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Chaos Control and Bifurcation Behavior for a Sprott E System with Distributed Delay Feedback

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Abstract: In this paper, the problem of controlling chaos in a Sprott E system with distributed delay feedback is considered. By analyzing the associated characteristic transcendental equation, we focus on the local stability and Hopf bifurcation nature of the Sprott E system with distributed delay feedback. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions are derived by using the normal form theory and center manifold theory. Numerical simulations for justifying the theoretical analysis are provided.

Keywords: Sprott E system, chaos control, stability, Hopf bifurcation, distributed delay.

1 Introduction

Since the pioneering work of Ott et al.[1], the topics of chaos and chaotic control are growing rapidly in many different fields such as ecological system, chemical systems and biological systems, and so forth[2−10]. We all know that chaotic systems have very complicated dynamical nature, which plays an important role in many fields such as secure communications, information processing, high-performance circuit design for telecommunications[11]. Chaos, which often causes irregular behaviors in practical system, is usually undesirable. In many cases, we wish to avoid and eliminate such behaviors. Recently, many schemes such as Ott et al.[1], feedback and non-feedback control[12−17], observer-based control[12], active control[13], adaptive control[18, 19], inverse optimal control[20], evolutionary algorithm[21], etc., have been presented to implement the chaos control. For more related work, one can see [22–26]. In 2012, Wang and Chen[27] reported the very surprising finding of the following new 3D autonomous chaotic Sprott E system with one stable node or stable focus

$$\begin{cases} \dot{x} = y(t)z(t) + a \\ \dot{y} = x^2(t) - y(t) \\ \dot{z} = 1 - 4x(t). \end{cases} \qquad (1)$$

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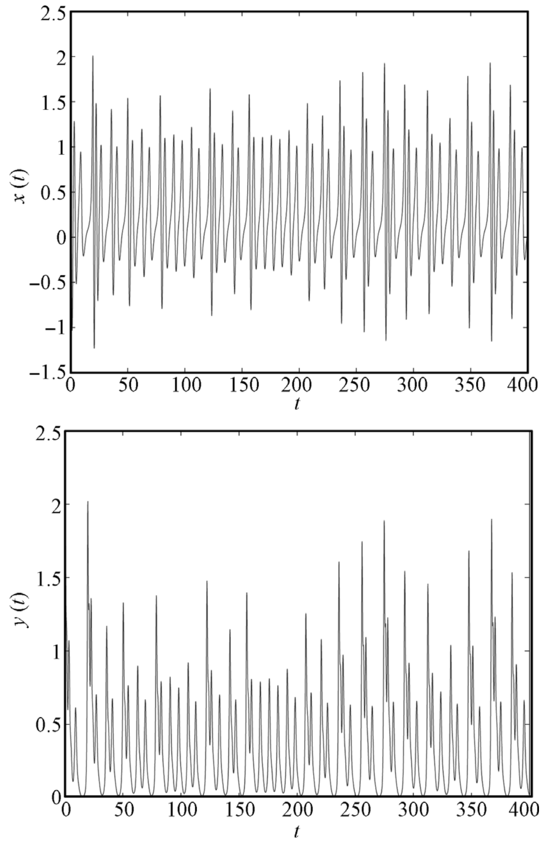
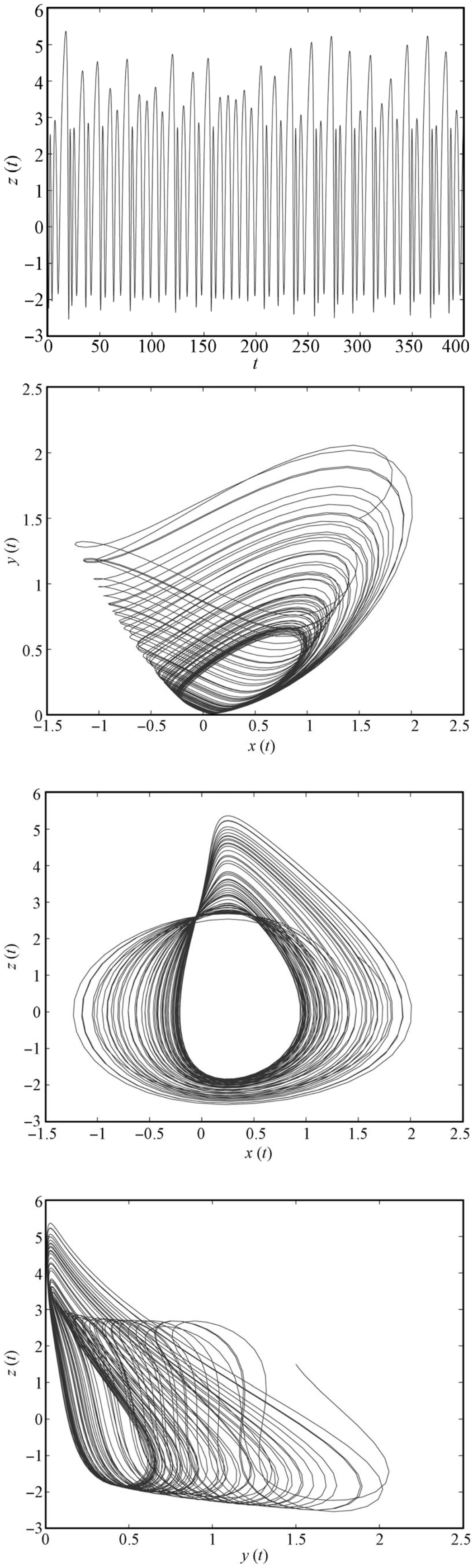
When $a = 0$, it is the Sprott E system[28]. When $a \neq 0$, the stability of the single equilibrium is fundamentally different[29]. Let $yz + a = 0, x^2 - y = 0, 1 - 4x = 0$, we can obtain that system (1) has only one stable equilibrium $E(x^*, y^*, z^*) = (\frac{1}{4}, \frac{1}{16}, 16a)$ if $a > 0$. Interestingly, Wang and Chen[29] found that system (1) can generate chaotic phenomenon which is shown in Fig. 1 (Fig. 1 shows the waveform portraits and the phase portraits of system (1)).





