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The Dynamics and Bifurcation Control of a Singular Biological Economic Model

Ning Li[1] Hai-Yi Sun[2] Qing-Ling Zhang[1]
[1]Institute of Systems Science, Northeastern University, Shenyang 110819, PRC
[2]College of Science, Shenyang Jianzhu University, Shenyang 110168, PRC

Abstract: The objective of this paper is to study systematically the dynamics and control strategy of a singular biological economic model that is described by a differential-algebraic equation. It is shown that when the economic profit passes through zero, this model exhibits the transcritical bifurcation, the Hopf bifurcation, and the limit cycle. In particular, the system undergoes the singularity induced bifurcation at the positive equilibrium, which can result in impulse. Then, state feedback controllers closer to the actual control strategies are designed to eliminate the unexpected singularity induced bifurcation and stabilize the positive equilibrium under the positive profit. Finally, numerical simulations verify the results and illustrate the effectiveness of the controllers. Also, the model with positive economic profit is shown numerically to have different dynamics.

Keywords: Differential-algebraic equation, transcritical bifurcation, Hopf bifurcation, limit cycle, singularity induced bifurcation, bifurcation control.

1 Introduction

Recently, the study of biological system has attracted much attention from both biologists and mathematicians[1−4]. In this paper, we consider the following predator-prey system with undercrowding effect and Holling II functional response[5, 6]

$$\begin{cases} \dot{\bar{x}}(\tau) = e\bar{x}(\bar{x}-L)\left(1-\dfrac{\bar{x}}{K}\right) - \dfrac{b\bar{x}\bar{y}}{1+h\bar{x}} \\ \dot{\bar{y}}(\tau) = -f\bar{y} + \dfrac{d\bar{x}\bar{y}}{1+h\bar{x}} \end{cases} \tag{1}$$

where $\bar{x}(\tau)$ and $\bar{y}(\tau)$ represent the density of prey and predator at time τ. The term $e\bar{x}(\bar{x}-L)(1-\bar{x}/K)$ denotes the growth of the prey with undercrowding effect, where e, L, and K stand for the intrinsic growth rate, the low threshold, and the carrying capacity of the prey population in the absence of predation, respectively. $b\bar{x}/(1+h\bar{x})$ is the Holling II functional response. f is the predator death rate, and d represents the conversion efficiency of prey into predators.

For mathematical simplicity, let us introduce the following non-dimensional variables[6]: $x = \bar{x}/K$, $y = \dfrac{b}{ehK^2}\bar{y}$, $\mathrm{d}t = eK/(a+x)\mathrm{d}\tau$, then (1) takes the form:

$$\begin{cases} \dot{x}(t) = x\left[(x-r)(a+x)(1-x)-y\right] \\ \dot{y}(t) = \alpha y(x-\beta) \end{cases} \tag{2}$$

where $r = L/K$, $a = 1/hK$, $\alpha = (d-fh)/ehK$, $\beta = fah/(d-fh)$.

In recent years, there has been a wide range of interest in the use of bioeconomic models to gain insight into the scientific management of renewable resources. In this paper, we focus on (2) with harvesting. Without loss of generality, we assume that the predator is continuously being harvested. Then (2) becomes

$$\begin{cases} \dot{x}(t) = x\left[(x-r)(a+x)(1-x)-y\right] \\ \dot{y}(t) = \alpha y(x-\beta) - Ey \end{cases} \tag{3}$$

where $E(t)$ represents the capture capability at time t, and $E(t)y(t)$ indicates that the harvest of predator is proportional to its density at time t.

In (3), the prey, the predator, and the harvest are in the same ecosystem without human control. Notice that the economic profit is not mentioned in (3); we will take the profit into consideration and model the biological economic system with a differential-algebraic equation.

To the best of our knowledge, the study of singular systems in biology is a relatively new research field. In [7, 8], passivity, optimal control, and dynamics analysis were studied, respectively. A class of state feedback controller is presented in [9, 10]. In this paper, for (3), using the economic theory of fishery resource[11], we study the following singular biological economic model:

$$\begin{cases} \dot{x}(t) = x\left[(x-r)(a+x)(1-x)-y\right] \\ \dot{y}(t) = \alpha y(x-\beta) - yE \\ \quad 0 = E(py-c) - m \end{cases} \tag{4}$$

where $p > 0$, $c > 0$, and $m \geqslant 0$ are the unit harvest reward, cost, and the economic profit, respectively. Then Epy and Ec are the total rewards and cost. Here, we consider the simplest condition that the harvest rewards and cost are all positive constants.

In (4), the first two equations indicate the increase rate of the prey and predator, and the last equation indicates the relation of the total rewards, cost, and the economic profit. It is obvious that (4) is a differential-algebraic system, in which x and y are differential variables, and E is an algebraic variable.

The objective of this paper is to study systematically the dynamical behaviors of (4) when the economic profit

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increases from negative to positive and passes through zero, then propose singular state feedback controllers to stabilize the bifurcation orbits. Compared with the controllers in [9, 10], here we present a class of feedback controllers that has a more generalized form and is more close to the control strategies existing in practice.

The organization of this paper is as follows: with an eye on the model with zero economic profit, in Section 2, equilibria and their stability are discussed. Section 3 studies the dynamics near each equilibrium. In Section 4, a class of feedback controllers is presented. Finally, numerical simulations are given.

