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Unknown Inputs Observer for a Class of Nonlinear Uncertain Systems: An LMI Approach

Kamel Mohamed[1] Mohammed Chadli[2] Mohamed Chaabane[1]
[1]Industrial Process Control Unit, National Engineering School of Sfax, Tunisia
[2]University of Picardie Jules Verne, MIS (EA 4290). 7 rue Moulin Neuf, 80000 Amiens, France

Abstract: This paper deals with the simultaneous estimation of states and unknown inputs for a class of Lipschitz nonlinear systems using only the measured outputs. The system is assumed to have bounded uncertainties that appear on both the state and output matrices. The observer design problem is formulated as a set of linear constraints which can be easily solved using linear matrix inequalities (LMI) technique. An application based on manipulator arm actuated by a direct current (DC) motor is presented to evaluate the performance of the proposed observer. The observer is applied to estimate both state and faults.

Keywords: Unknown inputs observer, uncertain nonlinear systems, Lipschitz condition, bounded uncertainties, linear matrix inequality (LMI).

1 Introduction

Over the last decades, tremendous research activities have been focused on observer design for linear or nonlinear systems with unknown inputs. This topic was motivated by the fact that state estimation together with unknown inputs can be used for control, diagnosis or supervision.

In the linear systems case, Johnson[1] and Meditch and Hostetter[2] suggest approximating the unknown inputs by the response of a known dynamic system. In [3–9], change of coordinates methods, from which a conventional observer has been designed, are proposed. In [10] an observer for the simultaneous estimation of the states and the unknown inputs is proposed. The observer design problem is solved via strong conditions, involving equalities and inequalities constraints. These constraints are then formulated as linear matrix inequality (LMI).

On the other hand, many works focus on particular classes of nonlinear systems. For the class of Lipschitz nonlinear systems, Ha and Trinh[11] have proposed an observer permitting the simultaneous estimation of the states and the unknown inputs. The observer design problem has been solved via the LMI technique. For a class of multi-inputs multi-outputs nonlinear systems with unknown inputs, the work of [12] has proposed a high gain observer to estimate states and unknown inputs. Using sliding mode technique, an LMI condition to design observer for nonlinear system in multiple model representation is given in [13] and in recent work in [14]. In [15], a robust unknown input observer for linear and nonlinear systems is proposed in LMI formulation. Other works consider different observers for descriptor systems[16], for neutral models[17] and for systems with actuator constraints[18].

However, few works which deal unknown inputs observer design for uncertain nonlinear systems exist in the literature. Moreover, to the best of our knowledge, no results consider uncertainties on output matrix.

In this study, the objective is to design an unknown input observer (UIO) for a class of nonlinear systems in the presence of uncertainties that appears on both the state and output matrices. For this purpose, a UIO to estimate both state and unknown inputs is derived in LMI terms. The paper is organized as follows. In Section 2, the considered class of nonlinear systems is introduced. Section 3 is devoted to the observer synthesis. Section 4 illustrates the obtained results using physical model. Finally, a conclusion is given in Section 5.

Notations. Throughout the paper, the following useful notation is used: X^{T} denotes the transpose of the matrix X, $X > 0$ means that X is a symmetric positive definite matrix and $\|\cdot\|$ represents the Euclidean norm for vectors and the spectral norm for matrices.

2 Problem statement

Consider the following model described by an uncertain system with Lipshitz constraint.

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + \phi(x, u) + Dv(t) \\ y(t) = (C + \Delta C)x(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the system state vector, $u(t) \in \mathbf{R}^s$ is the system input vector, $v(t) \in \mathbf{R}^m$ is the unknown input vector and $y(t) \in \mathbf{R}^p$ is the system output vector. A, C, and D are constant matrices with compatible dimensions. ΔA, ΔC are real-valued matrix functions which represent time-varying parameter uncertainties affecting the state and the output matrices, respectively.

In the following, we mention a useful lemma that will be used later in the proof of our results.

Lemma 1. For any positive scalar $\mu > 0$, the following inequality holds:

$$X^{\mathrm{T}}Y + Y^{\mathrm{T}}X \leqslant \mu X^{\mathrm{T}}X + \mu^{-1}Y^{\mathrm{T}}Y.$$

For the observer synthesis, the following assumptions are needed.

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Assumption 1. The nonlinear function $\Phi(\cdot,\cdot)$ satisfies the Lipschitz condition, i.e., for all x, $\hat{x}\in\mathbf{R}^n$,

$$\|\Phi(x,u^*)-\Phi(\hat{x},u^*)\|\leqslant k\,\|x-\hat{x}\| \quad (2)$$

where k is a Lipschitz constant. $\Phi(0,u^*)=0$, u^* is any admissible control signal.