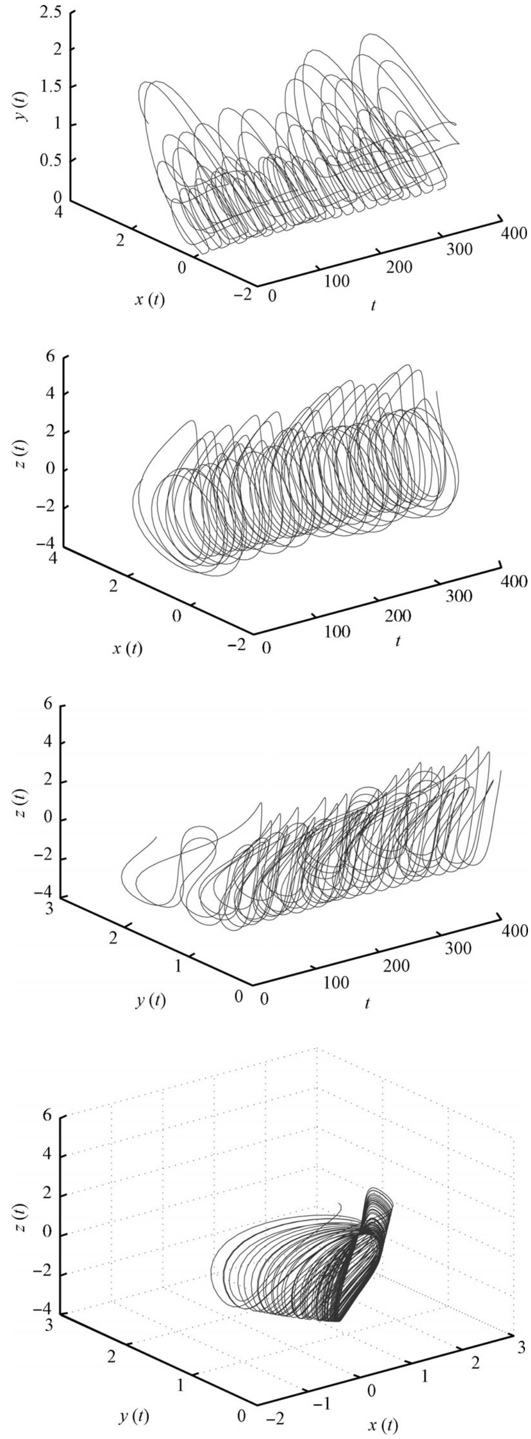


Fig. 1 Chaotic attractor of system (1) with $a = 0.006$. The initial value is (1.5,1.5,1.5)

In this paper, we study the stability, and the local Hopf bifurcation nature for system (1). To the best of our knowledge, it is the first try to introduce continuous distributed time-delayed feedback force to control the chaos of the Sprott E system.

The remainder of the paper is organized as follows. In Section 2, we investigate the stability of the equilibrium and the occurrence of local Hopf bifurcations of the Sprott E system with distributed delay feedback. In Section 3, the direction and stability of the local Hopf bifurcation are established. In Section 4, numerical simulations are carried out to show the validity of chaotic control. Some main conclusions are drawn in Section 5.

2 Stability and bifurcation analysis

In this section, we shall study the stability of the equilibrium and the existence of local Hopf bifurcations.

In order to apply feedback control, we add a continuous distributed time delayed force

$$b\int_{-\infty}^{0}[z(t)-z(t+s)]k(-s)\mathrm{d}s$$

to the third equation of system (1), then system (1) takes the form

$$\begin{cases}\dot{x}=y(t)z(t)+a\\ \dot{y}=x^2(t)-y(t)\\ \dot{z}=1-4x(t)+b\int_{-\infty}^{0}[z(t)-z(t+s)]k(-s)\mathrm{d}s\end{cases}\tag{2}$$

where $a,b>0$, $\int_{-\infty}^{0}k(s)\mathrm{d}s=1$, $\int_{-\infty}^{0}sk(s)\mathrm{d}s<+\infty$. Obviously, system (2) has the equilibrium point $E(x^*,y^*,z^*)=(\frac{1}{4},\frac{1}{16},-16a)$.

Let $\bar{x}(t)=x(t)-x^*,\bar{y}(t)=y(t)-y^*,\bar{z}(t)=z(t)-z^*$ and still denote $\bar{x}(t),\bar{y}(t)$ and $\bar{z}(t)$ by $x(t),y(t)$ and $z(t)$, respectively, then (2) becomes

$$\begin{cases}\dot{x}=z^*y(t)+y^*z(t)+y(t)z(t)\\ \dot{y}=2x^*x(t)-y(t)+x^2(t)\\ \dot{z}=-4x(t)+bz(t)-b\int_{-\infty}^{0}z(t+s)k(-s)\mathrm{d}s.\end{cases}\tag{3}$$

The linearization of (3) near $E(x^*,y^*,z^*)$ is given by

$$\begin{cases}\dot{x}=z^*y(t)+y^*z(t)\\ \dot{y}=2x^*x(t)-y(t)\\ \dot{z}=-4x(t)+bz(t)-b\int_{-\infty}^{0}z(t+s)k(-s)\mathrm{d}s\end{cases}\tag{4}$$

whose characteristic equation appears as

$$\lambda(\lambda+1)\left(\lambda-b+b\int_{-\infty}^{0}k(-s)\mathrm{e}^{\lambda s}\mathrm{d}s\right)+4y^*(\lambda+1)-\\ 2x^*z^*\left(\lambda-b+b\int_{-\infty}^{0}k(-s)\mathrm{e}^{\lambda s}\mathrm{d}s\right)=0.\tag{5}$$

In this paper, we consider the weak kernel case, i.e., $k(s)=\alpha\mathrm{e}^{-\alpha s}$, where $\alpha>0$. As to the general gamma kernel case, we can make a similar analysis. We give the initial condition of system (4) as

$$\begin{bmatrix}x(s)\\ y(s)\\ z(s)\end{bmatrix}=\begin{bmatrix}\phi_1(s)\\ \phi_2(s)\\ \phi_3(s)\end{bmatrix},\quad -\infty<s\leq 0.$$

The characteristic equation (5) with the weak kernel case takes the form

$$\lambda^4+\theta_1(\alpha)\lambda^3+\theta_2(\alpha)\lambda^2+\theta_3(\alpha)\lambda+\theta_4(\alpha)=0\tag{6}$$

where

$$\begin{cases}\theta_1(\alpha)=\alpha-b+1\\ \theta_2(\alpha)=\alpha-b-2x^*z^*+4y^*\\ \theta_3(\alpha)=4y^*(1+\alpha)-2x^*z^*(\alpha-b)\\ \theta_4(\alpha)=4y^*\alpha.\end{cases}\tag{7}$$

In view of the well known Routh-Hurwitz criterion, we can conclude that all the roots of (6) has negative real parts if the following conditions

$$\begin{cases}D_1(\alpha)=\theta_1(\alpha)=\alpha-b+1>0\\ D_2(\alpha)=\theta_1(\alpha)\theta_2(\alpha)-\theta_3(\alpha)=\\ \qquad(\alpha-b+1)(\alpha-b-2x^*z^*+4y^*)-\\ \qquad[4y^*(1+\alpha)-2x^*z^*(\alpha-b)]>0\\ D_3(\alpha)=\theta_3(\alpha)D_2(\alpha)-\theta_1^2(\alpha)\theta_4(\alpha)=\\ \qquad[4y^*(1+\alpha)-2x^*z^*(\alpha-b)]\times\\ \qquad\{(\alpha-b+1)(\alpha-b-2x^*z^*+4y^*)-\\ \qquad[4y^*(1+\alpha)-2x^*z^*(\alpha-b)]\}-\\ \qquad 4y^*\alpha(\alpha-b+1)^2>0\\ D_4(\alpha)=\theta_4(\alpha)D_3(\alpha)=4y^*\alpha D_3(\alpha)>0\end{cases}\tag{8}$$

hold true.

