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Statistic PID Tracking Control for Non-Gaussian Stochastic Systems Based on T-S Fuzzy Model

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Abstract: A new robust proportional-integral-derivative (PID) tracking control framework is considered for stochastic systems with non-Gaussian variable based on B-spline neural network approximation and T-S fuzzy model identification. The tracked object is the statistical information of a given target probability density function (PDF), rather than a deterministic signal. Following B-spline approximation to the integrated performance function, the concerned problem is transferred into the tracking of given weights. Different from the previous related works, the time delay T-S fuzzy models with the exogenous disturbances are applied to identify the nonlinear weighting dynamics. Meanwhile, the generalized PID controller structure and the improved convex linear matrix inequalities (LMI) algorithms are proposed to fulfil the tracking problem. Furthermore, in order to enhance the robust performance, the peak-to-peak measure index is applied to optimize the tracking performance. Simulations are given to demonstrate the efficiency of the proposed approach.

Keywords: Non-Gaussian systems, probability density function, statistic tracking control, T-S fuzzy model, proportional-integral-derivative control.

1 Introduction

Non-Gaussian variables exist in many complex stochastic systems due to the nature of the random input sources and system nonlinearity, which may possess asymmetric and multiple-peak stochastic distributions[1]. For non-Gaussian systems, mean and variance are insufficient to characterize the stochastic properties. On the other hand, motivated by several typical examples in practical systems, a group of control strategies that control the shape of the output probability density function (PDF) for general stochastic systems have been developed[1−8]. This kind of problem has been called the stochastic distribution control (SDC)[1, 2]. In this paper, we present a new type of stochastic tracking control framework for the non-Gaussian systems, called statistic tracking control (STC). Different from either the conventional stochastic tracking[9, 10] or SDC problem, the goal of control here is to ensure that the statistical information of the system output is made to follow that of a target PDF. The main results obtained in this paper have two features. First, since the mean and the variance are two commonly used control objectives for the Gaussian systems, our control objective is a reasonable generalization for the non-Gaussian systems. Second, technically the application of statistic tracking will eliminate the constraint resulting from the B-spline expansions for the output PDFs.

Currently, a well-known T-S fuzzy model[11] is recognized as a popular and powerful tool in approximating a complex nonlinear system. Recently, the analysis and synthesis of the T-S fuzzy model have been involved more complex nonlinear models, for example, the descriptor system[12], the time-delay model[13, 14], the stochastic system[15], and so on. So far, various techniques have been developed for the stability analysis of T-S fuzzy systems (see [13, 15, 16]). On the other hand, the nonlinear tracking control problems have been considered through the T-S fuzzy model as well[17−20]. In [19], the feedback linearization technique was proposed to design a fuzzy tracking controller. Variable structure control has been applied to solve the tracking problems in [17]. An observer-based fuzzy controller was developed in [20] to reduce the tracking errors based on linear matrix inequalities (LMI) approach.

Similar to the PDF tracking control problem, after using the B-spline approximation theory for the performance function, it is shown that the STC problem can be transformed into a tracking problem for the weight dynamics. Different from the previous works in [1, 2, 7, 8], the T-S fuzzy models are first utilized to describe the nonlinear weight dynamics that cannot be exactly modeled, which represents a significant extension to the previous results. In this paper, a robust tracking problem is studied for the more complex T-S fuzzy weight model where there exist non-zero equilibrium, time delay term, and exogenous disturbances. Meanwhile, a generalized proportional-integral-derivative (PID) controller can be obtained through improved LMI algorithms such that the stability, tracking performance, and robustness are guaranteed simultaneously. Furthermore, in order to enhance the robustness, the peak-to-peak measure is applied to optimize the tracking performance, which generalizes the corresponding result for linear systems with zero equilibrium in [21].

In the following, for a square matrix M, we denote $sym(M) = M + M^{\mathrm{T}}$. The norms $\|\cdot\|$ of a real vector function and a matrix are defined as their Euclidean norms. For a vector $V(t)$, it is denoted that $\|V(t)\| = \sqrt{V^{\mathrm{T}}(t)V(t)}$ and $\|V(t)\|_{\infty} = \sup_{t\geqslant 0}\sqrt{V^{\mathrm{T}}(t)V(t)}$.

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2 Formulation of STC problem

For a dynamic stochastic system, denote $u(t) \in \mathbf{R}^m$ as the control input, $\eta(t) \in [a, b]$ as the stochastic output, whose conditional PDF is denoted by $\gamma(y, u(t))$, where y is the variable in the sample interval $[\alpha, \beta]$. It is noted that the PDF $\gamma(y, u(t))$ is a dynamic function of y along with time variable t. In previous works, the PDF tracking problem has been studied with some effective design algorithms (see [1, 2, 7, 8]) where the B-spline expansions are used to approximate $\gamma(y, u(t))$ or $\sqrt{\gamma(y, u(t))}$ and the control objective is to find $u(t)$ such that $\gamma(y, u(t))$ is convergent to the target PDF $g(y)$ (e.g., [8]).

The idea results from a simple observation. It is well known that mean and variance can characterize the stochastic property of a Gaussian variable. Generally, the moments from the lower order to higher order can decide the shape of a non-Gaussian PDF. In addition, entropy has also been widely used in communication and control theories as a measure for the average information contained in a given PDF of a stochastic variable[3]. Thus, the PDF tracking can be achieved via the tracking of the above proper statistical information, which has motivated the so-called STC problem. To illustrate the design algorithm, in this paper, we consider a special STC problem. The considered performance index is $\int_\alpha^\beta \delta(\gamma, u(t))\mathrm{d}y$, in which

$$\delta(\gamma, u(t)) = Q_1\gamma(y, u(t))\ln(\gamma(y, u(t))) + Q_2 y\gamma(y, u(t)) \quad (1)$$

where Q_1 and Q_2 are two parameters. In (1) the integral of the first term is the entropy, and that of the second one is the mean of the output variable.