Assumption 2. If $v(t)$ is bounded by a known positive scalar, then there exists β such that $\|v\|\leqslant\beta$.

Assumption 3. $\mathrm{rang}(CD)=m, m<p$, with D^+ is the left inverse of D, i.e., $D^+D=I$

Assumption 4. If ΔA and ΔC are bounded by known positives scalars, then there exist σ_1 and σ_2 such that $\|\Delta A\|<\sigma_1$ and $\|\Delta C\|<\sigma_2$.

3 Main result

In this section, we aim to design an observer for a class of nonlinear system in the presence of uncertainties affecting both the state and the output matrices. Let us consider the following observer:

$$\begin{cases}\dot{\hat{x}}(t)=A\hat{x}(t)+\Phi(\hat{x},u)+Dv_e(t)+\\ \qquad L(y(t)-y_e(t))+s(t)+m(t)\\ y_e(t)=C\hat{x}(t)\\ v_e(t)=\gamma G(y(t)-y_e(t))\end{cases} \quad (3)$$

where γ is a positive scalar, L, $s(t)$, $m(t)$, and G are the observer′s parameters to be determined and $\hat{x}$, v_e are the estimates of x, v, respectively.

To ensure the observer′s stability, the following theorem is given.

Theorem 1. Suppose that there exist matrices $P>0$, K, L and G, positive scalars μ_1, μ_2, η_i , $i=1,2,3$ satisfying the following linear conditions:

$$\begin{bmatrix}\Omega & K & P & P\\ K^{\mathrm{T}} & -\mu_2 & 0 & 0\\ P & 0 & -\mu_1 & 0\\ P & 0 & 0 & -\dfrac{1}{k^2}\end{bmatrix}<0 \quad (4)$$

$$D^{\mathrm{T}}P=GC$$

with

$$\Omega=A^{\mathrm{T}}P-C^{\mathrm{T}}K^{\mathrm{T}}+PA-KC+I+\sigma_1^2\eta_1+\\ \sigma_2^2\eta_2+\sigma_2^2\eta_3 \quad (5)$$

and

$$L=P^{-1}K. \quad (6)$$

Then, for any $\varepsilon>0$ such that $\lim_{t\to\infty}\sup\|x(\mathrm{t})-\hat{x}(\mathrm{t})\|\leqslant\varepsilon$, the observer (2) is defined by

$$\gamma\geqslant\frac{\beta}{\|D^{\mathrm{T}}P\|\,\varepsilon} \quad (7)$$

$$s(t)=\begin{cases}\dfrac{1}{2}\sigma_1^2\mu_1\left(1+\dfrac{\mu_1}{\eta_1-\mu_1}\right)\dfrac{P^{-1}C^{\mathrm{T}}r(t)}{r^{\mathrm{T}}(t)r(t)}\hat{x}^{\mathrm{T}}(t)\hat{x}(t),\\ \qquad\text{if } r(t)\neq0\\ s(t)=0,\quad\text{if } r(t)=0.\end{cases} \quad (8)$$

Proof. see Appendix. □

Equality (5) can be solved using existing numerical tools. It can be also approximated by an LMI formulation with minimization of an additional variable δ:

$$\min\ \delta\\ \begin{bmatrix}\delta I & D^{\mathrm{T}}P-GC\\ (D^{\mathrm{T}}P-GC)^{\mathrm{T}} & \delta I\end{bmatrix}\geqslant0. \quad (9)$$

Design steps. The estimation error cannot converge to zero but to a small neighbourhood of zero depending on the choice of $\varepsilon>0$ such that $\lim_{t\to\infty}\sup\|x(\mathrm{t})-\hat{x}(\mathrm{t})\|\leqslant\varepsilon$.

1) Solve LMI conditions (4)/(10) and compute unknown variables: $P>0$, K, L, G, μ_1, μ_2, η_i , $i=1,2,3$.

2) Deduce gains observer L, γ, $s(t)$, $m(t)$ and G from (7)–(9).

Notice that when the system is assumed without uncertainties on the output matrix, i.e., $\Delta C=0$, the observer can be written as follow:

$$\begin{cases}\dot{\hat{x}}(t)=A\hat{x}(t)+\Phi(\hat{x},u)+Dv_e(t)+\\ \qquad L(y(t)-y_e(t))+s(t)\\ y_e(t)=C\hat{x}(t)\\ v_e(t)=\gamma G(y(t)-y_e(t))\end{cases} \quad (10)$$

with (11), and the following design observer conditions are given.