Based on the analysis above, we can easily obtain the following Theorem 1.

Theorem 1. The equilibrium $E(x^*,y^*,z^*)$ of system (2) with the weak kernel is locally asymptotically stable if the following conditions are fulfilled:

$$\begin{cases}\alpha-b+1>0\\ (\alpha-b+1)(\alpha-b-2x^*z^*+4y^*)-\\ \qquad[4y^*(1+\alpha)-2x^*z^*(\alpha-b)]>0\\ [4y^*(1+\alpha)-2x^*z^*(\alpha-b)]\times\\ \qquad\{(\alpha-b+1)(\alpha-b-2x^*z^*+4y^*)-\\ \qquad[4y^*(1+\alpha)-2x^*z^*(\alpha-b)]\}-4y^*\alpha(\alpha-b+1)^2>0.\end{cases}$$

Let λ_i $(i=1,2,3,4)$ be the roots of (6), then we have

$$\begin{cases}\lambda_1+\lambda_2+\lambda_3+\lambda_4=-\theta_1(\alpha)\\ \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_1\lambda_4+\lambda_2\lambda_3+\lambda_2\lambda_4+\lambda_3\lambda_4=\theta_2(\alpha)\\ \lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2\lambda_4+\lambda_1\lambda_3\lambda_4+\lambda_2\lambda_3\lambda_4=-\theta_3(\alpha)\\ \lambda_1\lambda_2\lambda_3\lambda_4=\theta_4(\alpha).\end{cases}\tag{9}$$

If there exists an $\alpha_0\in\mathbf{R}^+$ such that $D_3(\alpha_0)=0$ and $\frac{\mathrm{d}D_3(\alpha)}{\mathrm{d}\alpha}|_{\alpha=\alpha_0}\neq 0$, then by the Routh-Hurwitz criterion, there exists a pair of purely imaginary roots, say $\lambda_1=\bar{\lambda}_2=\mathrm{i}\omega_0\,(\omega\neq 0)$, and the other two roots λ_3,λ_4 satisfy: if λ_3,λ_4 are real, then $\lambda_3<0,\lambda_4<0$; if λ_3,λ_4 are complex conjugate, then $\mathrm{Re}\{\lambda_3\}=\mathrm{Re}\{\lambda_4\}=-\frac{\theta_1(\alpha)}{2}$. It is easy to calculate that

$$\frac{\mathrm{d}(\mathrm{Re}(\lambda_1))}{\mathrm{d}\alpha}=-\frac{\theta_1(\alpha)}{2[\theta_1^3(\alpha)\theta_3(\alpha)+(\theta_1(\alpha)\theta_2(\alpha)-2\theta_3(\alpha))^2]}\times\\ \left.\frac{\mathrm{d}D_3(\alpha)}{\mathrm{d}\alpha}\right|_{\alpha=\alpha_0}\tag{10}$$

thus the Hopf bifurcation occurs near $E(x^*,y^*,z^*)$ when α passes through α_0.

3 Direction and stability of bifurcating periodic solutions

In this section, by using techniques from normal form and center manifold theory[30], we shall investigate the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $E(x^*, y^*, z^*)$. Let $\mu = \alpha - \alpha_0$, then system (3) undergoes the Hopf bifurcation at the equilibrium $E(x^*, y^*, z^*)$ for $\mu = 0$, and then $\pm i\omega_0$ are purely imaginary roots of the characteristic equation at the equilibrium $E(x^*, y^*, z^*)$. System (3) can be written as an functional differential equation (FDE) in $\mathbf{C} = \mathbf{C}([-\infty, 0]), \mathbf{R}^3)$ as

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t \tag{11}$$

where $u(t) = [x(t), y(t), z(t)]^{\mathrm{T}} \in \mathbf{C}$ and $u_t(\theta) = u(t+\theta) = [x(t+\theta), y(t+\theta), z(t+\theta)]^{\mathrm{T}} \in \mathbf{C}$, and A and R are given by

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & -\infty \le \theta < 0 \\ L\phi(\theta) + \int_{-\infty}^{0} F(s)\phi(s)\mathrm{d}s, & \theta = 0 \end{cases} \tag{12}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} [0,0,0]^{\mathrm{T}}, & -\infty \le \theta < 0 \\ [f_1, f_2, 0]^{\mathrm{T}}, & \theta = 0 \end{cases} \tag{13}$$

respectively, where $\phi(\theta) = [\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)]^{\mathrm{T}} \in \mathbf{C}$ and $f_1 = \phi_2(0)\phi_3(0), f_2 = \phi_1^2(0)$.

For $\psi \in \mathbf{C}([0, +\infty], (\mathbf{R}^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0, +\infty] \\ L^{\mathrm{T}}\psi(0) + \int_{-\infty}^{0} F^{\mathrm{T}}(t)\psi(-t)\mathrm{d}s, & s = 0. \end{cases}$$

For $\phi \in \mathbf{C}([-\infty, 0], \mathbf{R}^3)$ and $\psi \in \mathbf{C}([0, +\infty], (\mathbf{R}^3)^*)$, define the bilinear form

$$\langle \psi, \phi \rangle = \bar{\psi}(0)\phi(0) - \int_{-\infty}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)\mathrm{d}\eta(\theta)\phi(\xi)\mathrm{d}\xi$$

where $\eta(\theta) = \eta(\theta, 0)$, the $A = A(0)$ and A^* are adjoint operators. By the discussions in Section 2, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. Assume that $q(\theta) = (1, a_1, a_2)^{\mathrm{T}}e^{i\omega_0\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0$, then we have $A(0)q(0) = i\omega_0 q(0)$, i.e.,

$$Lq(0) + \int_{-\infty}^{0} F(s)q(s)\mathrm{d}s = \begin{bmatrix} 0 & z^* & y^* \\ 2x^* & -1 & 0 \\ -4 & 0 & b \end{bmatrix}\begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} +$$

$$\int_{-\infty}^{0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & bk(-s) \end{bmatrix}\begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} e^{i\omega_0 s}\mathrm{d}s =$$

$$\begin{bmatrix} a_1 z^* + a_2 y^* \\ 2x^* - a_1 \\ -4 + ba_2 - ba_2\chi^{(1)} \end{bmatrix} = \begin{bmatrix} i\omega_0 \\ i\omega_0 a_1 \\ i\omega_0 a_2 \end{bmatrix}$$

where

$$\chi^{(1)} = \int_{-\infty}^{0} k(-s)e^{i\omega_0 s}\mathrm{d}s = \frac{\alpha}{\alpha + i\omega_0}.$$