2 Equilibria and their stability

When the economic profit increases through zero, (4) can be written as:

$$\begin{cases} \dot{x}(t) = x\left[(x-r)(a+x)(1-x)-y\right] \\ \dot{y}(t) = \alpha y(x-\beta) - yE \\ \quad 0 = E(py-c). \end{cases} \tag{5}$$

First, let us begin to determine the number and location of the equilibria of (5) in the closed first quadrant $\mathbf{R}^2_+ = \{(x,y)|x \geqslant 0, y \geqslant 0\}$. It is clear that (5) always has boundary equilibria $P_1(0,0,0)$, $P_2(r,0,0)$, and $P_3(1,0,0)$ for all parameters. For the case $r<\beta<1$, (5) also has a boundary equilibrium $P_4(\beta, (\beta-r)(1-\beta)(a+\beta), 0)$. Here we suppose that (5) has a positive equilibrium $P^*(x^*, y^*, E^*)$, where x^* is the positive real root of the equation $(x-r)(1-x)(a+x)-c/p=0$, $y^*=c/p$, and $E^*=\alpha(x^*-\beta)$.

Next, we consider the stability of (5) in the neighborhood of each equilibrium. We have

Theorem 1. Assume $0<r<1$.

1) P_1 is a stable node.

2) If $r<\beta$, then P_2 is a saddle; if $r>\beta$, then P_2 is an unstable node.

3) If $\beta<1$, then P_3 is a saddle; if $\beta>1$, then P_3 is a stable node.

4) Suppose that $r<\beta<1$, and let $W=-3\beta^2+2(1+r-a)\beta+a+ar-r$. If $W<0$, then P_4 is a stable node (or focus); if $W>0$, then P_4 is an unstable node (or focus).

Proof. The Jacobian matrix of (5) at $P_1(0,0,0)$ is

$$\boldsymbol{J}_{P_1} = \begin{bmatrix} -ar & 0 & 0 \\ 0 & -\alpha\beta & 0 \\ 0 & 0 & -c \end{bmatrix}$$

then the characteristic polynomial is

$$\det(\lambda\boldsymbol{M} - \boldsymbol{J}_{P_1}) = c(\lambda+ar)(\lambda+\alpha\beta)$$

where

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then the eigenvalues are $\lambda_1=-ar<0$, $\lambda_2=-\alpha\beta<0$. It is obvious that the conclusion (1) holds. Using the same method, the other conclusions can be proved. □

Straightforward computation shows that $P^*(x^*,y^*,E^*)$ is at the singular surface[12], and the dynamics near $P^*(x^*,y^*,E^*)$ will be analyzed in the next section.

3 Dynamics

From Theorem 1 we can see that the equilibria P_2 and P_3 exchange their stability when r increases through 1, i.e., one real eigenvalue at P_2 moves from $\mathbf{C}^+$ to $\mathbf{C}^-$, while the corresponding real eigenvalue at P_3 moves from $\mathbf{C}^-$ to $\mathbf{C}^+$. Moreover, P_2 and P_3 become one point when $r=1$. Therefore, choosing r as the bifurcation parameter, we have

Theorem 2. If $\beta \neq 1$, then (5) undergoes the transcritical bifurcation at $P_3(1,0,0)$, and $r=1$ is the bifurcation value.

Proof. For simplicity, let

$$f(X,E,m) = \begin{bmatrix} x[(x-r)(1-x)(a+x)-y] \\ y[\alpha(x-\beta)-E] \end{bmatrix}$$
$$g(X,E,m) = E(py-c) - m$$

where $X=(x,y)^{\mathrm{T}}$.

We choose r as the bifurcation parameter. If $m=0$, then $(D_E g)_{P_3} = -c \neq 0$, where D denotes the partial differential operator. Therefore, by the implicit function theorem, in a neighborhood of $P_3(1,0,0)$, (5) can be reduced to

$$\begin{cases} \dot{x}(t) = x\left[(x-r)(a+x)(1-x)-y\right] \\ \dot{y}(t) = \alpha y(x-\beta). \end{cases} \tag{6}$$

We denote (6) by $\dot{X} = f_R(X,r)$.

It is clear that $P_3^{'}(1,0)$ is an equilibrium of (6), and when $r=1$, the Jacobian matrix of (6) at $P_3^{'}(1,0)$

$$(D_X f_R)_{P_3^{'}} = \begin{bmatrix} 0 & -1 \\ 0 & \alpha(1-\beta) \end{bmatrix}$$

has a simple zero eigenvalue with right eigenvector $\Psi = (1,0)^{\mathrm{T}}$ and left eigenvector $\Omega = (\alpha(1-\beta)/[1+\alpha^2(1-\beta)^2]^{1/2}, 1/[1+\alpha^2(1-\beta)^2]^{1/2})$, and

$$(\Omega\,(D_X D_r f_R)\,\Psi)_{P_3^{'}} = \left[\frac{\alpha(1-\beta)}{[1+\alpha^2(1-\beta)^2]^{1/2}}, \frac{1}{[1+\alpha^2(1-\beta)^2]^{1/2}}\right] \times \begin{bmatrix} a+1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\alpha(\alpha+1)(1-\beta)}{[1+\alpha^2(1-\beta)^2]^{1/2}} \neq 0$$

$$\left(\Omega D_X^2 f_R(\Psi,\Psi)\right)_{P_3^{'}} = \left(\Omega \sum_{i=1}^{2}\left(e_i \Psi^{\mathrm{T}} D_X\left(D_X f_i\right)^{\mathrm{T}}\Psi\right)\right)_{P_3^{'}} = \frac{-2\alpha(a+1)(1-\beta)}{[1+\alpha^2(1-\beta)^2]^{1/2}} \neq 0.$$