Corollary 1. Suppose that there exist matrices $P>0$, K, L and G, positive scalars μ_1 and η_1 satisfying the following linear conditions:

$$\begin{bmatrix}\Phi & P & P\\ P & -\mu_1 & 0\\ P & 0 & -\dfrac{1}{k^2}\end{bmatrix}<0 \quad (12)$$

$$D^{\mathrm{T}}P=GC \quad (13)$$

with

$$\Phi=A^{\mathrm{T}}P-C^{\mathrm{T}}K^{\mathrm{T}}+PA-KC+I+\sigma_1^2\eta_1.$$

$$r(t)=Ce(t)$$

$$m(t)=\begin{cases}\dfrac{1}{2}\sigma_2^2\left(\mu_2\left(1+\dfrac{\mu_2}{\eta_2-\mu_2}\right)+\mu_3\left(1+\dfrac{\mu_2}{\eta_2-\mu_2}\right)\right)\dfrac{P^{-1}C^{\mathrm{T}}r(t)}{r^{\mathrm{T}}(t)r(t)}\hat{x}^{\mathrm{T}}(t)\hat{x}(t)+\\ \dfrac{1}{2}\gamma\dfrac{\eta_2}{\eta_3\mu_2}\|G\|^4P^{-1}C^{\mathrm{T}}r(t)\quad\text{if } r(t)\neq0,\\ 0\quad\text{if } r(t)=0\end{cases} \quad (11)$$

Then, for any $\varepsilon > 0$, such that $\lim_{t\to\infty} \sup \|x(\mathrm{t}) - \hat{x}(\mathrm{t})\| \leqslant \varepsilon$, the observer (11) is defined by (6)–(9) and

$$s(t) = \begin{cases} \dfrac{1}{2}\sigma_1^2\mu_1\left(1+\dfrac{\mu_1}{\eta_1-\mu_1}\right)\dfrac{P^{-1}C^{\mathrm{T}}r(t)}{r^{\mathrm{T}}(t)r(t)}\hat{x}^{\mathrm{T}}(t)\hat{x}(t), \\ \qquad \text{if } r(t) \neq 0 \\ 0, \quad \text{if } r(t) = 0. \end{cases} \tag{14}$$

Remark 1. It is important to note that a potential problem arises in the implementation of the given observer. Thus when the output estimation error $r(t) = Ce(t)$ tends towards zero, the magnitude of $s(t)$ and $m(t)$ may increase without bound. This problem is overcome as follows: the terms $s(t)$ and $m(t)$ are fixed to zero when the output estimation error is such that $\|r(t)\| \leqslant \lambda$ where λ is a small positive number chosen by the user. It goes without saying that the estimation error converges to a small neighbourhood of zero. Notice also that the choice of large values of gain γ produces a large sensitivity to the noises.

4 Example: manipulator arm actuated by a DC motor

The proposed approach is tested on a real process of a manipulator arm with revolute joints actuated by a DC motor[19] as shown in Fig. 1. The nonlinear model of this system is given by

$$\begin{cases} \dot{\theta}_m(t) = \omega_m(t) \\ \dot{\omega}_m(t) = \dfrac{k}{J_m}(\theta_l(t) - \theta_m(t)) - \dfrac{B}{J_m}\omega_m(t) + \dfrac{k_\tau}{J_m}u(t) \\ \dot{\theta}_l(t) = \omega_l(t) \\ \dot{\omega}_l(t) = -\dfrac{k}{J_l}(\theta_l(t) - \theta_m(t)) - \dfrac{mgh}{J_l}\sin(\theta_l(t)) \end{cases} \tag{15}$$

where $\theta_m(t)$ stands for the angular rotation of the motor, $\omega_m(t)$ is the angular velocity of the motor, $\theta_l(t)$ is the angular position of the link, and $\omega_l(t)$ is the angular velocity of the link. The input signal is given by $u(t) = \sin(t)$, and the initial condition is $x_0 = 0$ for the system and $\hat{x}_0 = 1$ for the observer.

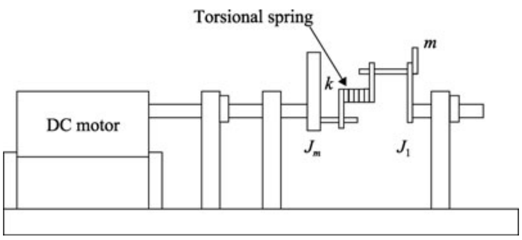

Fig. 1 Manipulator arm actuated by a DC motor

The parameters definition and their values are given in Table 1.