Based on the previous PDF control theory, we construct the B-spline expansions to approximate $\delta(\gamma, u(t))$ as follows

$$\delta(\gamma, u(t)) = C(y)V(t) \quad (2)$$

$$C(y) = [B_1(y) \cdots B_n(y)], \quad V(t) = [v_1(t) \cdots v_n(t)]^{\mathrm{T}} \quad (3)$$

where $B_i(y)$ $(i = 1, 2, \cdots, n)$ are pre-specified basis functions, and $v_i(u(t)) = v_i(t)$ $(i = 1, 2, \cdots, n)$ are the corresponding weights. Based on (1) and (2), for the target PDF (in many cases it can be (but not required) a Gaussian one), we can find the corresponding weights, which can be denoted as $V_g(t)$. That is, $\delta(g, u(t)) = C(y)V_g(t)$. The tracking objective is to find $u(t)$ such that $\delta(\gamma, u(t)) - \delta(g, u(t)) = C(y)e(t)$ converges to 0, where $e(t) = V(t) - V_g(t)$.

3 PID controller design based on T-S fuzzy weight model

Once B-spline expansions have been made for the integrated performance function in the definition of the entropy and mean, the next step is to find the dynamic relationships between the control input and the weight vectors related to the integrated function. However, most published results only concerned linear precise models[1, 7], while practically the relationships from control input $u(t)$ to weight vectors $V(t)$ should be nonlinear dynamics. The nonlinear models are actually difficult to obtain through traditional identification approaches.

Recently, the T-S fuzzy model has been proved to be a very good representation for a certain class of nonlinear dynamic systems in control systems and signal processing. Therefore, we consider the nonlinear weighting dynamics which could be described by the following T-S fuzzy model.

Plant rule i. If $\theta_1(t)$ is M_{i1} and $\cdots$ and $\theta_p(t)$ is M_{ip}, then

$$\begin{aligned}\dot{V}(t) =& A_{0i}V(t) + F_{0i}V_\tau(t) + B_{01i}u(t) + \\ & B_{02i}u_\tau(t) + E_{0i}w(t).\end{aligned} \quad (4)$$

where $u(t)$ and $w(t)$ represent the control input and the exogenous perturbation, respectively. $V_\tau(t) = V(t - \tau(t))$ represents the time delay weight vectors, and $u_\tau(t) = u(t - \tau(t))$ is the control input with time-delay term. $A_{0i}, F_{0i}, B_{01i}, B_{02i}$, and E_{0i} are known coefficient matrices with compatible dimensions. $\theta_j(t)$ and M_{ij} $(i = 1, \cdots, r,\ j = 1, \cdots, p)$ are respectively the premise variables and the fuzzy sets, r is the number of the If-Then rules, and p is the number of the premise variables. The time-varying delays $\tau(t)$ satisfy $0 < \dot{\tau}(t) < \beta < 1$, where β is a known positive constant.

By fuzzy blending, the fuzzy model is inferred as

$$\begin{aligned}\dot{V}(t) =& \frac{1}{\sum_{i=1}^r \omega_i(\theta(t))}[\sum_{i=1}^r \omega_i(\theta(t))(A_{0i}V(t) + F_{0i}V_\tau(t) + \\ & B_{01i}u(t) + B_{02i}u_\tau(t) + E_{0i}w(t))] = \\ & \sum_{i=1}^r h_i(\theta(t))(A_{0i}V(t) + F_{0i}V_\tau(t) + B_{01i}u(t) + \\ & B_{02i}u_\tau(t) + E_{0i}w(t))\end{aligned} \quad (5)$$

where $\theta(t) = [\theta_1(t), \cdots, \theta_p(t)]$, and

$$\omega_i(\theta(t)) = \prod_{j=1}^p M_{ij}(\theta_j(t)), h_i(\theta(t)) = \frac{\omega_i(\theta(t))}{\sum_{i=1}^r \omega_i(\theta(t))} \quad (6)$$

in which $M_{ij}(\theta_j(t))$ is the grade of membership of $\theta_j(t)$ in M_{ij}. Some basic properties are $\omega_i(\theta(t) \geqslant 0$ and $\sum_{i=1}^r \omega_i(\theta(t)) > 0$. It is obvious that $h_i(\theta(t)) \geqslant 0$ and $\sum_{i=1}^r h_i(\theta(t)) = 1$ can be satisfied.