By the theorems in [12], (6) undergoes the transcritical bifurcation at the equilibrium $P_3^{'}(1,0)$, that is, (5) undergoes the transcritical bifurcation at the equilibrium $P_3(1,0,0)$. □

From Theorem 1, if $r < \beta < 1$, and $W = 0$, then the Jacobian matrix at P_4 has a pair of purely imaginary eigenvalues. We have

Theorem 3. If the parameters satisfy: $r < \beta < 1$, $p(\beta - r_0)(1-\beta)(a+\beta) - c \neq 0$, where $r_0 = [3\beta^2 - 2(1-a)\beta - a]/(-1+a+2\beta) > 0$, then for some r in the neighborhood of r_0, there exists a periodic solution of (5) near the equilibrium $P_4(x_4, y_4, 0)$ with period $2\pi/\omega$, where $x_4 = \beta, y_4 = (\beta - r_0)(1-\beta)(a+\beta)$, $\omega = (\alpha\beta y_4)^{1/2}$, and $r = r_0$ is the Hopf bifurcation value.

Proof. If $r = r_0 < \beta < 1$, then (5) has a boundary equilibrium $P_4(x_4, y_4, 0)$, where $x_4 = \beta$, $y_4 = (\beta - r_0)(1-\beta)(a+\beta)$. Similar to the proof of Theorem 2, (5) can be reduced to (6) $\dot{X} = f_R(X, r)$ in a neighborhood of P_4. It is clear that $P_4'(x_4, y_4)$ is an equilibrium point of (6). Moreover, when $r = r_0$,

$$(\mathrm{tr}\,(D_X f_R))_{P_4'} = 0, \quad (\det(D_X f_R))_{P_4'} = \alpha\beta y_4 > 0.$$

Suppose

$$(D_X f_R)_{P_4'}(x_4, y_4, r) = (D_X f_R)_{P_4'}(x_4, y_4, r_0) + (r - r_0)B(x_4, y_4, r)$$

then

$$\mathrm{tr}B(x_4, y_4, r_0) = \mathrm{tr}\lim_{r\to r_0} B(x_4, y_4, r) =$$

$$\mathrm{tr}\lim_{r\to r_0} \frac{(D_X f_R)_{P_4'}(x_4, y_4, r) - (D_X f_R)_{P_4'}(x_4, y_4, r_0)}{r - r_0} =$$

$$\lim_{r\to r_0} \frac{\mathrm{tr}\,(D_X f_R)_{P_4'}(x_4, y_4, r) - \mathrm{tr}\,(D_X f_R)_{P_4'}(x_4, y_4, r_0)}{r - r_0} =$$

$$\lim_{r\to r_0} \frac{\beta(1-\beta)(a+\beta) - \beta(\beta - r)(a+\beta) + \beta(\beta - r)(1-\beta)}{r - r_0} =$$

$$\beta(-1+a+2\beta) \neq 0.$$

By the bifurcation theorems in [13], the conclusion holds. □

Next we consider the limit cycle of (5). Let $x = \xi + x_4$, $y = \eta + y_4$, then (6) becomes

$$\begin{cases} \dot{\xi} = a_{11}\eta + a_{12}\xi\eta + a_{13}\xi^2 + a_{14}\xi^3 \\ \dot{\eta} = a_{21}\xi + a_{22}\xi\eta \end{cases} \tag{7}$$

where

$$\begin{aligned} &a_{11} = -x_4, && a_{12} = -1, \\ &a_{13} = x_4(-3x_4 + r_0 + 1 - a), && a_{14} = -4x_4 + r_0 + 1 - a, \\ &a_{21} = \alpha y_4, && a_{22} = \alpha. \end{aligned}$$

The Jacobian matrix of (7) at $(0,0)$ has a pair of purely imaginary eigenvalues $\lambda_{1,2}(r_0) = \pm \mathrm{j}\omega$, where $\omega = (\alpha\beta y_4)^{1/2}$. Perform the transformation near $(0,0)$:

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{(\alpha\beta y_4)^{\frac{1}{2}}}{\beta} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

we can obtain

$$\begin{cases} \dot{u} = b_{11}v + b_{12}uv + b_{13}u^2 + b_{14}u^3 \\ \dot{v} = b_{21}u + b_{22}uv \end{cases} \tag{8}$$

where

$$\begin{aligned} &b_{11} = -(\alpha\beta y_4)^{\frac{1}{2}}, && b_{12} = -(\alpha\beta y_4)^{\frac{\frac{1}{2}}{\beta}}, \\ &b_{13} = x_4(-3x_4 + r_0 + 1 - a), && b_{14} = -4x_4 + r_0 + 1 - a, \\ &b_{21} = (\alpha\beta y_4)^{\frac{1}{2}}, && b_{22} = \alpha. \end{aligned}$$

Theorem 4. System (5) has at least one stable limit cycle near $P_4(x_4, y_4, 0)$ if $r > r_0$ and $(r - r_0)$ is small enough, and they move towards $P_4(x_4, y_4, 0)$ as r tends to r_0.

Proof. Denote the solutions of (8), which satisfy the initial conditions $u(0) = c$, $v(0) = 0$ are $u = u(t)$, $v = v(t)$, and the time they need to return to u axis is T, then $T = 2\pi/\omega\left(1 + h_2c^2 + h_3c^3 + \cdots\right)$, where $h_2, h_3, \cdots$, are constants.