Table 1 Model parameters of manipulator arm actuated by a DC motor

Parameter	Meaning	Value
J_m	Motor inertia	3.7×10^{-3} kg·m^2
J_l	Link inertia	9.3×10^{-3} kg·m^2
m	Pointer mass	2.1×10^{-1} kg
h	Link length	3.1×10^{-1} m
k	Torsional spring constant	1.8×10^{-1} N·m·rad^{-1}
B	Viscous friction coefficient	4.6×10^{-2} N·m·V^{-1}
k_τ	Amplifier gain	8×10^{-2} N·m·V^{-1}

This system can be submitted to faults that are considered as unknown inputs[20]. Then the system (15) can be rewritten as follows:

$$\begin{cases} \dot{x}(t) = (A + \Delta A)x(t) + Bu(t) + \Phi(x) + Dv(t) \\ y(t) = (C + \Delta C)x(t). \end{cases} \tag{16}$$

The state vector is $x(t) = [\theta_m(t)\omega_m(t)\theta_l(t)\omega_l(t)]^{\mathrm{T}}$, the measured outputs vector is $y(t) = [\theta_m(t)\omega_m(t)]^{\mathrm{T}}$ and the considered matrices:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 1.95 & 0 & -1.95 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \end{bmatrix}$$

and $\Phi(x) = \begin{bmatrix} 0 & 0 & -3.33\sin(x_3(t)) & 0 \end{bmatrix}^{\mathrm{T}}$. It follows that the Lipschitz constant of $\Phi(x)$ is $k = 0.333$.

For simulations, the considered uncertainties are assumed bounded (Assumption 4) as follows: $\|\Delta A\| < \sigma_1$ and $\|\Delta C\| < \sigma_2$ with $\sigma_1 = \sigma_2 = 0.2$.

The proposed approach assumes that the estimation error cannot converge to zero asymptotically but to a small positive number (ε) chosen by the user (see Remark 1). The simulation results are carried out with $\|r(t)\| \leqslant \lambda$ with $\lambda = 0.01$.

4.1 Simulation results

By using the *feasp* and *mincx* functions of the LMI Toolbox of Matlab, the resolution of (4) and (10) lead to the following data:

$$L = \begin{bmatrix} 9.9278 & -1.5049 \\ -6.8376 & 61.9656 \\ 9.5740 & 20.5173 \\ 20.9922 & 43.2715 \end{bmatrix} \qquad G = \begin{bmatrix} 30.6892\ 9.3064 \end{bmatrix};$$

$$P = \begin{bmatrix} 13.3537 & 1.9909 & -6.6508 & 0.1017 \\ 1.9909 & 2.6623 & -3.3772 & 0.0434 \\ -6.6508 & -3.3772 & 15.3582 & -1.9655 \\ 0.1017 & 0.0434 & -1.9655 & 0.8718 \end{bmatrix}$$

$\mu_1 = 273.9789$, $\mu_2 = 487.4580$

$\eta_1 = 0.0056$, $\eta_2 = 26.6006$, $\eta_3 = 26.6006$.

Figs. 2 and 3 show the estimation of state and fault (unknown input) using the designed observer with $\gamma = 1.2$. We note a good estimation of the non-measured outputs $\theta_l(t)$ and $\omega_l(t)$ of the system as well as the unknown input despite the presence of the uncertainties on state and on the output matrices.

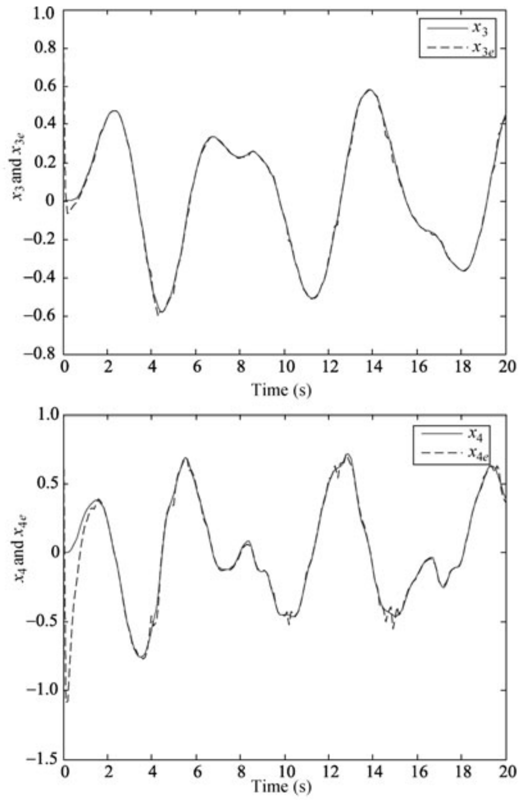

Fig. 2 The non-measured outputs $\theta_l(t)$ and $\omega_l(t)$ and their estimates