We can obtain

$$q(\theta) = [1, a_1, a_2]^{\mathrm{T}}e^{i\omega_0\theta}$$

where

$$a_1 = \frac{2x^*}{i\omega_0 + 1}, \quad a_2 = \frac{4}{b - b\chi^{(1)} - i\omega_0}.$$

Assume that $q^*(s) = D[1, a_1^*, a_2^*]^{\mathrm{T}}e^{i\omega_0 s} (0 \le s < +\infty)$ is the eigenvector of $A^*(0)$ corresponding to $-i\omega_0$, then we have $A^*(0)q^*(0) = i\omega_0 q^*(0)$, i.e.,

$$L^{\mathrm{T}}q^*(0) + \int_{-\infty}^{0} F^*(s)q^*(-s)\mathrm{d}s =$$

$$\begin{bmatrix} 0 & 2x^* & -4 \\ z^* & -1 & 0 \\ y^* & 0 & b \end{bmatrix}\begin{bmatrix} D \\ Da_1^* \\ Da_2^* \end{bmatrix} +$$

$$\int_{-\infty}^{0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & bk(-s) \end{bmatrix}\begin{bmatrix} D \\ Da_1^* \\ Da_2^* \end{bmatrix} e^{-i\omega_0 s}\mathrm{d}s =$$

$$\begin{bmatrix} 2x^* a_1^* - 4a_2^* \\ z^* - a_1^* \\ y^* + ba_2^* - b\chi^{(2)} \end{bmatrix} = \begin{bmatrix} -i\omega_0 D \\ -i\omega_0 a_1^* D \\ -i\omega_0 a_2^* D \end{bmatrix}$$

where

$$\chi^{(2)} = \int_{-\infty}^{0} k(-s)e^{-i\omega_0 s}\mathrm{d}s = \frac{\alpha}{\alpha - i\omega_0}.$$

We can obtain

$$q^*(s) = D[1, a_1^*, a_2^*]^{\mathrm{T}}e^{i\omega_0 s}$$

where

$$a_1^* = \frac{z^*}{1 - i\omega_0}, \quad a_2^* = \frac{b\chi^{(2)} - y^*}{b + i\omega_0}.$$

If we choose

$$D = \frac{1}{1 + a_1^*\bar{a}_1 + a_2^*\bar{a}_2 + ba_2^*\bar{a}_2\displaystyle\int_{-\infty}^{0}\theta e^{-i\omega_0\theta}k(-\theta)\mathrm{d}\theta}$$

then $< q^*(s), q(\theta) \ge 1$ and $< q^*(s), \bar{q}(\theta) \ge 0$.

Next, we use the same notations as those in Hassard[30] and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of (11) when $\mu = 0$.

Define

$$z(t) = \langle q^*, u_t \rangle W(t, \theta) = u_t(\theta) - 2\mathrm{Re}\{z(t)q(\theta)\} \tag{14}$$

on the center manifold C_0, and we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) \tag{15}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots \tag{16}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if u_t is real, we consider only real solutions. For solutions $u_t \in C_0$ of (11),

$$\dot{z}(t) = \mathrm{i}\omega_0 z + \bar{q}^*(\theta) f[0, W(z,\bar{z},\theta)] + 2\mathrm{Re}\{zq(\theta)\} \stackrel{\mathrm{def}}{=} \mathrm{i}\omega_0 z + \bar{q}^*(0)[f_1, f_2, 0]^{\mathrm{T}}.$$

That is

$$\dot{z}(t) = \mathrm{i}\omega_0 z + g(z,\bar{z})$$

where

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots.$$

Let $f_0 = [f_1, f_2]^{\mathrm{T}}$. Hence, we have

$$g(z,\bar{z}) = \bar{q}^*(0)f_0(z,\bar{z}) = f(0,u_t) = \bar{D}\left(a_1a_2 + \bar{a}_1^*\right)z^2 + \bar{D}(2\mathrm{Re}\{a_1\bar{a}_2\} + 2\bar{a}_1^*)z\bar{z} + \bar{D}(\bar{a}_1\bar{a}_2 + \bar{a}_1^*)\bar{z}^2 + \bar{D}\left[\frac{1}{2}\bar{a}_1 W_{20}^{(3)}(0) + \frac{1}{2}\bar{a}_2 W_{20}^{(3)}(0) + \bar{a}_1 W_{11}^{(3)}(0) + \bar{a}_2 W_{11}^{(2)}(0) + \bar{a}_1^*\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right)\right]z^2\bar{z} + \text{high order term}.$$

Then, we get

$$g_{20} = 2\bar{D}\left(a_1a_2 + \bar{a}_1^*\right)$$

$$g_{11} = 2\bar{D}(\mathrm{Re}\{a_1\bar{a}_2\} + \bar{a}_1^*)$$

$$g_{02} = 2\bar{D}(\bar{a}_1\bar{a}_2 + \bar{a}_1^*)$$

$$g_{21} = 2\bar{D}\left[\frac{1}{2}\bar{a}_1 W_{20}^{(3)}(0) + \frac{1}{2}\bar{a}_2 W_{20}^{(3)}(0) + \bar{a}_1 W_{11}^{(3)}(0) + \bar{a}_2 W_{11}^{(2)}(0) + \bar{a}_1^*\left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0)\right)\right].$$

Since there exist unknowns $W_{20}^{(1)}(0)$, $W_{20}^{(3)}(0)$, $W_{11}^{(1)}(0), W_{11}^{(2)}(0), W_{11}^{(3)}(0)$ in g_{21}, we still need to compute them.