Let $t/T = \tau/2\pi$, then (8) becomes

$$\begin{cases} \dot{u}(\tau) = \left(-v + \dfrac{b_{12}}{\omega}uv + \dfrac{b_{13}}{\omega}u^2 + \dfrac{b_{14}}{\omega}u^3\right)\left(1 + h_2c^2 + \right. \\ \qquad\qquad \left. h_3c^3 + \cdots\right) \\ \dot{v}(\tau) = \left(u + \dfrac{b_{22}}{\omega}uv\right)\left(1 + h_2c^2 + h_3c^3 + \cdots\right). \end{cases} \tag{9}$$

If $\tau = 0$, then $t = 0$. Hence, the initial conditions are still $u(0) = c$, $v(0) = 0$. Considering the analyticity of $u(\tau)$, $v(\tau)$ to c, for small c, $u(\tau)$ and $v(\tau)$ can be denoted by

$$\begin{cases} u(\tau) = u_1(\tau)c + u_2(\tau)c^2 + \cdots \\ v(\tau) = v_1(\tau)c + v_2(\tau)c^2 + \cdots \end{cases} \tag{10}$$

and the initial conditions of u_i and v_i are as follows

$$\begin{cases} u_1(0) = 1 \\ u_2(0) = u_3(0) = \cdots = v_1(0) = v_2(0) = \cdots = 0. \end{cases} \tag{11}$$

Carry (10) in (9), and compare the coefficients of c, c^2, and c^3, we can obtain

$$\begin{cases} \dot{u}_1(\tau) = -v_1 \\ \dot{v}_1(\tau) = u_1 \end{cases} \tag{12}$$

$$\begin{cases} \dot{u}_2(\tau) = -v_2 + \dfrac{b_{12}}{\omega}u_1v_1 + \dfrac{b_{13}}{\omega}u_1^2 = -v_2 + P_2 \\ \dot{v}_2(\tau) = u_2 + \dfrac{b_{22}}{\omega}u_1v_1 = u_2 + Q_2 \end{cases} \tag{13}$$

$$\begin{cases} \dot{u}_3(\tau) = -v_3 - h_2v_1 + \dfrac{b_{12}}{\omega}(u_1v_2 + u_2v_1) + \\ \qquad \dfrac{2b_{13}}{\omega}u_1u_2 + \dfrac{b_{14}}{\omega}u_1^3 = \\ \qquad -v_3 - h_2v_1 + P_3 \\ \dot{v}_3(\tau) = u_3 + h_2u_1 + \dfrac{b_{22}}{\omega}(u_1v_2 + u_2v_1) = \\ \qquad u_3 + h_2u_1 + Q_3. \end{cases} \tag{14}$$

For (12), by the initial conditions (11), we can get

$$u_1(\tau) = \cos\tau, \qquad v_1(\tau) = \sin\tau.$$

It is clear that u_1 and v_1 are periodic 2π functions.

Because

$$I_2 = \int_0^{2\pi} (P_2 \cos\tau + Q_2 \sin\tau)\mathrm{d}\tau = 0$$

the solutions of (13) u_2 and v_2 are also periodic 2π functions, where

$$\begin{cases} u_2(\tau) = \dfrac{b_{12}}{3\omega}\cos\tau - \dfrac{(b_{22}-b_{13})}{3\omega}\sin\tau + \dfrac{(b_{22}+2b_{13})}{6\omega}\sin 2\tau - \\ \qquad \dfrac{b_{12}}{3\omega}\cos 2\tau \\ v_2(\tau) = \dfrac{b_{12}}{3\omega}\sin\tau + \dfrac{(b_{22}-b_{13})}{3\omega}\cos\tau - \dfrac{(2b_{22}+b_{13})}{6\omega}\cos 2\tau - \\ \qquad \dfrac{b_{12}}{6\omega}\sin 2\tau + \dfrac{b_{13}}{2\omega}. \end{cases}$$

By computation we can get

$$I_3 = \int_0^{2\pi} (P_3 \cos\tau + Q_3 \sin\tau)\,\mathrm{d}\tau = \frac{-3\beta\pi}{\omega} < 0$$

which indicates that $(0, 0)$ is a stable fine focus of (8) with multiplicity one, that is, $P_4^{'}(x_4, y_4)$ is a stable fine focus of (6) with multiplicity one.

By Theorem 1, we know that as r increases through r_0, and $(r - r_0)$ is small enough, $P_4^{'}(x_4, y_4)$ is still the focus of (6) and the stability of $P_4^{'}(x_4, y_4)$ changes, i.e., from stable to unstable. According to the theorems in [13, 14], the conclusion holds. □

Next, we will investigate the dynamics near the positive equilibrium $P^*(x^*, y^*, E^*)$. The following notations are introduced as in [15]: for any vector $\boldsymbol{v} \in \mathbf{R}^p$, $\langle \boldsymbol{v} \rangle = \{s\boldsymbol{v} : s \in \mathbf{R}\}$, and $N_\delta(\boldsymbol{v})$ denotes a neighborhood of $\boldsymbol{v}$ with radius δ; $\mathbf{L} : \mathbf{R}^p \to \mathbf{R}^p$ is a linear mapping, then $N(\mathbf{L})$ and $R(\mathbf{L})$ denotes its null space and range, respectively.