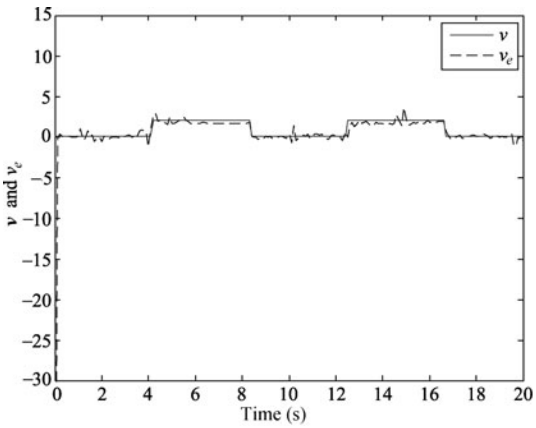

Fig. 3 The unknown input and its estimate

5 Conclusions

In this work, a class of uncertain nonlinear systems is considered. The system is assumed to have bounded uncertainties affecting both the state and output matrices. An unknown input observer is developed to estimate the states and the unknown inputs simultaneously. The stability conditions of such observer are expressed in terms of LMI. The given designed observer can be used for fault detection procedure. The performance of the proposed observer has been demonstrated in simulation through a manipulator arm actuated by a DC motor.

Appendix

Proof of Corollary 1. The estimation error is defined by

$$e(t) = x(t) - \hat{x}(t). \tag{A1}$$

From (1) and (2), the dynamics of the estimation error is

$$\begin{aligned} \dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = \\ (A - LC)e(t) + \Delta A x(t) - L\Delta C x(t) + \Phi(x, u) - \\ \Phi(\hat{x}, u)\dot{e}(t) + Dv(t) - Dv_e(t) - s(t) - m(t). \end{aligned} \tag{A2}$$

That can be rewritten as follow

$$\begin{aligned} \dot{e}(t) = \dot{x}(t) - \dot{x}_e(t) = \\ A_e e(t) + (\Delta A - L\Delta C)x(t) + \Delta\Phi + \\ Dv(t) - Dv_e(t) - s(t) - m(t) \end{aligned} \tag{A3}$$

with $\Delta\Phi = \Phi(x, u) - \Phi(\hat{x}, u)$ and $A_e = A - LC$.

Consider Lyapunov function candidate

$$Ve(t) = e^{\mathrm{T}}(t)Pe(t), \quad P > 0 \tag{A4}$$

the derivative of along the system (1) yields to

$$\begin{aligned} \dot{V} = \dot{e}^{\mathrm{T}}(t)Pe(t) + e^{\mathrm{T}}(t)P\dot{e}(t) \ = \\ e^{\mathrm{T}}(\mathrm{t})(A_e^{\mathrm{T}}P + PA_e)e(\mathrm{t}) + 2e^{\mathrm{T}}(\mathrm{t})P\Delta\Phi + \\ 2e^{\mathrm{T}}(\mathrm{t})PDv(\mathrm{t}) - 2e^{\mathrm{T}}PDv_e(\mathrm{t}) + \\ 2e^{\mathrm{T}}P(\Delta A - L\Delta C)x(\mathrm{t}) - 2e^{\mathrm{T}}Ps(\mathrm{t}) - 2e^{\mathrm{T}}Pm(\mathrm{t}) = \\ e^{\mathrm{T}}(\mathrm{t})(A_e^{\mathrm{T}}P + PA_e)e(\mathrm{t}) + 2e^{\mathrm{T}}(\mathrm{t})P\Delta\Phi + \\ 2e^{\mathrm{T}}(\mathrm{t})PDv(\mathrm{t}) - 2\gamma e^{\mathrm{T}}(\mathrm{t})PDGCe(\mathrm{t}) - 2e^{\mathrm{T}}(\mathrm{t})Ps(\mathrm{t}) + \\ 2e^{\mathrm{T}}(\mathrm{t})P\Delta Ax(\mathrm{t}) - 2e^{\mathrm{T}}(\mathrm{t})Pm(\mathrm{t}) - \\ 2\gamma e^{\mathrm{T}}(\mathrm{t})PDG\Delta Cx(\mathrm{t}) - 2e^{\mathrm{T}}(\mathrm{t})PL\Delta Cx(\mathrm{t}). \end{aligned}$$

