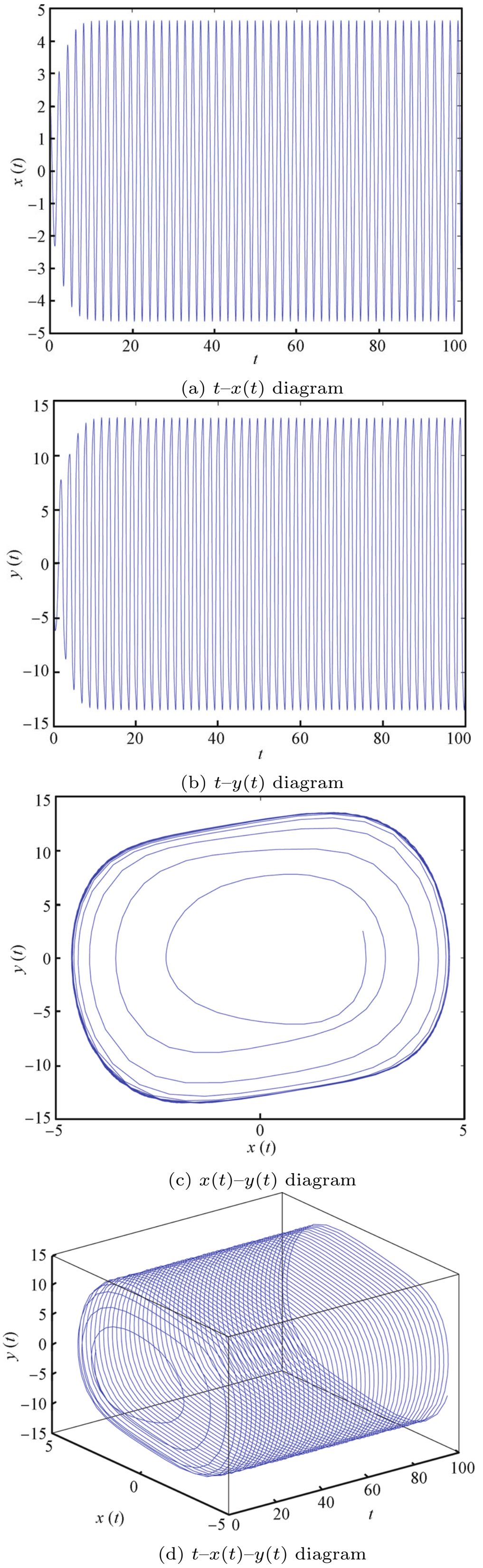


Fig. 8 Response of state variables and phase portrait of system (34) with $\tau = 1.2 > \tau^* \approx 0.619$. Hopf bifurcation occurs from the equilibrium.

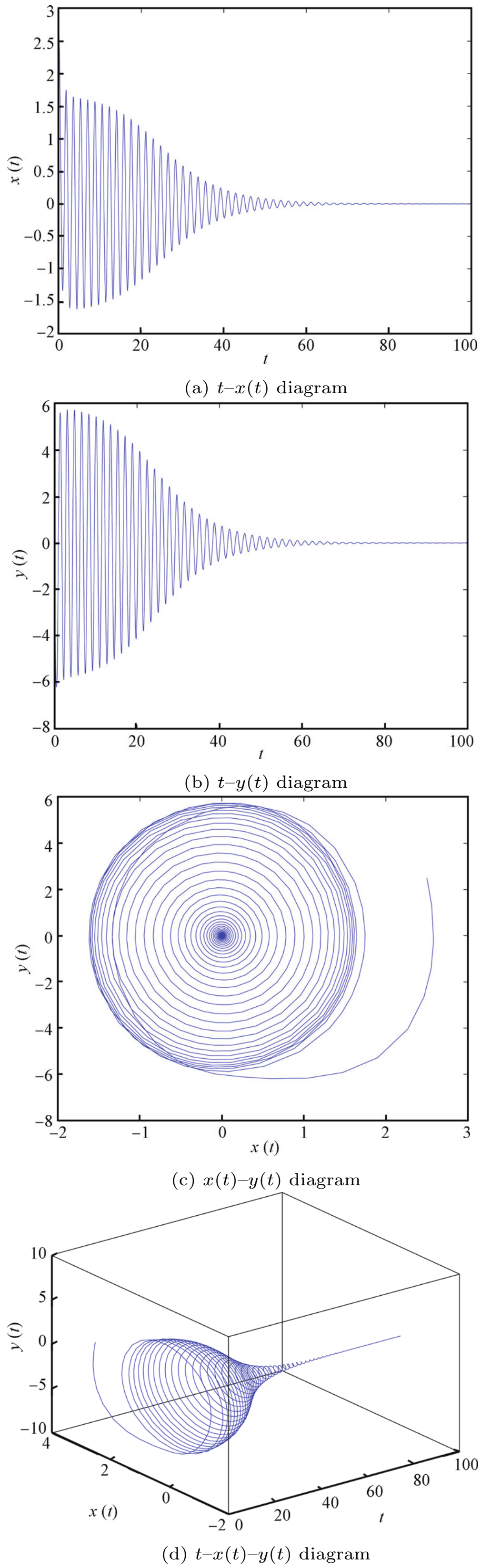
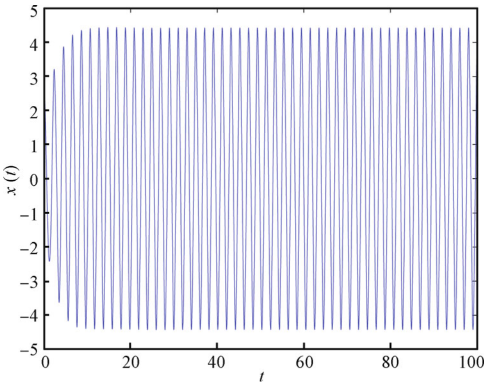
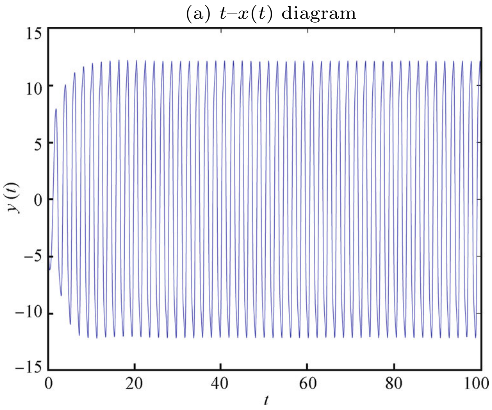