Theorem 5. If $A(x^*, y^*) \neq 0$, where $A(x, y) = (x - r)(1 - x)(a + x) + x(1 - x)(a + x) - x(x - r)(a + x) + x(x - r)(1 - x) - y$, then near $P^*(x^*, y^*, E^*)$, there exists a locus of eigenvalue $\lambda(m) = pE^{*2}y^*/m + \phi(m)$ of (4), where $\phi(m)$ is a continuous function of m. Moreover, there is a $\delta > 0$, such that for all $m \in N_\delta(0) \setminus \{0\}$, the other eigenvalue remains in some compact set that does not contain the origin, that is, it does not affect the stability of $P^*(x^*, y^*, E^*)$. Hence, (4) undergoes the singularity induced bifurcation (SIB) at $P^*(x^*, y^*, E^*)$, and $m = 0$ is the bifurcation value. Moreover, a stability switching occurs as m increases through 0.

Proof. Choose m as the bifurcation parameter, the Jacobian matrix of (4) at the equilibrium $(x(m), y(m), E(m))$ can be denoted by

$$\mathbf{J}(\mathrm{m}) = \left[\begin{array}{cc|c} A(x(m), y(m)) & -x(m) & 0 \\ \alpha y(m) & \alpha(x(m) - \beta) - E(m) & -y(m) \\ \hline 0 & pE(m) & py(m) - c \end{array}\right] = \begin{bmatrix} \boldsymbol{A}(m) & \boldsymbol{B}(m) \\ \boldsymbol{C}(m) & \boldsymbol{D}(m) \end{bmatrix}$$

where $A(x(m), y(m)) = (x(m) - r)(1 - x(m))(a + x(m)) + x(m)(1 - x(m))(a + x(m)) - x(m)(x(m) - r)(a + x(m)) + x(m)(x(m) - r)(1 - x(m)) - y(m)$.

We suppose that (5) has a positive equilibrium $P^*(x^*, y^*, E^*)$, and we have

$$\mathbf{J}(0) = \left[\begin{array}{cc|c} A(x^*, y^*) & -x^* & 0 \\ \alpha y^* & 0 & -y^* \\ \hline 0 & pE^* & 0 \end{array}\right] = \begin{bmatrix} \boldsymbol{A}(0) & \boldsymbol{B}(0) \\ \boldsymbol{C}(0) & \boldsymbol{D}(0) \end{bmatrix}$$

then

$$N(\boldsymbol{D}(0)) = N(\boldsymbol{D}(0)^{\mathrm{T}}) = \mathbf{R} = \langle 1 \rangle, \quad R(\boldsymbol{D}(0)) = \{0\}.$$

If $E \neq 0$, then $py - c = m/E$, thus we have $\boldsymbol{D}^{'}(0) = 1/E^* \notin R(\boldsymbol{D}(0))$,

$$\boldsymbol{C}(0)\boldsymbol{B}(0) = (0, pE^*)\begin{pmatrix} 0 \\ -y^* \end{pmatrix} = -pE^*y^* \notin R(\boldsymbol{D}(0))$$

and $\boldsymbol{J}(0)$ is invertible.

By the theorems in [15], the conclusion holds. □

Theorem 5 implies that when the economic profit increases through zero, there exists one real eigenvalue of the linearization of (4) about a parameterized equilibrium locus, which moves from one open half of the complex plane to the other by diverging to infinity. Biologically, these may bring impulse, i.e., rapid expansion of the population, which may lead to an imbalance in the ecosystem and disaster for humans. Moreover, the positive equilibrium $P^*(x^*, y^*, E^*)$ becomes unstable when m is positive.

4 Feedback control of SIB

In this section, the control strategy will be designed to eliminate the unexpected singularity induced bifurcation and stabilize the positive equilibrium when the economic profit is positive.

Let the feedback control have the form of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [0, 0, k] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix}$, where k is the gain, then the closed-loop system is

$$\begin{cases} \dot{x}(t) = x\left[(x - r)(a + x)(1 - x) - y\right] \\ \dot{y}(t) = \alpha y(x - \beta) - yE \\ 0 \quad\ = E(py - c) + k(E - E^*). \end{cases} \tag{15}$$

It is easy to verify that (15) is locally controllable.

Theorem 6. Let the feedback control have the form of $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot [0, 0, k] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix}$, if the gain k satisfies: $A + pE^*y^*/k < 0$, $\alpha x^*y^* + pE^*Ay^*/k > 0$, then P^* (x^*, y^*, E^*) is stable.

Proof. The Jacobian matrix of (15) at $P^*(x^*, y^*, E^*)$ is

$$\boldsymbol{J}_{P^*} = \begin{bmatrix} A & -x^* & 0 \\ \alpha y^* & 0 & -y^* \\ 0 & pE^* & k \end{bmatrix}$$

where the definition of A is the same as in Theorem 5. Then the characteristic equation is

$$k\lambda^2 - (kA + pE^*y^*)\,\lambda + k\alpha x^*y^* + pE^*Ay^* = 0.$$

If $P^*(x^*, y^*, E^*)$ is stable, then

$$\lambda_1 + \lambda_2 = \frac{kA + pE^*y^*}{k} < 0$$

$$\lambda_1\lambda_2 = \frac{k\alpha x^*y^* + pE^*Ay^*}{k} > 0$$

that is, k satisfies

$$\begin{cases} A + \dfrac{pE^*y^*}{k} < 0 \\ \alpha x^*y^* + \dfrac{pE^*Ay^*}{k} > 0. \end{cases} \tag{16}$$

□

The above theorem tells us that when the economic profit increases through zero, with an eye on long-term development, we should adopt some suitable policy to control the harvest and adjust the gain in a proper range to eliminate the singularity induced bifurcation.