Using the condition (5), we get

$$\begin{aligned} \dot{V} = e^{\mathrm{T}}(t)(A_e^{\mathrm{T}}P + PA_e)e(t) + 2e^{\mathrm{T}}(t)P\Delta\Phi + \\ 2e^{\mathrm{T}}(t)PDv(t) - 2\gamma e^{\mathrm{T}}(t)PDD^{\mathrm{T}}Pe(t) - 2e^{\mathrm{T}}(t)Ps(t) - \\ 2e^{\mathrm{T}}(t)Pm(t) - 2\gamma e^{\mathrm{T}}(t)PDG\Delta Cx(t) + \\ 2e^{\mathrm{T}}(t)P\Delta Ax(t) - 2e^{\mathrm{T}}(t)PL\Delta Cx(t) = \\ e^{\mathrm{T}}(t)(A_e^{\mathrm{T}}P + PA_e)e(t) + 2e^{\mathrm{T}}(t)P\Delta\Phi + \\ 2\|D^{\mathrm{T}}Pe\|\,\|v\| - 2\gamma\left\|D^{\mathrm{T}}Pe\right\|^2 - 2e^{\mathrm{T}}(t)Ps(t) - \\ 2e^{\mathrm{T}}(t)Pm(t) - 2\gamma e^{\mathrm{T}}(t)PDG\Delta Cx(t) + \\ 2e^{\mathrm{T}}(t)P\Delta Ax(t) - 2e^{\mathrm{T}}(t)PL\Delta Cx(t). \end{aligned}$$

Using again Assumption 2, we obtain

$$\begin{aligned} \dot{V} \leqslant e^{\mathrm{T}}(t)(A_e^{\mathrm{T}}P + PA_e)e(t) + 2e^{\mathrm{T}}(t)P\Delta\Phi + \\ 2\|\beta D^{\mathrm{T}}Pe\| - 2\gamma\|D^{\mathrm{T}}Pe\|^2 - 2e^{\mathrm{T}}Ps(t) - \\ 2e^{\mathrm{T}}Pm(t) - 2\gamma e^{\mathrm{T}}(t)PDG\Delta Cx(t) + \\ 2e^{\mathrm{T}}(t)P\Delta Ax(t) - 2e^{\mathrm{T}}(t)PL\Delta Cx(t). \end{aligned} \tag{A5}$$

According to Assumption 1, we have $2e^{\mathrm{T}}P\Delta\Phi \leqslant 2k\,\|Pe\|\,\|e\|$ and from Lemma 1, we have

$$2e^{\mathrm{T}}P\Delta\Phi \leqslant k^2 e^{\mathrm{T}}PPe + e^{\mathrm{T}}e. \tag{A6}$$

Using Lemma 1, for any positive scalars μ_1, μ_2, μ_3, μ'_1 and μ'_2, the following inequalities hold:

$$\begin{aligned}2e^{\rm T}(t)P\Delta Ax(t) \leqslant{}& \mu_1 x^{\rm T}(t)(\Delta A)^{\rm T}\Delta Ax(t) + \\ & \mu_1^{-1}e^{\rm T}(t)PPe(t) = \\ & \mu_1\sigma_1^2 x^{\rm T}(t)x(t) + \mu_1^{-1}e^{\rm T}(t)PPe(t) = \\ & \mu_1\sigma_1^2(\hat{x}^{\rm T}+e^{\rm T})(\hat{x}+e) + \mu_1^{-1}e^{\rm T}(t)PPe(t) = \\ & \mu_1\sigma_1^2\hat{x}^{\rm T}\hat{x} + \mu_1\sigma_1^2(\hat{x}^{\rm T}e + e^{\rm T}\hat{x}) + \\ & \mu_1\sigma_1^2 e^{\rm T}e + \mu_1^{-1}e^{\rm T}(t)PPe(t) = \\ & \mu_1\sigma_1^2\hat{x}^{\rm T}\hat{x} + \mu_1\sigma_1^2(\mu'_1\hat{x}^{\rm T}\hat{x} + \mu_1'^{-1}e^{\rm T}e) + \\ & \mu_1\sigma_1^2 e^{\rm T}e + \mu_1^{-1}e^{\rm T}(t)PPe(t) = \\ & \mu_1\sigma_1^2(1+\mu'_1)\hat{x}^{\rm T}\hat{x} + \mu_1\sigma_1^2(1+\mu_1'^{-1})e^{\rm T}e + \\ & \mu_1^{-1}e^{\rm T}(t)PPe(t)\end{aligned} \tag{A7}$$