Based on the above mentioned T-S fuzzy model (6), we introduce a new state variable $x(t) = [V^{\mathrm{T}}(t), \int_0^t e^{\mathrm{T}}(\tau)\mathrm{d}\tau, \dot{V}^{\mathrm{T}}(t)]^{\mathrm{T}}$. Then, the following augmented system with disturbance $w(t)$ and reference input $V_g(t)$ can be established as a descriptor form

$$\begin{cases} E\dot{x}(t) = \sum\limits_{i=1}^r h_i(\theta(t))(A_i x(t) + F_i x_\tau(t) + B_{1i}u(t) + \\ \qquad B_{2i}u_\tau(t) + E_i w(t) + HV_g) \\ z(t) = \sum\limits_{i=1}^r h_i(\theta(t))(C_i x(t) + D_i w(t)) \\ x(t) = \phi(t), \qquad t \in [-\tau(t), 0] \end{cases} \quad (7)$$

where $z(t)$ is the controller output, and

$$A_i = \begin{bmatrix} 0 & 0 & I \\ I & 0 & 0 \\ A_{0i} & 0 & -I \end{bmatrix}, \quad F_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F_{0i} & 0 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 \\ -I \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_i = \begin{bmatrix} 0 \\ 0 \\ E_{0i} \end{bmatrix}, \quad B_{ki} = \begin{bmatrix} 0 \\ 0 \\ B_{0ki} \end{bmatrix}, \quad k = 1, 2.$$

The classical PID control is unavailable for PDF tracking since $e(y,t)$ cannot be used for feedback[8]. As a result, we adopt the following generalized PID control structure.

Plant rule j. If θ_1 is M_{j1} and $\cdots$ and θ_p is M_{jp}, then

$$u(t) = \sum_{j=1}^{r} h_i(\theta(t))(K_{Pj}V(t) + K_{Ij}\int_0^t e(\tau)\mathrm{d}\tau + K_{Dj}\dot{V}(t)) \tag{8}$$

where K_{Pj}, K_{Ij}, and K_{Dj} are controller gains to be determined.

With such an augmented system (7), the tracking problem can be further reduced to a stabilization framework because the PID controller described by (8) can be formulated as

$$u(t) = \sum_{j=1}^{r} h_j(\theta(t))[K_j x(t)]$$
$$K_j = [K_{Pj} \quad K_{Ij} \quad K_{Dj}]. \tag{9}$$

Remark 1. Based on previous results in [1, 3, 7, 8], PDF $\gamma(y, u(t))$ is assumed measurable, so the statistic performance function $\delta(y, u(t))$ is also measurable. Meanwhile, because $C(y)$ is the pre-specified basis function, the weight vectors $V(t)$ are measurable. Furthermore, the derivative of $V(t)$ in (8) is also measurable. Similar to the PID controller structure in [7, 22, 23], once the following convex LMIs are obtained, the PID controller gains are ready in hand and no additional computation is needed.

Remark 2. It should be pointed out that although some nonlinear tracking approaches have been provided in the last decade, few results can be applied to the above T-S fuzzy model[17, 20]. They do not consider non-zero equilibrium and exogenous disturbance simultaneously. On the other hand, it is noted that weight modeling procedures may result in modeling errors which were also ignored previously[1, 7, 8]. In the following, an improved PID controller will be obtained through solving improved LMIs such that stability, tracking performance, and robustness can be guaranteed simultaneously.

4 Stability analysis with peak-to-peak performance

L_1 performance index is the measure used to describe the level of disturbance attenuation, which is also called peak-to-peak performance[21].

Definition 1. The peak-to-peak control gain for a nonlinear system is defined by $\sup_{\|w\|_\infty \leqslant 1}\|z(t)\|_\infty$. The peak-to-peak control problem is to find controller $u(t)$ such that the peak-to-peak gain is minimized or satisfies

$$\sup_{\|w\|_\infty \leqslant 1} \|z(t)\|_\infty < \gamma \tag{10}$$

or $\sup_{0\leqslant\|w\|\leqslant\infty}\|z(t)\|_\infty/\|w(t)\|_\infty < \gamma$, where $\gamma > 0$ is a given constant. Since $V_g(t)$ can be seen as a known vector, we denote $y_d = \|V_g(t)\|^2$. The following result provides a criterion for the L_1 performance problem of the unforced system of (7).

Theorem 1. For the known parameters μ_i $(i = 1, 2, 3)$, $\alpha > 0$, and $\gamma > 0$, if there exist matrices $U > 0$, $T > 0$, and P such that the matrix inequalities (11)–(13) at the bottom are solvable, then the unforced system of (7) (set $u(t) = 0$) is stable, and $\sup_{0\leqslant\|w\|\leqslant\infty}\|z(t)\|_\infty/\|w(t)\|_\infty < \gamma$ holds.

Proof. Defining a Lyapunov-Krasovskii function as

$$S_1(x(t), t) = x^{\mathrm{T}}(t)P^{\mathrm{T}}Ex(t) + \int_{t-\tau(t)}^{t} x^{\mathrm{T}}(\beta)Ux(\beta)\mathrm{d}\beta. \tag{14}$$

Obviously, it is noted that $S_1(x(t), t) \geqslant 0$. Furthermore, it can be seen that