Meanwhile, in practice, only controlling the harvest is not enough, and measures of controlling the density of the population such as releasing the species are also adopted. Therefore, we consider the generalized feedback control $\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot [k_1, 0, 0] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} \cdot [0, k_2, 0] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} \cdot [0, 0, k_3] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix}$, where k_1, k_2 and k_3 are the gains, then the closed-loop system is

$$\begin{cases} \dot{x}(t) = x\left[(x - r)(a + x)(1 - x) - y\right] + k_1(x - x^*) \\ \dot{y}(t) = \alpha y(x - \beta) - yE + k_2(y - y^*) \\ 0 \quad = E(py - c) + k_3(E - E^*). \end{cases} \tag{17}$$

Using the same approach, we can get

Theorem 7. Let the feedback control have the form of $\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot [k_1, 0, 0] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} \cdot [0, k_2, 0] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} \cdot [0, 0, k_3] \cdot \begin{bmatrix} x(t) - x^* \\ y(t) - y^* \\ E(t) - E^* \end{bmatrix}$, if the gain k_1, k_2 and k_3 satisfy: $A + k_1 + k_2 + pE^*y^*/k_3 < 0$, $k_2(A + k_1) + \alpha x^*y^* + pE^*y^*(A + k_1)/k_3 > 0$, where the definition of A is the same as in Theorem 5, then $P^*(x^*, y^*, E^*)$ is stable.

When the economic profit m is positive, we suppose (4) has a positive equilibrium $\tilde{P^*}(\tilde{x^*}, \tilde{y^*}, \tilde{E^*})$, where $\tilde{x^*}$ is the positive real root of the equation $(x - \beta)[p(x - r)(a + x)(1 - x) - c] = m/\alpha$, $\tilde{y^*} = (\tilde{x^*} - r)(a + \tilde{x^*})(1 - \tilde{x^*})$, $\tilde{E^*} = \alpha(\tilde{x^*} - \beta)$. From Theorem 5 we know that $\tilde{P^*}(\tilde{x^*}, \tilde{y^*}, \tilde{E^*})$ is unstable. Using similar methods, we have

Theorem 8. Let the feedback control have the form of $\begin{bmatrix}0\\0\\1\end{bmatrix} \cdot [0, 0, k] \cdot \begin{bmatrix} x(t) - \tilde{x}^* \\ y(t) - \tilde{y}^* \\ E(t) - \tilde{E}^* \end{bmatrix}$, if the gain k satisfies: $A + p\tilde{E^*}\tilde{y^*}/k < 0$, $\alpha\tilde{x^*}\tilde{y^*} + pA\tilde{E^*}\tilde{y^*}/k > 0$, where the definition of A is the same as in Theorem 5, then $\tilde{P^*}(\tilde{x^*}, \tilde{y^*}, \tilde{E^*})$ is stable.

Theorem 9. Let the feedback control have the form of $\begin{bmatrix}1\\0\\0\end{bmatrix} \cdot [k_1, 0, 0] \cdot \begin{bmatrix} x(t) - \tilde{x^*} \\ y(t) - \tilde{y^*} \\ E(t) - \tilde{E^*} \end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix} \cdot [0, k_2, 0] \cdot \begin{bmatrix} x(t) - \tilde{x^*} \\ y(t) - \tilde{y^*} \\ E(t) - \tilde{E^*} \end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} \cdot [0, 0, k_3] \cdot \begin{bmatrix} x(t) - \tilde{x^*} \\ y(t) - \tilde{y^*} \\ E(t) - \tilde{E^*} \end{bmatrix}$, if the gain k_1, k_2 and k_3 satisfy: $A + k_1 + k_2 + p\tilde{E^*}\tilde{y^*}/k_3 < 0$, $k_2(A + k_1) + \alpha\tilde{x^*}\tilde{y^*} + p\tilde{E^*}\tilde{y^*}(A + k_1)/k_3 > 0$, where the definition of A is the same as in Theorem 5, then $\tilde{P^*}(\tilde{x^*}, \tilde{y^*}, \tilde{E^*})$ is stable.

5 Numerical simulations

In this section, four examples are given. The first two examples verify the above results and illustrate the effectiveness of the control strategy. The next two examples study (4) numerically to show the different dynamics when the economic profit is positive.

Example 1. Let $a = 1.8$, $\beta = 0.6$, $\alpha = 1$, $p = 10$, $c = 1$, and $m = 0$, then the parameters satisfy the conditions of Theorem 3. Therefore, there exists a periodic solution near the equilibrium point $P_4'(0.6, 0.4608)$ with period $2\pi/[(0.6 \times 0.4608)^{1/2}] = 11.9495$, and $r_0 = 0.12$ is a Hopf bifurcation value. This is depicted in Fig. 1.