$$\begin{aligned}2e^{\rm T}(t)P\Delta Cx(t) \leqslant{}& \mu_2 x^{\rm T}(t)(\Delta C)^{\rm T}\Delta Cx(t) + \\ & \mu_2^{-1}e^{\rm T}(t)PLL^{\rm T}Pe(t) = \\ & \mu_2\sigma_2^2 x^{\rm T}(t)x(t) + \mu_2^{-1}e^{\rm T}(t)PLL^{\rm T}Pe(t) = \\ & \mu_2\sigma_2^2(\hat{x}^{\rm T}+e^{\rm T})(\hat{x}+e) + \mu_2^{-1}e^{\rm T}(t)PLL^{\rm T}Pe(t) = \\ & \mu_2\sigma_2^2\hat{x}^{\rm T}\hat{x} + \mu_2\sigma_2^2(\hat{x}^{\rm T}e + e^{\rm T}\hat{x}) + \\ & \mu_2\sigma_2^2 e^{\rm T}e + \mu_2^{-1}e^{\rm T}(t)PLL^{\rm T}Pe(t) = \\ & \mu_2\sigma_2^2\hat{x}^{\rm T}\hat{x} + \mu_2\sigma_2^2(\mu'_2\hat{x}^{\rm T}\hat{x} + \mu_2'^{-1}e^{\rm T}e) + \\ & \mu_2\sigma_2^2 e^{\rm T}e + \mu_2^{-1}e^{\rm T}(t)PLL^{\rm T}Pe(t) = \\ & \mu_2\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} + \mu_2\sigma_2^2(1+\mu_2'^{-1})e^{\rm T}e + \\ & \mu_2^{-1}e^{\rm T}(t)PLL^{\rm T}Pe(t)\end{aligned} \tag{A8}$$

and

$$\begin{aligned}& 2\gamma e^{\rm T}PDG\Delta Cx \leqslant \\ & \quad \mu_3 x^{\rm T}(\Delta C)^{\rm T}\Delta Cx + \gamma\mu_3^{-1}e^{\rm T}PDGG^{\rm T}D^{\rm T}Pe = \\ & \quad \mu_3\sigma_2^2 x^{\rm T}(t)x(t) + \gamma\mu_3^{-1}e^{\rm T}PDGG^{\rm T}D^{\rm T}Pe = \\ & \quad \mu_3\sigma_2^2(\hat{x}^{\rm T}+e^{\rm T})(\hat{x}+e) + \gamma\mu_3^{-1}e^{\rm T}PDGG^{\rm T}D^{\rm T}Pe = \\ & \quad \mu_3\sigma_2^2\hat{x}^{\rm T}\hat{x} + \mu_3\sigma_2^2(\hat{x}^{\rm T}e + e^{\rm T}\hat{x}) + \mu_3\sigma_2^2 e^{\rm T}e + \\ & \quad \gamma\mu_3^{-1}e^{\rm T}PDGG^{\rm T}D^{\rm T}Pe = \\ & \quad \mu_3\sigma_2^2\hat{x}^{\rm T}\hat{x} + \mu_3\sigma_2^2(\mu'_2\hat{x}^{\rm T}\hat{x} + \mu_2'^{-1}e^{\rm T}e) + \mu_3\sigma_2^2 e^{\rm T}e + \\ & \quad \gamma\mu_3^{-1}e^{\rm T}PDGG^{\rm T}D^{\rm T}Pe = \\ & \quad \mu_3\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} + \mu_3\sigma_2^2(1+\mu_2'^{-1})e^{\rm T}e + \\ & \quad \gamma\mu_3^{-1}e^{\rm T}PDGG^{\rm T}D^{\rm T}Pe.\end{aligned}$$

Using condition (5) with $r(t) = Ce(t)$, one gets

$$\begin{aligned}2\gamma e^{\rm T}PDG\Delta Cx \leqslant{}& \mu_3\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} + \\ & \mu_3\sigma_2^2(1+\mu_2'^{-1})e^{\rm T}e + \gamma\mu_3^{-1}e^{\rm T}C^{\rm T}G^{\rm T}GG^{\rm T}GCe \\ & \mu_3\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} + \mu_3\sigma_2^2(1+\mu_2'^{-1})e^{\rm T}e + \\ & \gamma\mu_3^{-1}\,\|G\|^4\, r^{\rm T}r.\end{aligned} \tag{A9}$$