It follows from (11) and (14) that

$$W' = \begin{cases} AW - 2\mathrm{Re}\{\bar{q}^*(0)[f_1, f_2, 0]^{\mathrm{T}}q(\theta)\}, -\infty \le \theta < 0 \\ AW - 2\mathrm{Re}\{\bar{q}^*(0)[f_1, f_2, 0]^{\mathrm{T}}q(\theta)\} + [f_1, f_2, 0]^{\mathrm{T}}, \\ \theta = 0 \end{cases}$$

$$\stackrel{\mathrm{def}}{=} AW + H(z,\bar{z},\theta) \qquad (17)$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \qquad (18)$$

Comparing the coefficients, we obtain

$$(AW - 2\mathrm{i}\omega_0)W_{20} = -H_{20}(\theta) \qquad (19)$$

$$AW_{11}(\theta) = -H_{11}(\theta). \qquad (20)$$

For $\theta \in [-\infty, 0)$,

$$H(z,\bar{z},\theta) = -\bar{q}^*(0)f_0 q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \qquad (21)$$

Comparing the coefficients of (21) with (18) gives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) \qquad (22)$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \qquad (23)$$

From (19), (22) and the definition of A, we get

$$\dot{W}_{20}(\theta) = 2\mathrm{i}\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \bar{g_{02}}\bar{q}(\theta). \qquad (24)$$

Noting that $q(\theta) = q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta}$, we have

$$W_{20}(\theta) = \frac{\mathrm{i}g_{20}}{\omega_0}q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\bar{g}_{02}}{3\omega_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\theta} + E_1\mathrm{e}^{2\mathrm{i}\omega_0\theta} \qquad (25)$$

where $E_1 = [E_1^{(1)}, E_1^{(2)}, E_1^{(3)}] \in \mathbf{R}^3$ is a constant vector. Similarly, from (20), (23) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g_{11}}\bar{q}(\theta) \qquad (26)$$

$$W_{11}(\theta) = -\frac{\mathrm{i}g_{11}}{\omega_0}q(0)\mathrm{e}^{\mathrm{i}\omega_0\theta} + \frac{\mathrm{i}\bar{g}_{11}}{\omega_0}\bar{q}(0)\mathrm{e}^{-\mathrm{i}\omega_0\theta} + E_2 \qquad (27)$$

where $E_2 = [E_2^{(1)}, E_2^{(2)}, E_2^{(3)}] \in \mathbf{R}^3$ is a constant vector.

In what follows, we shall seek appropriate E_1, E_2 in (25), (27), respectively. It follows from the definition of A and (22), (23) that

$$\int_{-1}^{0} \mathrm{d}\eta(\theta)W_{20}(\theta) = 2\mathrm{i}\omega_0 W_{20}(0) - H_{20}(0) \qquad (28)$$

and

$$\int_{-1}^{0} \mathrm{d}\eta(\theta)W_{11}(\theta) = -H_{11}(0) \qquad (29)$$

where $\eta(\theta) = \eta(0,\theta)$.

From (22) and (23), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\begin{bmatrix} a_1a_2 \\ 1 \\ 0 \end{bmatrix} \qquad (30)$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}(0)\bar{q}(0) + 2\begin{bmatrix} \mathrm{Re}\{a_1\bar{a}_2\} \\ 1 \\ 0 \end{bmatrix}. \qquad (31)$$

Noting that

$$\left(\mathrm{i}\omega_0 I - \int_{-1}^{0}\mathrm{e}^{\mathrm{i}\omega_0\theta}\mathrm{d}\eta(\theta)\right)q(0) = 0$$

$$\left(-\mathrm{i}\omega_0 I - \int_{-1}^{0}\mathrm{e}^{-\mathrm{i}\omega_0\theta}\mathrm{d}\eta(\theta)\right)\bar{q}(0) = 0$$

and substituting (25) and (30) into (28), we have

$$\left(2\mathrm{i}\omega_0 I - \int_{-1}^{0}\mathrm{e}^{2\mathrm{i}\omega_0\theta}\mathrm{d}\eta(\theta)\right)E_1 = 2\begin{bmatrix} a_1a_2 \\ 1 \\ 0 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 2\mathrm{i}\omega_0 & -z^* & -y^* \\ 2x^* & 2\mathrm{i}\omega_0+1 & 0 \\ 4 & 0 & 2\mathrm{i}\omega_0-b+b\chi^{(3)} \end{bmatrix} E_1 = 2\begin{bmatrix} a_1a_2 \\ 1 \\ 0 \end{bmatrix}$$

where

$$\chi^{(3)} = \int_{-\infty}^{0} k(-s)\mathrm{e}^{2\mathrm{i}\omega_0 s}\mathrm{d}s = \frac{\alpha}{\alpha+2\mathrm{i}\omega_0}.$$

It follows that

$$E_1^{(1)} = \frac{\Delta_{11}}{\Delta_1},\ E_1^{(2)} = \frac{\Delta_{12}}{\Delta_1},\ E_1^{(3)} = \frac{\Delta_{13}}{\Delta_1} \tag{32}$$

where

$$\Delta_1 = \det\begin{bmatrix} 2\mathrm{i}\omega_0 & -z^* & -y^* \\ 2x^* & 2\mathrm{i}\omega_0+1 & 0 \\ 4 & 0 & 2\mathrm{i}\omega_0-b+b\chi^{(3)} \end{bmatrix}$$

$$\Delta_{11} = 2\det\begin{bmatrix} a_1a_2 & -z^* & -y^* \\ 1 & 2\mathrm{i}\omega_0+1 & 0 \\ 0 & 0 & 2\mathrm{i}\omega_0-b+b\chi^{(3)} \end{bmatrix}$$

$$\Delta_{12} = 2\det\begin{bmatrix} 2\mathrm{i}\omega_0 & a_1a_2 & -y^* \\ 2x^* & 1 & 0 \\ 4 & 0 & 2\mathrm{i}\omega_0-b+b\chi^{(3)} \end{bmatrix}$$

$$\Delta_{13} = 2\det\begin{bmatrix} 2\mathrm{i}\omega_0 & -z^* & a_1a_2 \\ 2x^* & 2\mathrm{i}\omega_0+1 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Similarly, substituting (26) and (31) into (29), we have

$$\left(\int_{-1}^{0} \mathrm{d}\eta(\theta)\right) E_2 = 2\begin{bmatrix} \mathrm{Re}\{a_1\bar{a}_2\} \\ 1 \\ 0 \end{bmatrix}.$$

That is

$$\begin{bmatrix} 0 & z^* & y^* \\ 2x^* & -1 & 0 \\ -4 & 0 & b+b\chi^{(1)} \end{bmatrix} E_2 = 2[-\mathrm{Re}\{a_1\bar{a}_2\}, -1, 0]^{\mathrm{T}}.$$