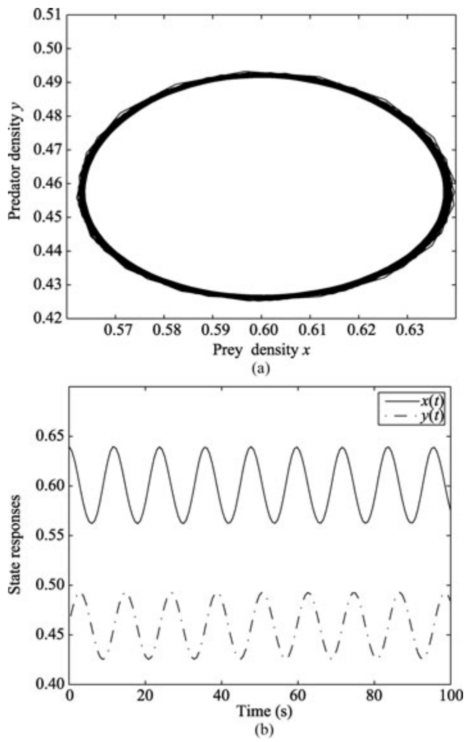

Fig. 1 Phase trajectories (a) and time responses (b) of state variables

Example 2. Let $r = 0.2$, $a = 5$, $\beta = 0.5$, $\alpha = 2.5$, $p = 1$, $c = 0.69$, and $m = 0$, then system (4) has a unique positive equilibrium $P^*(x^*, y^*, E^*)$=(0.8027,0.69,0.7568). By Theorem 5, near $P^*(x^*, y^*, E^*)$, there exists an eigenvalue $\lambda(m) = 0.3952/m + \phi(m)$ which moves from $\mathbf{C}^-$ to $\mathbf{C}^+$ by diverging to ∞ as m passes 0 from $\mathbf{R}^-$ to $\mathbf{R}^+$ along the real axis.

By Theorem 7, if the gain k_1, k_2, and k_3 satisfy: $-1.7928 + k_1 + k_2 + 0.5222/k_3 < 0$, $k_2(-1.7928 + k_1) + 1.3847 + 0.5222(-1.7928 + k_1)/k_3 > 0$, then the SIB is eliminated and $P^*(x^*, y^*, E^*)$ is stable. This is illustrated in Fig. 2.

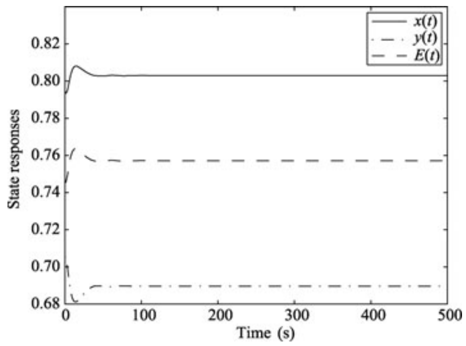

Fig. 2 Time responses for $k_1 = 0.5, k_2 = 0.5, k_3 = 1$

Let $m = 0.01 > 0$, then system (4) has a positive equilibrium $\tilde{P^*}(\tilde{x^*}, \tilde{y^*}, \tilde{E^*}) = (0.7966, 0.7035, 0.7414)$. By Theorem 9, if the gain k_1, k_2 and k_3 satisfy: $-1.7191 + k_1 + k_2 + 0.5216/k_3 < 0$, $k_2(-1.7191 + k_1) + 1.401 + 0.5216(-1.7191 + k_1)/k_3 > 0$, then $\tilde{P^*}(\tilde{x^*}, \tilde{y^*}, \tilde{E^*})$ is stable. This is illustrated in Fig. 3.

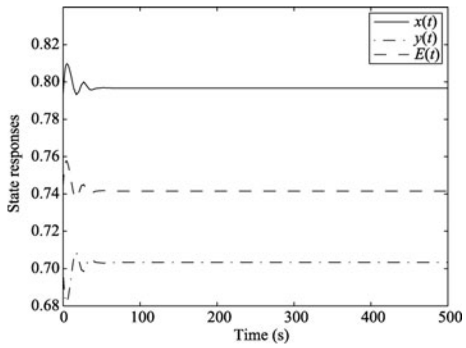

Fig. 3 Time responses for $k_1 = 0.5, k_2 = 0.5, k_3 = 1$

Example 3. We consider another bifurcation, saddle-node bifurcation. Let $r = 0.5$, $a = 1$, $\beta = 0.5$, $\alpha = 1$, $p = 10$, and $c = 1$. If $m \neq 0$, (4) is locally homeomorphic to

$$\begin{cases} \dot{x}(t) = x\left[(x - 0.5)(1 + x)(1 - x) - y\right] \\ \dot{y}(t) = y(x - 0.5) - \dfrac{my}{10y - 1} \end{cases} \tag{18}$$

which is denoted by $\dot{X}(t) = f_R(X, m) = (f_1(X, m), f_2(X, m))^{\rm T}$.

Choose m as the bifurcation parameter. There exist two positive equilibria when $m < m^* = 0.02705$, and no positive real equilibrium when $m > m^*$. For $m = m^*$, there is a unique positive equilibrium $P^*(0.7772, 0.1098, 0.2772)$. Furthermore,

$$(\boldsymbol{D}_X f_R)_{P^*} = \begin{bmatrix} -0.0273 & -0.7772 \\ 0.1098 & 3.1188 \end{bmatrix}$$

has a geometrically simple zero eigenvalue with the right eigenvector $\Psi = (-28.4713, 1)^{\rm T}$ and the left eigenvector $\Omega = (4.0204, 1)$. Also,

$$(\Omega \boldsymbol{D}_m f_R)_{P^*} = -1.1248 \neq 0$$

$$\left(\Omega \boldsymbol{D}_X^2 f_R(\Psi, \Psi)\right)_{P^*} = \left(\Omega \sum_{i=1}^{2} \left(e_i \Psi^{\rm T} \boldsymbol{D}_X (\boldsymbol{D}_X f_i)^{\rm T} \Psi\right)\right)_{P^*} = 1.0138 \neq 0$$

where e_i, $i = 1, 2$, is the unit vector in $\mathbf{R}^2$.