$$\begin{aligned}
\dot{S}_1 = {} & 2x^{\mathrm{T}}(t)P^{\mathrm{T}}E\dot{x}(t) + x^{\mathrm{T}}(t)Ux(t) - (1-\dot{\tau}(t))x_\tau^{\mathrm{T}}(t)Ux_\tau(t) = \\
& \sum_{i=1}^{r} h_i(\theta)x^{\mathrm{T}}(t)(sym(P^{\mathrm{T}}A_i) + U)x(t) + 2x^{\mathrm{T}}(t)P^{\mathrm{T}}HV_g - \\
& (1-\dot{\tau}(t))x_\tau^{\mathrm{T}}(t)Ux_\tau(t) + 2\sum_{i=1}^{r} h_i(\theta)x^{\mathrm{T}}(t)P^{\mathrm{T}}F_i x_\tau(t) + \\
& 2\sum_{i=1}^{r} h_i(\theta)x^{\mathrm{T}}(t)P^{\mathrm{T}}E_i w(t) \leqslant \\
& \sum_{i=1}^{r} h_i x^{\mathrm{T}}(t)(\Upsilon_i + \frac{1}{\mu_2^2}P^{\mathrm{T}}E_iE_i^{\mathrm{T}}P + \frac{1}{\mu_3^2}P^{\mathrm{T}}HH^{\mathrm{T}}P)x(t) + \\
& 2\sum_{i=1}^{r} h_i(\theta)x^{\mathrm{T}}(t)P^{\mathrm{T}}F_i x_\tau(t) - (1-\dot{\tau}(t))x_\tau^{\mathrm{T}}(t)Ux_\tau(t) + \\
& \|\mu_2 w(t)\|^2 + \|\mu_3 V_g(t)\|^2 \leqslant \\
& \sum_{i=1}^{r} h_i(\theta)\zeta^{\mathrm{T}}(t)\Phi_i\zeta(t) + \|\mu_2 w(t)\|^2 + \mu_3^2 y_d
\end{aligned} \tag{15}$$

$$P^{\mathrm{T}}E = EP \geqslant 0, \quad \begin{bmatrix} sym(A_i^{\mathrm{T}}P) + \mu_1^2 T + U & P^{\mathrm{T}}F_i & P^{\mathrm{T}}E_i & P^{\mathrm{T}}H \\ F_i^{\mathrm{T}}P & -(1-\beta)U & 0 & 0 \\ E_i^{\mathrm{T}}P & 0 & -\mu_2^2 I & 0 \\ H^{\mathrm{T}}P & 0 & 0 & -\mu_3^2 I \end{bmatrix} < 0, \quad i = 1, 2, \cdots, r \tag{11}$$

$$\begin{bmatrix} \mu_1^2 T & 0 & \frac{1}{2}(C_i^{\mathrm{T}} + C_j^{\mathrm{T}}) \\ 0 & (\gamma - \mu_2^2 - \mu_3^2 y_d)I & \frac{1}{2}(D_i^{\mathrm{T}} + D_j^{\mathrm{T}}) \\ \frac{1}{2}(C_i + C_j) & \frac{1}{2}(D_i + D_j) & \gamma I \end{bmatrix} > 0, \quad i, j = 1, 2, \cdots, r \tag{12}$$

$$\begin{bmatrix} \alpha I & T \\ T & T \end{bmatrix} > 0, \quad \begin{bmatrix} T & 0 & \frac{1}{2}(C_i^{\mathrm{T}} + C_j^{\mathrm{T}}) \\ 0 & (\gamma - \alpha x_m^{\mathrm{T}} x_m)I & \frac{1}{2}(D_i^{\mathrm{T}} + D_j^{\mathrm{T}}) \\ \frac{1}{2}(C_i + C_j) & \frac{1}{2}(D_i + D_j) & \gamma I \end{bmatrix} > 0, \quad i, j = 1, 2, \cdots, r. \tag{13}$$

where

$$\Phi_i = \begin{bmatrix} \Upsilon_i + \frac{1}{\mu_2^2}P^{\mathrm{T}}E_iE_i^{\mathrm{T}}P + \frac{1}{\mu_3^2}P^{\mathrm{T}}HH^{\mathrm{T}}P & P^{\mathrm{T}}F_i \\ F_i^{\mathrm{T}}P & -(1-\beta)U \end{bmatrix}$$

$$\zeta(t) = [x^{\mathrm{T}}(t),\ \ x_\tau^{\mathrm{T}}(t)]^{\mathrm{T}}, \quad \Upsilon_i = A_i^{\mathrm{T}}P + P^{\mathrm{T}}A_i + U. \quad (16)$$

Based on Schur complement formula, (11) implies that for any $w(t)$ satisfying $\|w(t)\|_\infty \leqslant 1$, $\Phi_i < -\mu_1^2 T$ holds. With (15), it can be seen that

$$\frac{\mathrm{d}S_1(x(t),t)}{\mathrm{d}t} \leqslant -\mu_1^2 x^{\mathrm{T}}(t)Tx(t) + \mu_2^2 + \mu_3^2 y_d \quad (17)$$

where $\mu_3^2 y_d$ can be considered as a known parameter.

Thus, $\mathrm{d}S_1(x(t),t)/\mathrm{d}t < 0$, if $x^{\mathrm{T}}(t)Tx(t) > \mu_1^{-2}(\mu_2^2+\mu_3^2 y_d)$ holds.

Thus, for any $x(t)$, it can be verified that

$$x^{\mathrm{T}}(t)Tx(t) \leqslant \max\{x_m^{\mathrm{T}}Tx_m, \mu_1^{-2}(\mu_2^2+\mu_3^2 y_d)\}$$

$$\|x_m\| = \sup_{-\tau(t)\leqslant t\leqslant 0}\|x(t)\| \quad (18)$$

which also implies that the unforced system of (7) is stable.

From (18), we can get that $x^{\mathrm{T}}(t)Tx(t) \leqslant x_m^{\mathrm{T}}Tx_m$ or $x^{\mathrm{T}}(t)Tx(t) \leqslant \mu_1^{-2}(\mu_2^2+\mu_3^2 y_d)$.