It follows that

$$E_2^{(1)} = \frac{\Delta_{21}}{\Delta_2},\ E_2^{(2)} = \frac{\Delta_{22}}{\Delta_2},\ E_2^{(3)} = \frac{\Delta_{23}}{\Delta_2} \tag{33}$$

where

$$\Delta_2 = \det\begin{bmatrix} 0 & z^* & y^* \\ 2x^* & -1 & 0 \\ -4 & 0 & b+b\chi^{(1)} \end{bmatrix}$$

$$\Delta_{21} = 2\det\begin{bmatrix} -\mathrm{Re}\{a_1\bar{a}_2\} & z^* & y^* \\ -1 & -1 & 0 \\ 0 & 0 & b+b\chi^{(1)} \end{bmatrix}$$

$$\Delta_{22} = 2\det\begin{bmatrix} 0 & -\mathrm{Re}\{a_1\bar{a}_2\} & y^* \\ 2x^* & -1 & 0 \\ -4 & 0 & b+b\chi^{(1)} \end{bmatrix}$$

$$\Delta_{23} = 2\det\begin{bmatrix} 0 & z^* & -\mathrm{Re}\{a_1\bar{a}_2\} \\ 2x^* & -1 & -1 \\ -4 & 0 & 0 \end{bmatrix}.$$

From (25), (27), (32) and (33), we can calculate g_{21} and derive the following values:

$$\begin{cases} c_1(0) = \dfrac{\mathrm{i}}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \dfrac{|g_{02}|^2}{3}\right) + \dfrac{g_{21}}{2} \\ \mu_2 = -\dfrac{\mathrm{Re}\ \{c_1(0)\}}{\mathrm{Re}\{\lambda'(\alpha_0)\}} \\ \beta_2 = 2\mathrm{Re}(c_1(0)) \\ T_2 = -\dfrac{\mathrm{Im}\{c_1(0)\} + \mu_2\mathrm{Im}\{\lambda'(\alpha_0)\}}{\omega_0} \end{cases} \tag{34}$$

which determine the quantities of bifurcating periodic solutions on the center manifold C_0 at the critical value α_0, namely, we have the following result.

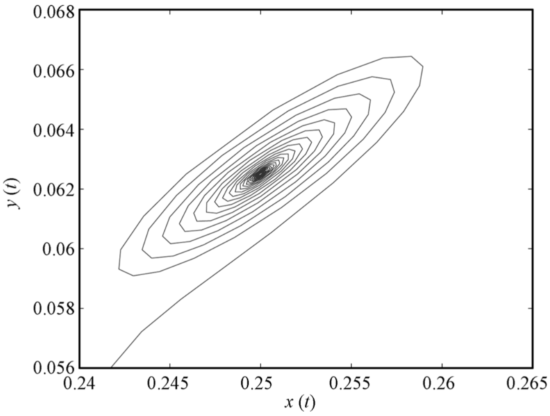
Theorem 2. The periodic solution is supercritical (subcritical) if $\mu_2 > 0$ ($\mu_2 < 0$); the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase (unstable) if $\beta_2 < 0$ ($\beta_2 > 0$); the periods of the bifurcating periodic solutions increase (decrease) if $T_2 > 0$ ($T_2 < 0$).