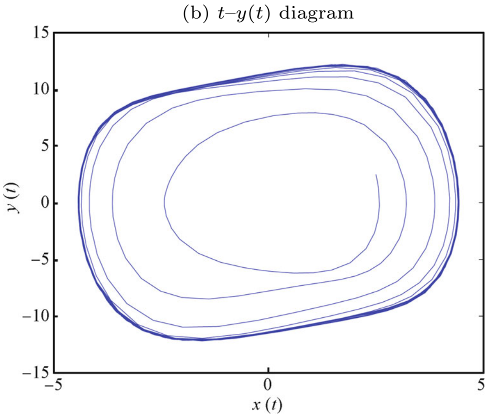
Fig. 9 Response of state variables and phase portrait of system (35) with $\tau = 0.62 < \tau^* \approx 0.63$. The equilibrium is asymptotically stable.



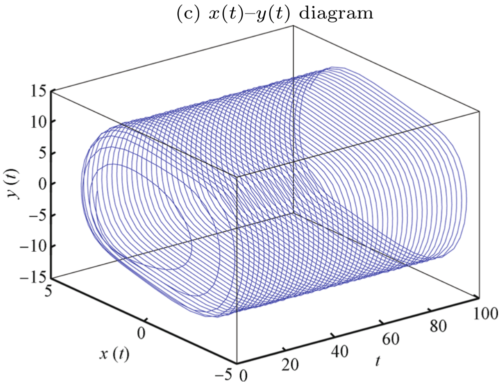
(a) t–$x(t)$ diagram



(b) t–$y(t)$ diagram



(c) $x(t)$–$y(t)$ diagram



(d) t–$x(t)$–$y(t)$ diagram

Fig. 10 Response of state variables and phase portrait of system (35) with $\tau = 1.3 > \tau^* \approx 0.63$. Hopf bifurcation occurs from the equilibrium.

4 Conclusions

In this paper, we deal with a Duffing oscillator model with delayed velocity feedback. Some sufficient conditions which ensure the stability and the existence of Hopf bifurcation of the model are obtained by using the time delayed feedback control method and delayed diffierential equation theory. By choosing the delay as bifurcation parameter and analyzing the associated characteristic equation, the existence of bifurcation parameter point is determined. It is shown that if the time delay is chosen as a bifurcation parameter, Hopf bifurcation occurs when the time delay passes through a series of critical values. Numerical simulations show that the designed feedback controllers not only delay the onset of Hopf bifurcation, but also enlarge the stability region for the model. The control method can be applied to control Hopf bifurcation of other delayed model.

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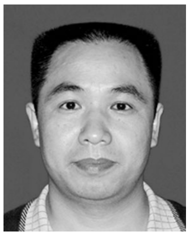

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