Substituting (A6)–(A9) in (A5), we obtain:

$$\begin{aligned}\dot{V} \leqslant{}& e^{\rm T}\{A_e^{\rm T}P + PA_e + k^2PP + \mu_1^{-1}(t)PP + \\ & \mu_2^{-1}(t)PLL^{\rm T}P + I + \mu_1\sigma_1^2(1+\mu_1'^{-1}) + \\ & \mu_2\sigma_2^2(1+\mu_2^{-1}) + \mu_3\sigma_2^2(1+\mu_2'^{-1})\}e(t) + \\ & 2\|\beta D^{\rm T}Pe\| - 2\gamma\|D^{\rm T}Pe\|^2 - 2e^{\rm T}Ps(t) - \\ & 2e^{\rm T}(t)Pm(t) + \mu_1\sigma_1^2(1+\mu'_1)\hat{x}^{\rm T}\hat{x} + \\ & \mu_2\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} + \mu_3\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} + \\ & \gamma\mu_3^{-1}\|G\|^4 r^{\rm T}r \leqslant \\ & e^{\rm T}(t)\Omega e(t) + 2\|\beta D^{\rm T}Pe\| - 2\gamma\|D^{\rm T}Pe\|^2 - \\ & 2e^{\rm T}Ps(t) - 2e^{\rm T}(t)Pm(t) + \\ & \mu_1\sigma_1^2(1+\mu'_1)\hat{x}^{\rm T}\hat{x} - \mu_2\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} - \\ & \mu_3\sigma_2^2(1+\mu'_2)\hat{x}^{\rm T}\hat{x} - \gamma\mu_3^{-1}\|G\|^4 r^{\rm T}r\end{aligned} \tag{A10}$$

where

$$\begin{aligned}\Omega ={}& A_e^{\rm T}P + PA_e + k^2PP + \mu_1^{-1}(t)PP + \\ & \mu_2^{-1}(t)PLL^{\rm T}P + I + \mu_1\sigma_1^2(1+\mu_1'^{-1}) + \\ & \mu_2\sigma_2^2(1+\mu_2'^{-1}) + \gamma\mu_3\sigma_2^2(1+\mu_3'^{-1}) = \\ & A^{\rm T}P + PA - L^{\rm T}P - PL + k^2PP + \\ & \mu_1^{-1}(t)PP + \mu_2^{-1}(t)PLL^{\rm T}P + I + \sigma_1^2\eta_1 + \\ & \sigma_2^2\eta_2 + \sigma_2^2\eta_3\end{aligned}$$

with $\eta_1 = \mu_1(1+\mu_1'^{-1}), \eta_2 = \mu_2(1+\mu_2'^{-1})$ and $\eta_3 = \mu_3(1+\mu_2'^{-1})$. Then with the definitions (8) and (9) we get

$$\dot{V} \leqslant e^{\rm T}(t)\Omega e(t) + 2\|\beta D^{\rm T}Pe\| - 2\gamma\|D^{\rm T}Pe\|^2. \tag{A11}$$

For any $\varepsilon > 0$ such that $\lim_{t\to\infty}\sup\|e(t)\| \leqslant \varepsilon$, we show that with the following condition

$$\gamma \geqslant \frac{\beta}{\|D^{\rm T}P\|\,\varepsilon} \tag{A12}$$

we get $\dot{V} \leqslant e^{\rm T}(t)\Omega e(t)$. The schur complement of (4) guarantees $\Omega < 0$ and then $\dot{V} > 0$. □

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Kamel Mohamed received his engineer′s degree from the University of Tunis in 1992 and the Ph. D. degree in automatic control and computer engineering from the University of Sfax, Tunisia in 2011.

His research interests include observer for nonlinear systems, fault detection and isolation (FDI), and application to industrial process.

E-mail: kamel68@yahoo.fr

Mohammed Chadli received the master degree (DEA) from National Institute of Applied Sciences, France in 1999 and the Ph. D. degree from National Polytechnic Institute of Lorraine of Nancy-France in 2002. From 1999 to 2004, he was an associate researcher in National Polytechnic Institute of Lorraine. Since 2004, he has been associate professor at the University of Picardie Jules Verne and a researcher in the ′Modélisation, Information et Systèmes′ Laboratory (MIS) in Amiens, France.

His research interests include theoretical side, analysis and control of singular (switched) systems, analysis and control of fuzzy/multiple model approach, robust control, fault detection and isolation (FDI), fault tolerant control (FTC), analysis and control via LMI optimization techniques and Lyapunov methods. On the application side he is mainly interested in automotive control.

E-mail: mohammed.chadli@u-picardie.fr (Corresponding author)

Mohamed Chaabane is working as a professor in automatic control at National School of Engineers of Sfax (ENIS), Tunisia. He received his doctorate degree in electrical engineering from the University of Nancy, France in 1991. Since 1997, he has held a research position at Automatic Control Unit, ENIS.

His research interests include robust and optimal control, delay systems, descriptor systems.

E-mail: chaabane@yahoo.fr