Defining $\eta(t) = [x^{\mathrm{T}}(t), w^{\mathrm{T}}(t)]^{\mathrm{T}}$, $H_i = [C_i, D_i]$, and $H_j = [C_j, D_j]$, we have

$$\|z(t)\|^2 = \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j(C_ix(t)+D_iw(t))^{\mathrm{T}}(C_jx(t)+D_jw(t))$$

$$= \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\eta^{\mathrm{T}}(t)(H_i^{\mathrm{T}}H_j + H_j^{\mathrm{T}}H_i)\eta(t) \leqslant$$

$$\frac{1}{4}\sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\eta^{\mathrm{T}}(H_i+H_j)^{\mathrm{T}}(H_i+H_j)\eta. \quad (19)$$

From (12), it can be seen that

$$\begin{bmatrix} \mu_1^2 T & 0 \\ 0 & (\gamma-\mu_2^2-\mu_3^2 y_d)I \end{bmatrix} -$$

$$\frac{1}{4\gamma}\begin{bmatrix} C_i^{\mathrm{T}}+C_j^{\mathrm{T}} \\ D_i^{\mathrm{T}}+D_j^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} C_i+C_j & D_i+D_j \end{bmatrix} > 0$$

which guarantees that under $x^{\mathrm{T}}(t)Tx(t) \leqslant \mu_1^{-2}(\mu_2^2+\mu_3^2 y_d)$ and $\|w(t)\|_\infty \leqslant 1$, we can get

$$\frac{1}{\gamma}\|z(t)\|^2 < \mu_1^2 x^{\mathrm{T}}(t)Tx(t) + (\gamma-\mu_2^2-\mu_3^2 y_d)w^{\mathrm{T}}(t)w(t) <$$

$$(\mu_2^2+\mu_3^2 y_d) + (\gamma-\mu_2^2-\mu_3^2 y_d)w^{\mathrm{T}}(t)w(t) = \gamma.$$

On the other hand, from (13), it can also be shown that

$$\begin{bmatrix} T & 0 \\ 0 & (\gamma-\alpha x_m^{\mathrm{T}}x_m)I \end{bmatrix} -$$

$$\frac{1}{4\gamma}\begin{bmatrix} C_i^{\mathrm{T}}+C_j^{\mathrm{T}} \\ D_i^{\mathrm{T}}+D_j^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} C_i+C_j & D_i+D_j \end{bmatrix} > 0.$$

Similar to the above proof, under $x^{\mathrm{T}}(t)Tx(t) \leqslant x_m^{\mathrm{T}}Tx_m$ and $\|w(t)\|_\infty \leqslant 1$, we can get

$$\frac{1}{\gamma}\|z(t)\|^2 < x^{\mathrm{T}}(t)Tx(t) + (\gamma-\alpha x_m^{\mathrm{T}}x_m)w^{\mathrm{T}}(t)w(t) <$$

$$\alpha x_m^{\mathrm{T}}x_m + (\gamma-\alpha x_m^{\mathrm{T}}x_m)w^{\mathrm{T}}(t)w(t) = \gamma.$$

Hence, the L_1 norm of the unforced system is less than γ. □

5 Peak-to-peak tracking performance

Considering the state feedback controller with PID structure, and substituting $u(t) = \sum_{j=1}^{r} h_j(\theta)[K_jx(t)]$ into (7), we can describe the corresponding nonlinear closed-loop system as

$$\begin{cases} E\dot{x}(t) = \sum\limits_{i=1}^{r} h_i(\theta)\sum\limits_{j=1}^{r} h_j(\theta)[(A_i+B_{1i}K_j)x(t)+ \\ \qquad (F_i+B_{2i}K_j)x_\tau(t) + E_iw(t) + HV_g(t)] \\ z(t) = \sum\limits_{i=1}^{r} h_i(\theta)[C_ix(t)+D_iw(t)] \end{cases} \quad (20)$$

where the coefficient matrices are given in (7). The following result provides a solution to the considered nonlinear tracking control problem with disturbance attenuation performance.

Theorem 2. For the known parameters $\mu_i\ (i=1,2,3)$, $\alpha > 0$, and $\gamma > 0$, if there exist matrices $U > 0$, $M = T^{-1} > 0$, and $Q = P^{-\mathrm{T}}$ such that the following matrix inequality (21)–(23) are solvable, then the closed-loop system (20) is stable and satisfies both $\lim_{t\to\infty} V(t) = V_g(t)$ and

$$EQ^{\mathrm{T}} = QE \geqslant 0, \quad \Theta_{ii} < 0, \quad i=1,\cdots,r, \quad \Theta_{ij}+\Theta_{ji} \leqslant 0, \quad i<j, \quad i,j=1,\cdots,r \quad (21)$$

$$\Theta_{ij} = \begin{bmatrix} sym(A_iQ^{\mathrm{T}}) + sym(B_{1i}R_j) + QUQ^{\mathrm{T}} & F_iQ^{\mathrm{T}} + B_{2i}R_j & E_i & H & Q \\ QF_i^{\mathrm{T}} + R_j^{\mathrm{T}}B_{2i}^{\mathrm{T}} & -(1-\beta)QUQ^{\mathrm{T}} & 0 & 0 & 0 \\ E_i^{\mathrm{T}} & 0 & -\mu_2^2 I & 0 & 0 \\ H^{\mathrm{T}} & 0 & 0 & -\mu_3^2 I & 0 \\ Q^{\mathrm{T}} & 0 & 0 & 0 & -\mu_1^{-2}M \end{bmatrix}$$