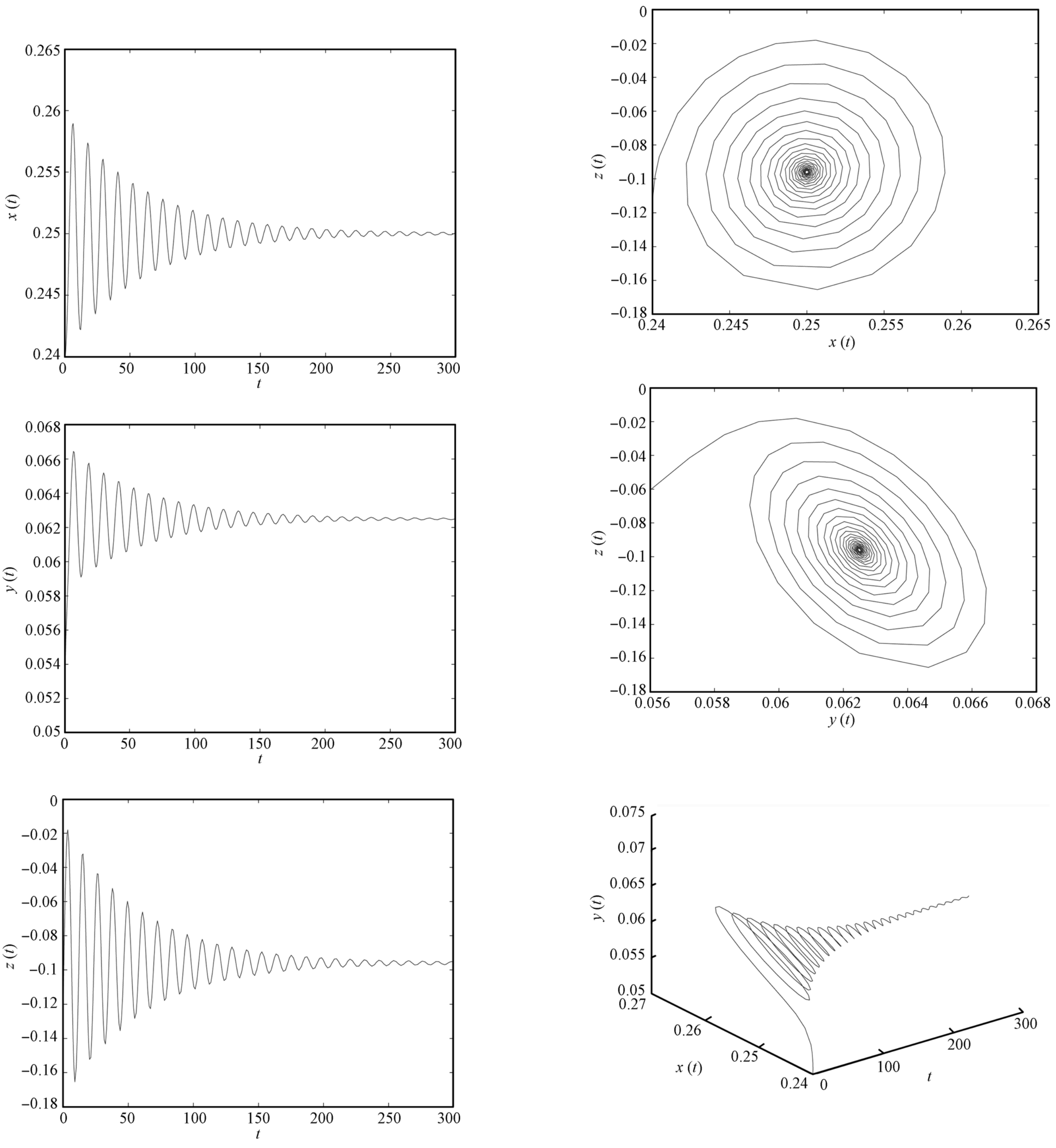
4 Computer simulations

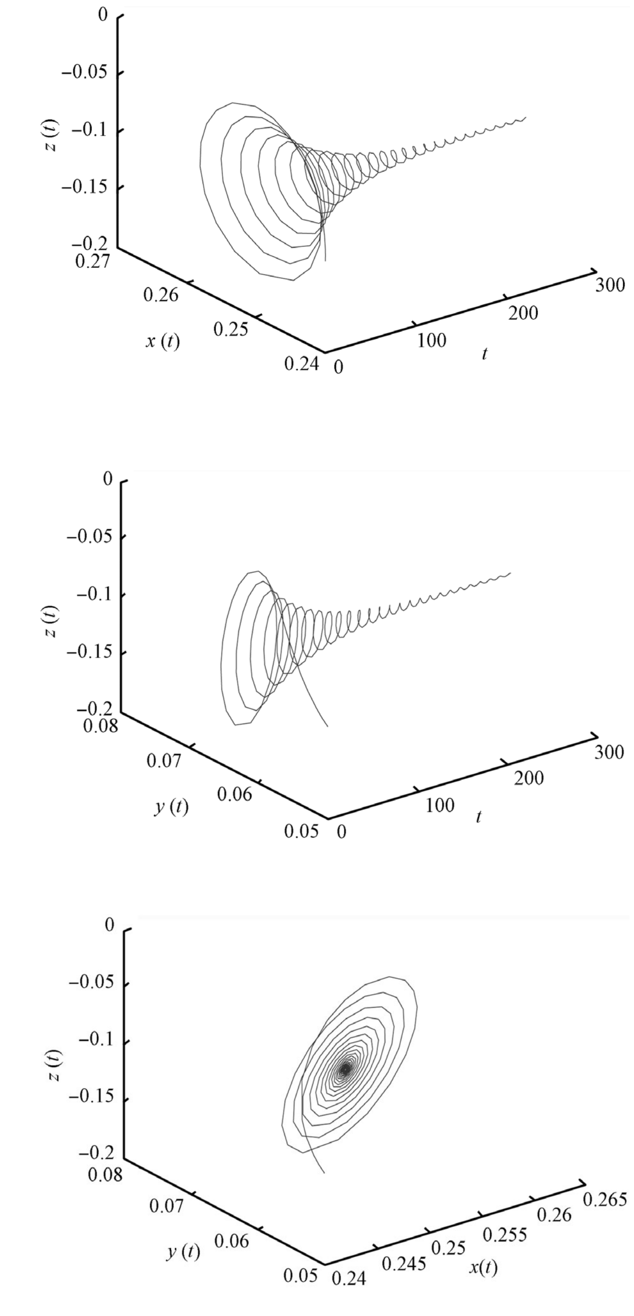
In this section, we present some numerical results of system (2) to verify the analytical predictions obtained in the previous section. From Section 3, we may determine the direction of a Hopf bifurcation and the stability of the bifurcation periodic solutions. Let us consider the following system:

$$\begin{cases} \dot{x} = y(t)z(t) + 0.006 \\ \dot{y} = x^2(t) - y(t) \\ \dot{z} = 1 - 4x(t) + 0.5\int_{-\infty}^{0}(z(t) - z(t+s))k(-s)\mathrm{d}s \end{cases} \tag{35}$$

where $k(s) = \alpha\mathrm{e}^{-\alpha s}, \alpha > 0$. Let $yz + 0.006 = 0, x^2 - y = 0, 1 - 4x = 0$. It is easy to see that system (35) has an equilibrium $\mathrm{E}(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$ and all the conditions indicated in Theorem 2 are satisfied. When $\tau = 0$, the equilibrium $E(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$ is asymptotically stable. By means of Matlab 7.0, we get $\alpha_0 \approx 0.4512$. Thus the equilibrium $E(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$ is stable when $\alpha < \alpha_0$ which is illustrated by the computer simulations (see Fig. 2). When α passes through the critical value α_0, the equilibrium $E(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$ loses its stability and a Hopf bifurcation occurs, i.e., a family of periodic solutions bifurcations from the equilibrium $E(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$. It follows from the formulae (34) presented in Section 3 that $\mu_2 > 0$ and $\beta_2 < 0$. Then the direction of the Hopf bifurcation is $\alpha > \alpha_0$, and these bifurcating periodic solutions from $E(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$ around α_0 are stable, which are depicted in Fig. 3.