By the theorems in [16], (4) undergoes a saddle-node bifurcation at the equilibrium $P^*(0.7772, 0.1098, 0.2772)$, and $m = m^* = 0.02705$ is the bifurcation value.

Example 4. We consider another bifurcation, the Bogdanov-Takens bifurcation. Let $a = 1$, $\alpha = 1$, $\beta = 0.4$, $p = 10$, and $c = 1$. Choose r and m as the bifurcation parameters. When $m \neq 0$, (4) is locally homeomorphic to

$$\begin{cases} \dot{x}(t) = x\left[(x - r)(1 + x)(1 - x) - y\right] \\ \dot{y}(t) = y(x - 0.4) - \dfrac{my}{10y - 1}. \end{cases} \tag{19}$$

When $r = 0.0022$, $m = 0.7953$, (19) has a unique positive equilibrium $(x_0, y_0) = (0.7458, 0.33)$, and the trace and the determinant of the variational matrix of (19) at (x_0, y_0) are zero. Next, we reduce (19) to canonical form by using normal form theory[17]. Let $x_1 = x - x_0$, $y_1 = y - y_0$, then (19) becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{y}_1(t) \end{bmatrix} = \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} + \mathcal{O}(\| X_1 \|^3) \tag{20}$$

where $\Pi_1 = -0.4962x_1 - 0.7458y_1 - x_1y_1 - 2.3324x_1^2$, $\Pi_2 = 0.33x_1 + 0.4962y_1 + x_1y_1 - 0.6537y_1^2$. $\mathcal{O}(\| X_1 \|^3)$ is C^∞ in all variables, at least to the third order with respect to $X_1 = (x_1, y_1)^{\rm T}$.

Making the linear change of variables:

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -0.4962 & -0.7458 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

then (20) becomes

$$\begin{bmatrix} \dot{x}_2(t) \\ \dot{y}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} + \mathcal{O}(\| X_2 \|^3) \tag{21}$$

where $\Gamma_1 = -1.6671x_2^2 + 1.3408x_2y_2$, $\Gamma_2 = 1.5392x_2^2 + 1.2045x_2y_2 + 0.8765y_2^2$, $X_2 = (x_2, y_2)^{\rm T}$.

By the theory of normal forms, there exists a smooth invertible transformation in a small neighborhood of $(0, 0)$:

$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} + \begin{bmatrix} -1.1087x_2^2 \\ -1.6671x_2^2 - 0.8765x_2y_2 \end{bmatrix}$$

then (21) becomes

$$\begin{bmatrix} \dot{x}_3(t) \\ \dot{y}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \Theta \end{bmatrix} + \mathcal{O}(\| X_3 \|^3) \tag{22}$$

where $\Theta = 1.5392x_3^2 - 2.1296y_3^2$, $X_3 = (x_3, y_3)^{\rm T}$.

Therefore, (x_0, y_0) is a codimension 2 cusp of (19). Moreover, after tedious computation, which is analogous to the above transformation, we can show that under certain conditions as in [17], r and m can be chosen as the bifurcation parameters such that (19) undergoes the Bogdanov-Takens bifurcation.

6 Conclusions and discussions

This paper has discussed the dynamics and control technique of a singular biological economic model, which is described by a differential-algebraic equation. It is shown that when the economic profit increases through zero, there exists the transcritical bifurcation, the Hopf bifurcation and the limit cycle near the regular equilibria. Specifically, near the singular point there exists the singularity induced bifurcation. Then the feedback controllers, which can eliminate the impulse and stabilize the bifurcation orbits, are presented. Biologically, this means that the constrained system behavior may become unpredictable, which can result in an imbalance of the ecosystem, and when proper harvesting policy and population density control strategy are adopted, the rapid expansion of the population and the subsequent disaster can be controlled.

Generally, when the biological system is described by a partial differential equation (PDE), the above theory should also be applicable for the modeling of biological economic system. If the system, described by a PDE and an algebraic equation, can be reduced to a PDE, then the analysis and the control can be carried out using the existing methods. The theoretical research for the system described by PDEs and an algebraic equation will be a new topic of future research.

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Ning Li received the B. Sc. and M. Sc. degrees in mathematics from Liaoning Normal University, PRC in 2001 and 2004, respectively, and the Ph. D. degree in control theory and control engineering from Northeastern University, PRC in 2010. Her research interests include bifurcations, chaos and the control in biological systems, and complex networks.

E-mail: lining80@163.com (Corresponding author)

Hai-Yi Sun received the M. Sc. degree in operational research and cybernetics from Northeastern University, PRC in 2006, and the Ph. D. degree in control theory and control engineering from Northeastern University, PRC in 2011. He is an associate professor of Shenyang Jianzhu University. His research interests include synchronization of chaos and complex networks. E-mail: shy_xx@163.com

Qing-Ling Zhang received the B. Sc. and M. Sc. degrees from the Mathematics Department and the Ph. D. degree from the Automatic Control Department of Northeastern University, PRC in 1982, 1986, and 1995, respectively. He is a professor of Northeastern University, PRC and a member of the University Teaching Advisory Committee of National Ministry of Education. His research interests include chaos control, robust control, networked control system, and singular system. E-mail: qlzhang@mail.neu.edu.cn