$$\begin{bmatrix} \mu_1^2 M & 0 & \frac{1}{2}(MC_i^{\mathrm{T}}+MC_j^{\mathrm{T}}) \\ 0 & (\gamma-\mu_2^2-\mu_3^2 y_d)I & \frac{1}{2}(D_i^{\mathrm{T}}+D_j^{\mathrm{T}}) \\ \frac{1}{2}(C_iM+C_jM) & \frac{1}{2}(D_i+D_j) & \gamma I \end{bmatrix} > 0, \quad i,j=1,\cdots,r \quad (22)$$

$$\begin{bmatrix} \alpha I & I \\ I & M \end{bmatrix} > 0, \quad \begin{bmatrix} M & 0 & \frac{1}{2}(MC_i^{\mathrm{T}}+MC_j^{\mathrm{T}}) \\ 0 & (\gamma-\alpha x_m^{\mathrm{T}}x_m)I & \frac{1}{2}(D_i^{\mathrm{T}}+D_j^{\mathrm{T}}) \\ \frac{1}{2}(C_iM+C_jM) & \frac{1}{2}(D_i+D_j) & \gamma I \end{bmatrix} > 0, \quad i,j=1,\cdots,r. \quad (23)$$

$\sup_{0\leqslant\|w\|\leqslant\infty}\|z(t)\|_\infty/\|w(t)\|_\infty < \gamma$. In this case, the PID control gain K_j can be solved via $R_j = K_jQ^{\mathrm{T}}$.

Proof. Based on Theorem 1 and Lyapunov-Krasovskii function (14), we can get

$$\begin{aligned}\dot S_1 = {} & 2x^{\mathrm{T}}(t)P^{\mathrm{T}}E\dot x(t)+x^{\mathrm{T}}(t)Ux(t)-(1-\dot\tau(t))x_\tau^{\mathrm{T}}(t)Ux_\tau(t) = \\ & \sum_{i=1}^r h_i\sum_{j=1}^r h_jx^{\mathrm{T}}(t)(\Upsilon_i+P^{\mathrm{T}}B_{1i}K_j+K_j^{\mathrm{T}}B_{1i}^{\mathrm{T}}P)x(t)+ \\ & 2\sum_{i=1}^r h_ix^{\mathrm{T}}(t)P^{\mathrm{T}}F_ix_\tau(t)-(1-\dot\tau(t))x_\tau^{\mathrm{T}}(t)Ux_\tau(t)+ \\ & 2\sum_{i=1}^r h_i\sum_{j=1}^r h_jx^{\mathrm{T}}(t)P^{\mathrm{T}}B_{2i}K_jx_\tau(t)+2x^{\mathrm{T}}(t)P^{\mathrm{T}}HV_g+ \\ & 2\sum_{i=1}^r h_ix^{\mathrm{T}}(t)P^{\mathrm{T}}E_iw(t)\leqslant \\ & \sum_{i=1}^r h_i\sum_{j=1}^r h_j\zeta^{\mathrm{T}}(t)\Omega_{ij}\zeta(t)+\|\mu_2w(t)\|^2+\mu_3^2y_d = \\ & \sum_{i=1}^r h_i^2\zeta^{\mathrm{T}}\Omega_{ii}\zeta+\sum_{i=1}^{r-1}\sum_{j=i+1}^r h_ih_j\zeta^{\mathrm{T}}(\Omega_{ij}+\Omega_{ji})\zeta+ \\ & \|\mu_2w(t)\|^2+\mu_3^2y_d\end{aligned} \tag{24}$$

where $\zeta(t) = [x^{\mathrm{T}}(t),\ \ x_\tau^{\mathrm{T}}(t)]^{\mathrm{T}}$ and

$$\Omega_{ij} = \begin{bmatrix} \Xi_{ij} & P^{\mathrm{T}}F_i+P^{\mathrm{T}}B_{2i}K_j \\ F_i^{\mathrm{T}}P+K_j^{\mathrm{T}}B_{2i}^{\mathrm{T}}P & -(1-\beta)U\end{bmatrix}$$

$$\begin{aligned}\Xi_{ij} = {} & \Upsilon_i+sym(K_j^{\mathrm{T}}B_{1i}^{\mathrm{T}}P)+\frac{1}{\mu_2^2}P^{\mathrm{T}}E_iE_i^{\mathrm{T}}P+ \\ & \frac{1}{\mu_3^2}P^{\mathrm{T}}HH^{\mathrm{T}}P.\end{aligned} \tag{25}$$

Based on Schur complement formula, by pre-multiplying both sides of Θ_{ij} by $\mathrm{diag}\{P^{\mathrm{T}},P^{\mathrm{T}},I,I,I\}$ and post-multiplying both sides of Θ_{ij} by $\mathrm{diag}\{P,P,I,I,I\}$, we can get

$$\Theta_{ij}<0\Leftrightarrow\Omega_{ij}\leqslant\begin{bmatrix}-\mu_1^2T & 0\\ 0 & 0\end{bmatrix}.$$

So for any $w(t)$ satisfying $\|w(t)\|_\infty\leqslant 1$, it can be seen that

$$\begin{aligned}\frac{\mathrm{d}S_1}{\mathrm{d}t}\leqslant & -\mu_1^2\sum_{i=1}^r h_i^2x^{\mathrm{T}}Tx-2\mu_1^2\sum_{i=1}^{r-1}\sum_{j=i+1}^r h_ih_jx^{\mathrm{T}}Tx+\mu_2^2+ \\ & \mu_3^2y_d\leqslant-\mu_1^2x^{\mathrm{T}}(t)Tx(t)+\mu_2^2+\mu_3^2y_d.\end{aligned} \tag{26}$$

Similar to the proof of Theorem 1, it can be seen that (18) still holds for the closed-loop system, which implies that (20) is still stable in the presence of $w(t)$ and V_g. Meanwhile, by pre-multiplying both sides of (22, 23) by $\mathrm{diag}\{T,I,I\}$ and $\mathrm{diag}\{I,T\}$ and post-multiplying both sides of (22, 23) by $\mathrm{diag}\{T,I,I\}$ and $\mathrm{diag}\{I,T\}$, it is easy to show that (22, 23) is equivalent to (12, 13) so that the closed-loop system (20) satisfies the peak-to-peak disturbance attenuation performance.