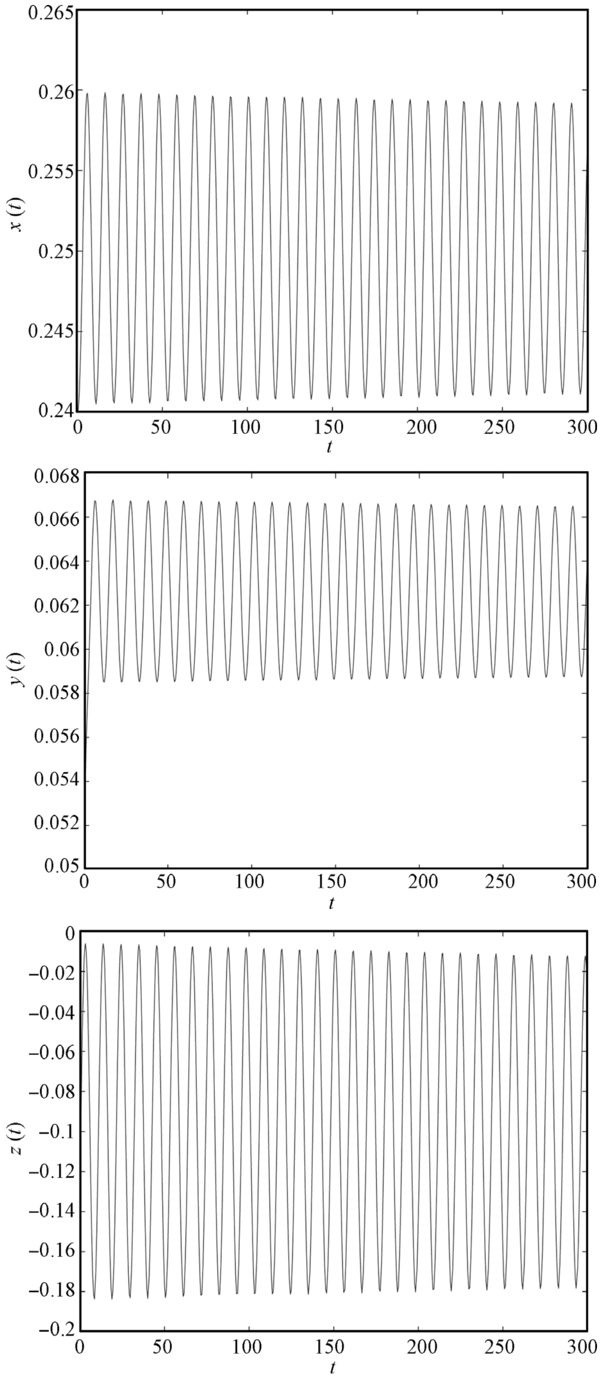
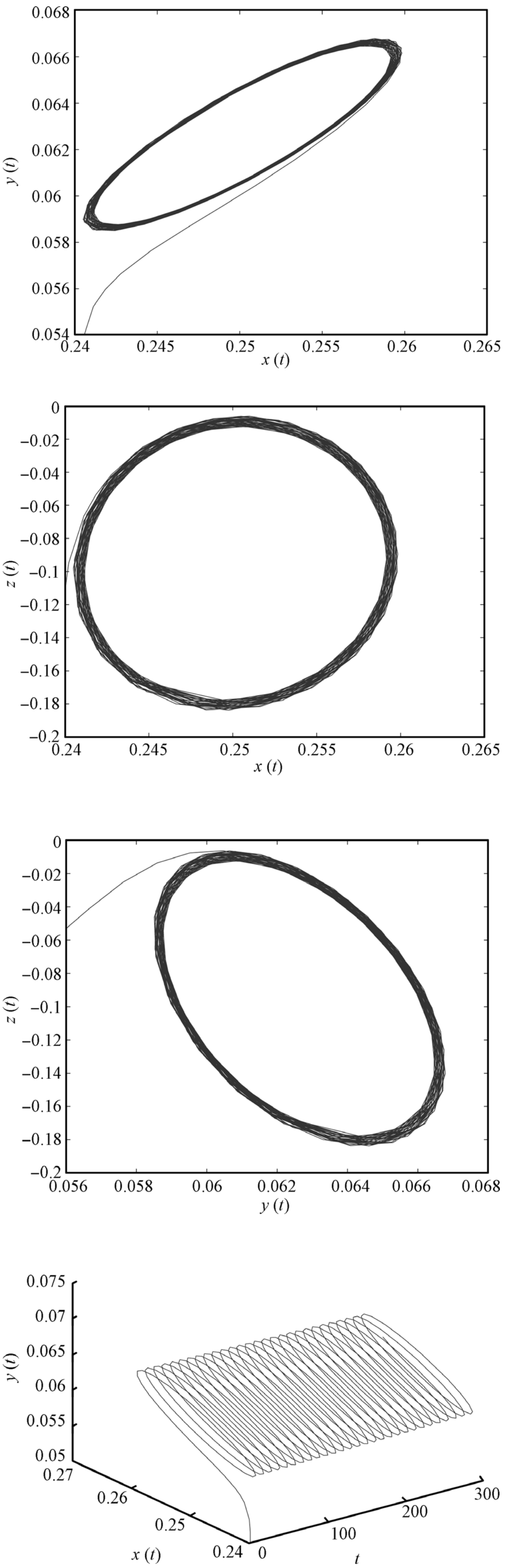


Fig. 2 Behavior and phase portraits of system (35) with $\alpha = 0.25 < \alpha_0 \approx 0.4512$. The equilibrium $E(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$ is asymptotically stable. The initial value is (1.5, 1.5, 1.5)

5 Conclusions

In this paper, a feedback control method is applied to suppress chaotic behavior of a Sprott E system within the chaotic attractor. By adding a continuous distributed time delayed force to the third equation of the Sprott E system, we have made a detailed discussion on the local stability of the equilibrium $E(x^*, y^*, z^*)$ and local Hopf bifurcation of the delayed Sprott E system model. We showed that if some conditions are fulfilled, then the Sprott E system is asymptotically stable for $\alpha < \alpha_0$ and when α passes through α_0, a sequence of Hopf bifurcations occur around the equilibrium $E(x^*, y^*, z^*)$, namely, a family of periodic orbits bifurcate from the the equilibrium $E(x^*, y^*, z^*)$, which implies the chaos of this system can be suppress. Some numerical simulations are included to visualize the theoretical findings. Moreover, the control method used in this paper can be applicable to other chaotic systems. We will carry our some related work in near future.



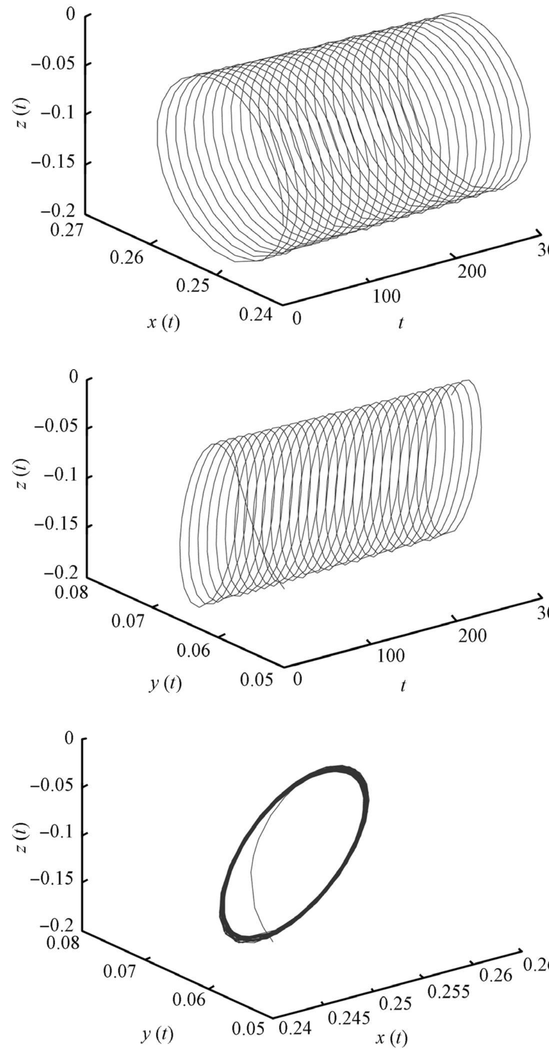


Fig. 3 Behavior and phase portraits of system (35) with $\alpha = 0.55 > \alpha_0 \approx 0.4512$. Hopf bifurcation occurs from the equilibrium $E(\frac{1}{4}, \frac{1}{16}, \frac{1}{4}, -\frac{12}{125})$. The initial value is (1.5,1.5,1.5)

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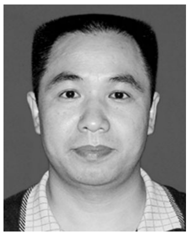

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