For a couple of $w(t)$ and $V_g(t)$, we suppose that $v_1(t)$ and $v_2(t)$ are two trajectories of the closed-loop system corresponding to a fixed initial condition. If $\sigma(t) = v_1(t)-v_2(t)$, then the dynamics for $\sigma(t)$ can be described as

$$\dot\sigma(t) = \sum_{i=1}^r h_i\sum_{j=1}^r h_j[(A_i+B_{1i}K_j)\sigma(t)+(F_i+B_{2i}K_j)\sigma_\tau(t)]. \tag{27}$$

Similar to (14), Lyapunov-Krasovskii function can be constructed as

$$S_2(\sigma(t),t) = \sigma^{\mathrm{T}}(t)P^{\mathrm{T}}E\sigma(t)+\int_{t-\tau(t)}^t\sigma^{\mathrm{T}}(\beta)U\sigma(\beta)\mathrm{d}\beta. \tag{28}$$

Inequality (21) implies that $\bar\Theta_{ii}<0$, $\ \bar\Theta_{ij}+\bar\Theta_{ji}\leqslant 0$, where

$$\bar\Theta_{ij} = \begin{bmatrix} sym(P^{\mathrm{T}}\bar A_{ij})+U+\mu_1^2T & P^{\mathrm{T}}(F_i+B_{2i}K_j)\\ (F_i+B_{2i}K_j)^{\mathrm{T}}P & -(1-\beta)U\end{bmatrix} \tag{29}$$

where $\bar A_{ij} = A_i+B_{1i}K_j$. Hence it can be seen that

$$\dot S_2(\sigma(t),t)\leqslant-\mu_1^2\sigma^{\mathrm{T}}(t)T\sigma(t)\leqslant-\mu_1^2\lambda_{\min}(T)\|\sigma(t)\|^2<0 \tag{30}$$

where $\lambda_{\min}(T)$ is denoted as the minimal eigenvalue of T. Thus, (27) is asymptotically stable with respect to $\sigma = 0$. This means that the closed-loop system (20) has also a unique stable equilibrium point. For this purpose, let us assume that x^* is one equilibrium of the descriptor system (20). Thus, we have $\lim_{t\to\infty}\frac{\mathrm{d}}{\mathrm{d}t}(\int_0^t e(\tau)\mathrm{d}\tau) = 0$, which shows that $\lim_{t\to\infty}V(t) = V_g(t)$. □

Remark 3. Using the LMI toolbox in Matlab, the above result shows that the design procedures can be reduced to a class of LMI algorithms with respect to Q and R_j, which is more beneficial than the previous results in [7, 8]. For example, the algorithms provided in [7] required to solve a class of nonlinear matrix inequalities (NLMIs), leading to a non-convex computation procedure.

6 Example

Suppose that the statistic information function of the output PDFs can be approximated using the following B-spline models described by (2) with $n = 3$, $y\in[0,1.5]$, and for $i = 1,2,3$

$$B_i(y) = \begin{cases}|\sin(2\pi y)| & \text{if}\ \ y\in[0.5(i-1);\ 0.5i]\\ 0 & \text{if}\ \ y\in[0.5(j-1);\ 0.5j],\quad i\neq j.\end{cases} \tag{31}$$

For simplicity, in the simulation, it is supposed that the target PDF can be denoted as

$$\gamma_g(y) = \frac{1}{\sqrt{2\pi}b}\exp(-\frac{(y-a)^2}{2b^2}) \tag{32}$$

where $a = 0.5+\ln\sqrt{2\pi}+1.2/\pi$, and $b = 1$. From (1), it can be obtained that

$$\sum_{i=1}^3(V_{gi}B_i(y)) = Q_1\gamma_g(y)\ln(\gamma_g(y))+Q_2y\gamma_g(y)$$

where $Q_1 = 1$ and $Q_2 = 1$. As a result, we have

$$\sum_{i=1}^3(V_{gi}\int_{-\infty}^{\infty}(B_i(y)\mathrm{d}y)) = (-\frac{1}{2})(1+\ln(2\pi b^2))+a. \tag{33}$$

Owing to $\int_{-\infty}^{\infty} B_i(y)\mathrm{d}y = 1/\pi, \quad i = 1, 2, 3$, it can be shown that the reference weights corresponding to the target statistical information of the target PDF satisfy the condition $\Sigma_{i=1}^{3}(V_{gi}) = 1.2$. Thus, the reference weights can be denoted as $V_g = [0.2 \;\; 0.4 \;\; 0.6]^{\mathrm{T}}$.

In this example, we choose the following member function:

$$M_i = \frac{\exp(\frac{-(x_2 \pm 1)^2}{\sigma^2})}{\exp(\frac{-(x_2+1)^2}{\sigma^2}) + \exp(\frac{-(x_2-1)^2}{\sigma^2})}, \quad i = 1, 2.$$

Consider a T-S fuzzy model with exogenous perturbation, together with $i = 1, 2$ (the data in simulation is omitted here to save space). Through solving the LMIs (21)–(23), the PID control gains can be computed as follows

$$K_{P1} = \begin{bmatrix} -50.3138 & 11.0482 & -10.8071 \\ -10.3492 & -77.5487 & 38.3444 \\ 13.4593 & -6.8202 & -28.8059 \end{bmatrix}$$

$$K_{I1} = \begin{bmatrix} -44.3963 & 8.2573 & -7.4896 \\ 2.0847 & -73.5738 & 11.7118 \\ 10.4367 & -5.1170 & -25.5716 \end{bmatrix}$$

$$K_{D1} = \begin{bmatrix} -4.4267 & 1.4682 & -0.9093 \\ 0.6915 & -7.3013 & 0.9554 \\ 1.5390 & -1.1110 & -2.5171 \end{bmatrix}$$

$$K_{P2} = \begin{bmatrix} -79.5300 & 8.2593 & 13.5175 \\ -3.3580 & -70.5243 & 37.4151 \\ 9.2556 & 7.5475 & -31.0943 \end{bmatrix}$$

$$K_{I2} = \begin{bmatrix} -68.9999 & 4.3229 & 11.3683 \\ 6.5455 & -67.1733 & 14.8545 \\ 5.3029 & 7.9659 & -25.5485 \end{bmatrix}$$

$$K_{D2} = \begin{bmatrix} -7.4555 & 1.5645 & 0.8604 \\ 1.1436 & -6.9858 & 1.2480 \\ 0.8359 & 0.4422 & -2.4532 \end{bmatrix}.$$

When the PID control input (8) and LMIs (21)–(23) are applied, the T-S fuzzy model responses for the dynamical weight vectors are shown in Fig. 1, where the dotted line, the dashed line, and the solid line stand for the different state vectors, respectively. In Fig. 1, the horizontal axis represents the time while the vertical one represents the value of the weight vectors. Compared with Fig. 2 in [7], the tracking performance can be enhanced by using the proposed method. The control input is shown in Fig. 2. Fig. 3 shows the 3D mesh plot of the statistic performance function with time t. Because the target PDF is a single peak Gaussian function, it can be shown that each states changes continuously with a single peak form and finally achieves stability.

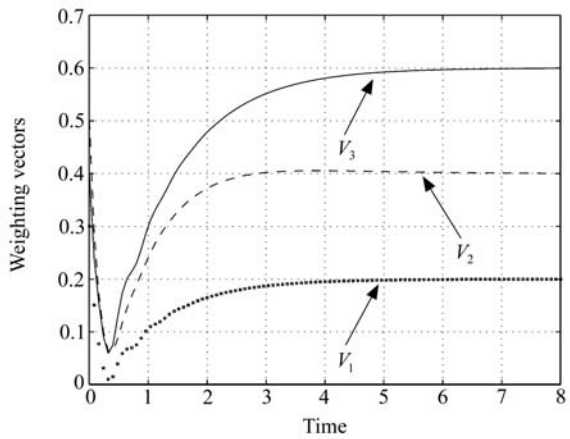

Fig. 1 Responses of the weight vectors

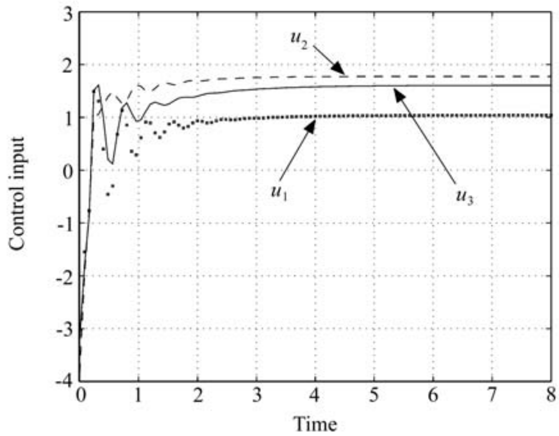

Fig. 2 Control inputs of PID controller

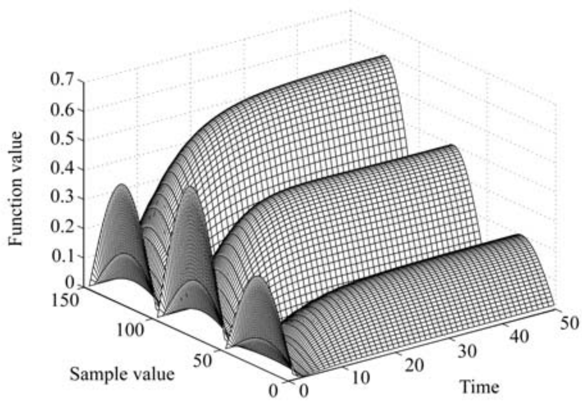

Fig. 3 3D mesh plot of the performance function

7 Conclusions

This paper considers the robust tracking control problem for the statistic information of non-Gaussian processes by using generalized PID controller. Compared with the previous works, the main results here have four features:

1) The T-S fuzzy model, as a system identifier, is firstly applied into STC problem.

2) Exogenous disturbance, non-zero equilibrium, and time-delay are all considered in the T-S fuzzy model tracking control problem.

3) Using the LMI methods, multiple control objectives including stabilization, tracking performance, and robustness can be guaranteed simultaneously.

4) To enhance the robustness, the peak-to-peak measure is applied to optimize the tracking performance.

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